ON NORMAL NUMBERS

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1. Introduction. A real number ξ, $0 \leqq \xi < 1$, is said to the *normal in the scale of* r (or *to base* r), if in $\xi = 0 \cdot a_1 a_2 \cdots$ expanded in the scale of r[1] every combination of digits occurs with the proper frequency. If $b_1 b_2 \cdots b_k$ is any combination of digits, and Z_N the number of indices i in $1 \leqq i \leqq N$ having

$$b_1 = a_i, \cdots, b_k = a_{i+k-1} ,$$

then the condition is that

$$\lim_{N\to\infty} Z_N N^{-1} = r^{-k} . \tag{1}$$

A number ξ is called *simply normal* in the scale of r if (1) holds for $k = 1$. A number is said to be *absolutely normal* if it is normal to every base r. It is well-known (see, for example, [6], Theorem 8.11) that almost every number ξ is absolutely normal.

We write $r \sim s$, if there exist integers n, m with $r^n = s^m$. Otherwise, we put $r \not\sim s$.

In this paper we solve the following problem. *Under what conditions on r, s is every number ξ which is normal to base r also normal to base s?* The answer is given by

THEOREM 1. *A Assume $r \sim s$. Then any number normal to base r is normal to base s.*

B If $r \not\sim s$, then the set of numbers ξ which are normal to base r but not even simply normal to base s has the power of the continuum.

The A-part of the Theorem is rather trivial, but I shall sketch a proof of it, since I could not find one in the literature.

Next, let I be an interval of length $|I|$ contained in the unit-interval $U = [0, 1]$. We write $M_N(\xi, r, I)$ for the number of indices i in $1 \leqq i \leqq N$ such that the fractional part $\{r^i \xi\}$ of $r^i \xi$ lies I. A sequence $\xi, r\xi, r^2\xi, \cdots$ *has uniform distribution modulo* 1 if

$$R_N(\xi, r, I) = M_N(\xi, r, I) - N|I| = o(N)$$

for any I. It was proved by Wall [8] (the most accessible proof in [6], Theorem 8.15) that ξ is normal to base r if and only if $\xi, r\xi, r^2\xi, \cdots$ has uniform distribution modulo 1.

Write $T_{s,t}$, where $1 < t < s$, for the following mapping in U: If $\xi = 0 \cdot a_1 a_2 \cdots$ in the scale of t, then $T_{s,t}\xi = 0 \cdot a_1 a_2 \cdots$ in the scale of s.

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[1] In case of ambiguity we take the representation with an infinity of a_i less then $r-1$. But this does not affect the property of ξ to be normal or not.

THEOREM 2. *Assume $r \not\sim s$. Then there exists a constant $\alpha_1 = \alpha_1(r, s, t) > 0$ such that for almost every ξ there exists a $N_0(\xi)$ with*

$$R_N(T_{s,t}\xi, r, I) \leqq N^{1-\alpha_1} \tag{2}$$

for every $N \geqq N_0(\xi)$ and any I.

Thus $T_{s,t}\xi$ is normal to base r for almost all ξ. Since $T_{s,t}\xi$ is not simply normal to base s part B of Theorem 1 follows. It does not follow immediately for $s = 2$, but instead of $T_{2,t}$, which does not exist, we may take $T_{4,t}$.

We can interpret our results as follows. Write $C_{s,t}$ for the image set $T_{s,t}U$ of the unit-interval U under the mapping $T_{s,t}$. $C_{s,t}$ is essentially a Cantor set. In $C_{s,t}$ we define a measure $\mu_{s,t}$ by

$$\int_{C_{s,t}} f(\xi)d\mu_{s,t} = \int_0^1 f(T_{s,t}\xi)d\xi \, , \tag{3}$$

where $f(\xi)$ is any real-valued function such that the integral on the right hand side of (3) exists. Then it follows from Theorem 2 that *with respect to $\mu_{s,t}$ almost every ξ in $C_{s,t}$ is normal in the scale of r.*

Throughout this paper, lower case italics stand for integers. $\alpha_1 = \alpha_1(r, s, t), \alpha_2, \alpha_3, \cdots$ will be positive constants depending on some or all the variables r, s, t.

