THE METRIZATION OF STATISTICAL METRIC SPACES

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In a previous paper on statistical metric spaces [3] it was shown that a statistical metric induces a natural topology for the space on which it is defined and that with this topology a large class of statistical metric (briefly, *SM*) spaces are Hausdorff spaces.

In this paper we show that this result (Theorem 7.2 of [3]) can be considerably generalized. In addition, as an immediate corollary of this generalization, we prove that with the given topology a large number of *SM* spaces are metrizable, i.e., that in numerous instances the existence of a statistical metric implies the existence of an ordinary metric.[1]

THEOREM 1.[2] *Let $(S, \mathscr{F})$ be a statistical metric space, $\mathscr{U}$ the two-parameter collection of subsets of $S \times S$ defined by*

$$\mathscr{U} = \{U(\varepsilon, \lambda); \varepsilon > 0,\ \lambda > 0\} ,$$

where

$$U(\varepsilon, \lambda) = \{(p, q);\ p, q \text{ in } S \text{ and } F_{pq}(\varepsilon) > 1 - \lambda\} ,$$

and T a two-place function from $[0, 1] \times [0, 1]$ to $[0, 1]$ satisfying $T(c, d) \geqq T(a, b)$ for $c \geqq a$, $d \geqq b$ and $\sup_{x<1} T(x, x) = 1$. Suppose further that for all p, q, r in S and for all real numbers x, y, the Menger triangle inequality.

$$F_{pr}(x + y) \geqq T(F_{pq}(x), F_{qr}(y)) \tag{1}$$

is satisfied. Then $\mathscr{U}$ is the basis for a Hausdorff uniformity on $S \times S$.

Proof. We verify that the $U(\varepsilon, \lambda)$ satisfy the axioms for a basis for a Hausdorff (or separated) uniformity as given in [2; p. 174–180] (or in [1; II, §1, n°1]).

(a) Let $\Delta = \{(p, p); p \in S\}$ and $U(\varepsilon, \lambda)$ be given. Since for any $p \in S$, $F_{pp}(\varepsilon) = 1$, it follows that $(p, p) \in U(\varepsilon, \lambda)$. Thus $\Delta \subset U(\varepsilon, \lambda)$.

(b) Since $F_{pq} = F_{qp}$, $U(\varepsilon, \lambda)$ is symmetric.

(c) Let $U(\varepsilon, \lambda)$ be given. We have to show that there is a $W \in \mathscr{U}$ such that $W \circ W \subset U$. To this end, choose $\varepsilon' = \varepsilon/2$ and λ' so small that $T(1 - \lambda', 1 - \lambda') > 1 - \lambda$. Suppose now that (p, q) and (q, r) belong to

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[1] These considerations have led to the study of *SM* spaces which are not metrizable as well as to other natural topologies for *SM* spaces, questions which will be investigated in a subsequent paper.

[2] The terminology and notation are as in [3].

$W(\varepsilon', \lambda')$ so that $F_{pq}(\varepsilon') > 1 - \lambda'$ and $F_{qr}(\varepsilon') > 1 - \lambda'$. Then, by (1),

$$F_{pr}(\varepsilon) \geqq T(F_{pq}(\varepsilon'), F_{qr}(\varepsilon')) \geqq T(1 - \lambda', 1 - \lambda') > 1 - \lambda .$$

Thus $(p, r) \in U(\varepsilon, \lambda)$. But this means that $W \circ W \subset U$.

(d) The intersection of $U(\varepsilon, \lambda)$ and $U(\varepsilon', \lambda')$ contains a member of $\mathscr{U}$, namely $U(\min(\varepsilon, \varepsilon'), \min(\lambda, \lambda'))$.

Thus $\mathscr{U}$ is the basis for a uniformity on $S \times S$.

(e) If p and q are distinct, there exists an $\varepsilon > 0$ such that $F_{pq}(\varepsilon) \neq 1$ and hence ε_0, λ_0 such that $F_{pq}(\varepsilon_0) = 1 - \lambda_0$. Consequently (p, q) is not in $U(\varepsilon_0, \lambda_0)$ and the uniformity generated by $\mathscr{U}$ is separated, i.e., Hausdorff.

Note that the theorem is true in particular for all Menger spaces in which $\sup_{x<1} T(x, x) = 1$. However, it is true as well for many *SM* spaces which are not Menger spaces.

COROLLARY. *If* $(S, \mathscr{F})$ *is an SM space satisfying the hypotheses of Theorem* 1, *then the sets of the form* $N_p(\varepsilon, \lambda) = \{q; F_{pq}(\varepsilon) > 1 - \lambda\}$ *are the neighborhood basis for a Hausdorff topology on* S.

Proof. These sets are a neighborhood basis for the uniform topology on S derived from $\mathscr{U}$.

THEOREM 2. *If an SM space satisfies the hypotheses of Theorem* 1, *then the topology determined by the sets* $N_p(\varepsilon, \lambda)$ *is metrizable.*

Proof. Let $\{(\varepsilon_n, \lambda_n)\}$ be a sequence that converges to $(0, 0)$. Then the collection $\{U(\varepsilon_n, \lambda_n)\}$ is a countable base for $\mathscr{U}$. The conclusion now follows from [2; p. 186].

Theorem 2 may be restated as follows: Under the hypotheses of Theorem 1, there exist numbers $\delta(p, q)$ which are determined by the distance distribution functions F_{pq} in such a manner that the function δ is an ordinary metric on S. Loosely speaking, if the statistical distances have certain properties, then certain numerical quantities associated with them have the properties of an ordinary distance. In a given particular case such quantities might be the means, medians, modes, etc.. For example, most of the particular spaces studied in [3] satisfy the hypotheses of Theorem 2, hence are metrizable. Indeed, as was shown in [3], in a simple space, the means (when they exist), medians, and modes (if unique) of the statistical distances each form metric spaces; and similarly, in a normal space, the means of the F_{pq} form a (generally discrete) metric space. What Theorem 2 now tells us is that in many (though not all!) *SM* spaces we can expect results of this general nature to hold.

References

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