1. The case $r \sim s$. First, it follows almost from the definition that any number normal to base s^n is normal to base s.

Next, assume ξ is normal to base r, we shall show it is normal in the scale of r^m. If $\xi = 0 \cdot a_1 a_2 \cdots$ in the scale of r, $b_1 \cdots b_{mk}$ is any combination of mk digits and $Z_N^{(1)}$ is the number of indices i in $1 \leqq i \leqq N$ with $i \equiv 1 \pmod m$ satisfying

$$b_1 = a_i, \cdots, b_{mk} = a_{i+mk-1} \, ,$$

then it was shown in [7] and in [3] that

$$\lim_{N\to\infty} Z_N^{(1)} N^{-1} = r^{-mk} m^{-1}$$

and hence

$$\lim_{N\to\infty} Z_{mN}^{(1)} N^{-1} = (r^m)^{-k} \, .$$

Thus ξ is normal to base r^m.

Combining the above remarks we obtain the A-part of Theorem 1.

2. The measure $\mu_{s,t}$. We define *numbers of order h* to be the number $0 \cdot a_1 \cdots a_h$ with $0 \leqq a_i < t$ in the scale of s. There are t^h numbers of order h, we denote them in ascending order by $\theta_1^{(h)}, \cdots, \theta_{t^h}^{(h)}$.

LEMMA 1. *Let $f(\xi)$ be a step-function, having a finite number of steps. Then*

$$\int_{C_{s,t}} f(\xi)d\mu_{s,t} = \int_0^1 f(T_{s,t}\xi)d\xi = \lim_{h\to\infty} t^{-h}\sum_{k=1}^{t^h} f(\theta_k^{(h)}) .$$

The integrals and the limit exist and are finite.

Proof. It will be sufficient to prove the lemma for $f(\xi) = \{\xi, \gamma\}$, where $0 \leqq \gamma \leqq 1$ and

$$\{\xi, \gamma\} = \begin{cases} 1, & \text{if } \{\xi\} < \gamma \\ 0 & \text{otherwise.} \end{cases}$$

$\xi_k^{(h)} = \int_0^1 \{T_{s,t}\xi, \theta_k^{(h)}\}d\xi$ is the least upper bound of numbers ξ having $T_{s,t}\xi \leqq \theta_k^{(h)}$. Thus if $\theta_k^{(h)} = 0\cdot a_1 \cdots a_h$ in the scale of s, then $\xi_k^{(h)} = 0\cdot a_1 \cdots a_h$ in the scale of t and therefore $\xi_k^{(h)} = (k-1)t^{-h}$.

Hence if $\theta_k^{(h)} \leqq \gamma \leqq \theta_{k+1}^{(h)}$, or if $\theta_k^{(h)} \leqq \gamma$ with $k = t^h$, then

$$\int_0^1 \{T_{s,t}\xi, \gamma\}d\xi = kt^{-h} - \varepsilon ,$$

where $0 \leqq \varepsilon \leqq t^{-h}$. We can rewrite this in the form

$$\int_0^1 \{T_{s,t}\xi, \gamma\}d\xi = t^{-h}\sum_{k=1}^{t^h} \{\theta_k^{(h)}, \gamma\} - \varepsilon ,$$

and Lemma 1 follows.

Particularly, for

$$\mu(\gamma, x) = \int_0^1 \{xT_{s,t}\xi, \gamma\}d\xi$$

$$\mu(\gamma, x, y) = \int_0^1 \{xT_{s,t}\xi, \gamma\}\{yT_{s,t}\xi, \gamma\}d\xi$$

we have

$$(4) \qquad \mu(\gamma, x) = \lim_{h\to\infty} t^{-h}\sum_{k=1}^{t^h} \{x\theta_k^{(h)}, \gamma\} ,$$

$$(5) \qquad \mu(\gamma, x, y) = \lim_{h\to\infty} t^{-h}\sum_{k=1}^{t^h} \{x\theta_k^{(h)}, \gamma\}\{y\theta_k^{(h)}, \gamma\} .$$

3. **Exponential sums.** Write $e(\xi)$ for $e^{2\pi i\xi}$. There exist ([5], pp. 91–92, 99) for any γ, $0 \leqq \gamma \leqq 1$, and any $\eta > 0$ functions $f_1(\xi), f_2(\xi)$ periodic in ξ with period 1, such that $f_1(\xi) \leqq \{\xi, \gamma\} \leqq f_2(\xi)$, having Fourier expansions

$$f_1(\xi) = \gamma - \eta + \sum_u{}' A_u^{(1)}e(u\xi)$$

$$f_2(\xi) = \gamma + \eta + \sum_u{}' A_u^{(2)} e(u\xi) ,$$

where the summation is over all $u \neq 0$ and $A_u^{(i)}$ is majorized by

$$(6) \qquad | A_u | \leqq \frac{1}{u^2 \eta} .$$

Applying this to (5) we obtain

$$\mu(\gamma, x, y) \leqq (\gamma + \eta)^2 + \overline{\lim_{h \to \infty}}\, t^{-h} \sum_{\substack{u,v \\ \neq 0,0}}{}' \left| A_u^{(2)} \right| \left| A_v^{(2)} \right| \left| \sum_{k=1}^{t^h} e((ux + vy)\theta_k^{(h)}) \right| ,$$

where we put $A_0^{(2)} = \gamma + \eta$ and take the sum over all pairs u, v of numbers not both being zero. Since

$$\left| t^{-h} \sum_{k=1}^{t^h} e((ux + vy)\theta_k^{(h)}) \right| \leqq 1 ,$$

and since the double sum over u, v is uniformly convergent in h, we may change the order of limit and summation and obtain

$$\mu(\gamma, x, y) \leqq (\gamma + \eta)^2 + \sum_{u,v}{}' | A_u^{(2)} || A_v^{(2)} | \overline{\lim_{h \to \infty}}\, t^{-h} \left| \sum_{k=1}^{t^h} e((ux + vy)\theta_k^{(h)}) \right| .$$

The numbers $\theta_k^{(h)}$ are the numbers

$$\frac{a_1}{s} + \frac{a_2}{s^2} + \cdots + \frac{a_h}{s^h} ,$$

where $0 \leqq a_i < t$. Hence

$$\sum_{k=1}^{t^h} e(w\theta_k^{(h)}) = \prod_{j=1}^{h} \left(1 + e\left(\frac{w}{s^j}\right) + e\left(\frac{2w}{s^j}\right) + \cdots + e\left(\frac{(t-1)w}{s^j}\right)\right) .$$

If we keep w fixed, and if j is large, then

$$\left| \left(1 + e\left(\frac{w}{s^j}\right) + \cdots + e\left(\frac{(t-1)w}{s^j}\right)\right) t^{-1} - 1 \right| < \frac{t\,|w|}{s^i} .$$

Therefore

$$(7) \qquad \Pi\,(s, t\,; w) = \prod_{j=1}^{\infty} \left| \left(1 + e\left(\frac{w}{s^j}\right) + \cdots + e\left(\frac{(t-1)w}{s^i}\right)\right) t^{-1} \right|$$

exists and

$$(8) \qquad \mu(\gamma, x, y) \leqq (\gamma + \eta)^2 + \sum_{u,v}{}' | A_u^{(2)} | | A_v^{(2)} | \,\Pi(s, t\,; ux + vy) .$$

The next three sections will be devoted to finding bounds for sums like

$$\sum_{N_1 < n, m \leqq N_2} \Pi(s, t\,; ur^n + vr^m) .$$

4. Two lemmas on digits.

LEMMA 2. *Write $w = c_g \cdots c_2 c_1$ in the scale of s. Assume there are at least z pairs of digits $c_{i+1}c_i$ with*

$$(9) \qquad 1 \leqq c_{i+1}c_i \leqq s^2 - 2 .$$

(Here $c_{i+1}c_i = sc_{i+1} + c_i$). Then

$$\Pi(s, t; w) \leqq \alpha_2^z ,$$

uhere $\alpha_2 = \alpha_2(s, t)$, $0 < \alpha_2 < 1$.

Proof. There are at least z numbers i having

$$\frac{1}{s^2} \leqq \left\{\frac{w}{s^i}\right\} \leqq 1 - \frac{1}{s^2} .$$

For such an i we have

$$\left|1 + e\left(\frac{w}{s^i}\right) + \cdots + e\left(\frac{(t-1)w}{s^i}\right)\right| \leqq \left|1 + e\left(\frac{1}{s^2}\right)\right| + t - 2 = t\alpha_2$$

and the Lemma is proved.

There exists an $\alpha_3(s)$, $0 < \alpha_3 < 1/4$, such that

$$\frac{(s^2-2)^{\alpha_3} 2^{1/2-\alpha_3}}{(2\alpha_3)^{\alpha_3}(1-2\alpha_3)^{1/2-\alpha_3}} < 2^{3/4} .$$

LEMMA 3. *If k is large, $k > \alpha_4(s)$, then the number of combinations of digits $c_k c_{k-1} \cdots c_1$ in the scale of s with less than $\alpha_3(s)k$ indices i satisfying (9) is not greater than $2^{(3/4)k}$.*

Proof. It will be sufficient to show that the number of combinations with less than $\alpha_3(s)k$ indices i satisfying both (9) and $i \equiv 1 \pmod 2$ is not greater than $2^{(3/4)k}$. We first assume k is even. There exist

$$\binom{\frac{k}{2}}{l}(s^2-2)^l 2^{k/2-l}$$

combinations $c_k \cdots c_1$ with exactly l indices i having both (9) and $i \equiv 1 \pmod 2$. Hence the number of combinations with less than $\alpha_3(s)k$ indices i satisfying (9) and $i \equiv 1 \pmod 2$ does not exceed

$$k\binom{\frac{k}{2}}{[\alpha_3 k]}(s^2-2)^{[\alpha_3 k]} 2^{(k/2)-[\alpha_3 k]} .$$

Using Stirling's formula for the binomial coefficient we obtain for large enough k the upper bound

$$\alpha_5(s)k\frac{(s^2-2)^{\alpha_3 k}2^{((1/2)-\alpha_3)k}}{(2\alpha_3)^{\alpha_3 k}(1-2\alpha_3)^{((1/2)-\alpha_3)k}} < 2^{(3/4)k} .$$

Actually, the expression on the left hand side is $< 2^{\alpha_6 k}$, where $\alpha_6 < 3/4$. This permits us to extend the result to odd k.

5. The order of r modulo p^k as a function of k.

LEMMA 4. *Assume p is a prime with $p \nmid r$. Then the order $o(r, p^k)$, of r modulo p^k satisfies*

$$o(r, p^k) \geqq \alpha_7(r, p)p^k .$$

COROLLARY. *Let n run through a residue system modulo p^k. Then at most $\alpha_8(r, p)$ of the numbers r^n will fall into the same residue class modulo p^k.*

Proof. Write

$$g = g(p) = \begin{cases} p-1\,, & \text{if } p \text{ is odd} \\ 2\,, & \text{if } p = 2. \end{cases}$$

There exists an $\alpha_9 = \alpha_9(r, p)$ such that

$$r^g \equiv 1 + qp^{\alpha_9 - 1} \pmod{p^{\alpha_9}} , \tag{10}$$

where $q \not\equiv 0 \pmod p$. We have necessarily $\alpha_9 > 1$ and even $\alpha_9 > 2$ if $p = 2$. If follows from (10) by standard methods (see, for instance, [4], § 5.5) that

$$r^{gp^e} \equiv 1 + qp^{\alpha_9 - 1 + e} \pmod{p^{\alpha_9 + e}}$$

for any $e \geqq 0$. Thus for $k \geqq \alpha_9$ we have

$$r^{gp^{k-\alpha_9}} \equiv 1 + qp^{k-1} \pmod{p^k}$$

and

$$o(r, p^k) \geqq gp^{k-\alpha_9} = \alpha_7(r, p)p^k .$$

Assume $r \not\sim s$. Write

$$r = p_1^{d_1}p_2^{d_2}\cdots p_h^{d_h}$$
$$s = p_1^{e_1}p_2^{e_2}\cdots p_h^{e_h} ,$$

where we may assume that never both $d_i = 0$, $e_i = 0$. We also may assume that the primes $p_1, \cdots, p_h$ are ordered in such a way that

$$\frac{e_1}{d_1} \geqq \frac{e_2}{d_2} \geqq \cdots \geqq \frac{e_h}{d_h} ,$$

where we put $(e_i/d_i) = +\infty$ if $d_i = 0$. Since $r \not\sim s$, we have

$$r_1 = \frac{r^{e_1}}{s^{d_1}} > 1 .$$

From now on, $p = p_1(r, s)$ is the prime defined above. We have $p \mid s$ but $p \nmid r_1$. For any $x \neq 0$, $y > 1$ we define two new numbers x_y and x'_y by $x = x_y x'_y$, where x_y is a power of y and $y \nmid x'_y$.

LEMMA 5. A. *Assume $r \not\sim s$, $v \neq 0$. Let m run through a system $K(s^k)$ of non-negative representatives modulo s^k. Then at most*

$$\alpha_{10}(r, s)\left(\frac{s}{2}\right)^k v_p$$

of the numbers

$$v(r^m)'_s$$

are in the same residue class modulo s^k.

B. *Assume $r \not\sim s$, furthermore $p \nmid r$. Suppose $u \neq 0$, $v \neq 0$, n are fixed. Then, if m runs through $K(s^k)$, at most*

$$\alpha_{11}(r, s)\left(\frac{s}{2}\right)^k v_p$$

of the numbers

$$ur^n + vr^m$$

will fall into the same residue class modulo s^k.

Proof. A. Write $m = m_1 e_1 + m_2$, $0 \leqq m_2 < e_1$. Then $r^m = r^{m_1 e_1 + m_2} = s^{m_1 d_1} r_1^{m_1} r^{m_2}$ and $v(r^m)'_s = v r_1^{m_1}(r^{m_2})'_s$. The equation

$$r_1^{m_1} \equiv a \pmod{p^k}$$

has for fixed a at most $e_1 \alpha_8(r_1, p)$ solutions in $m = m_1 e_1 + m_2$, if m runs through a system $K(p^k)$ of residues modulo p^k. This follows from the corollary of Lemma 4. The equation

$$av(r^{m_2})'_s \equiv b \pmod{p^k}$$

has for fixed b, m_2 at most

$$\text{g.c.d.}(v(r^{m_2})'_s, p^k) \leqq v_p r^{m_2}$$

solutions in a. Hence the number of solutions of

$$vr_1^{m_1}(r^{m_2})'_s \equiv b \pmod{p^k}$$

in $m = m_1e_1 + m_2 \in K(p^k)$ does not exceed

$$e_1\alpha_8 v_p(1 + r + \cdots + r^{e_1-1}) = \alpha_{10}(r, s)v_p \ .$$

But this implies that the number of solutions of

$$vr_1^{m_1}(r^{m_2})'_s \equiv b \pmod{s^k}$$

in $m = m_1e_1 + m_2 \in K(s^k)$ is not greater than

$$\alpha_{10}(r, s)v_p\left(\frac{s}{p}\right)^k \leqq \alpha_{10}(r, s)\left(\frac{s}{2}\right)^k v_p \ .$$

B. The equation

$$ur^n + vr^m \equiv b \pmod{p^k}$$

has according to the corollary of Lemma 4 at most

$$\alpha_8(r, p)v_p$$

solutions in $m \in K(p^k)$. The result follows as before.

The following conjecture seems related to our results: *Assume $r \not\sim s$. Then for any ε and k almost all the numbers $r, r^2, \cdots$ are (ε, k)-normal to the base s in the sense of* Besicovitch [1]; *that is, the number of $n \leqq N$ for which r^n is not (ε, k)-normal is $o(N)$ as $N \to \infty$ for fixed ε and k.*

6. Bounds for exponential sums.

LEMMA 6. A. *Let r, s, v be as in Lemma* 5A. *Then*

$$\sum_{m \in K(s^k)} \Pi(s, t\,;\, vr^m) \leqq \alpha_{12}v_p s^{(1-\alpha_{13})k}$$

B. *Let r, s, u, v, n be as in Lemma* 5B. *Then*

$$\sum_{m \in K(s^k)} \Pi(s, t\,;\, ur^n + vr^m) \leqq \alpha_{14}v_p s^{(1-\alpha_{15})k} \ .$$

Proof. A. Write $v(r^m)'_s = c_g \cdots c_k \cdots c_1$ in the scale of s. Lemma 5A implies that any digit combination $c_k c_{k-1} \cdots c_1$ will occur at most $\alpha_{10}(r, s)(s/2)^k v_p$ times. According to Lemma 3, there are for large k not more than $2^{(3/4)k}$ digit-combinations $c_k \cdots c_1$ with less than $\alpha_3 k$ indices i satisfying (9). Thus of all the numbers $v(r^m)'_s$, $m \in K(s^k)$, and hence of all the numbers vr^m there will be at most

$$\alpha_{10}(r, s)(s/2)^k v_p 2^{(3/4)k} = \alpha_{10}(r, s)v_p(s/2^{1/4})^k = \alpha_{10}(r, s)v_p s^{(1-\alpha_{16})k}$$

having less than $\alpha_3 k$ digits c_i in their expansion in the scale of s satisfying (9). Thus Lemma 2 yields

$$\Pi(s, t ; vr^m) \leqq \alpha_2^{k\alpha_3}$$

for all but at most

$$\alpha_{10}(r, s) v_p s^{(1-\alpha_{16})k}$$

numbers $m \in K(s^k)$. This gives

$$\sum_{m \in K(s^k)} \Pi(s, t ; vr^m) \leqq s^k \alpha_2^{k\alpha_3} + \alpha_{10} v_p s^{(1-\alpha_{16})k} \leqq \alpha_{12} v_p s^{(1-\alpha_{13})k} .$$

B is proved similarly, using Lemma 5B.

LEMMA 7. A. *Assume* $r \not\sim s$, $v \neq 0$. *Then*

$$\sum_{N_1 < n \leqq N_2} \Pi(s, t ; vr^m) \leqq \alpha_{17}(N_2 - N_1)^{1-\alpha_{18}} v_p . \tag{11}$$

B. Assume $r \not\sim s$, $u \neq 0$, $v \neq 0$. Then

$$\sum_{N_1 < n, m \leqq N_2} \Pi(s, t ; ur^n + vr^m) \leqq \alpha_{19}(N_2 - N_1)^{2-\alpha_{20}} \max(u_p, v_p) . \tag{12}$$

Proof. A. There exists a k having $s^{2k} \leqq N_2 - N_1 < s^{2(k+1)}$, hence there exists a w satisfying $s^k w \leqq N_2 - N_1 < s^k(w+1)$, where $s^k \leqq w < s^{k+2}$. Thus if m runs from N_1 to N_2, then m runs through w systems $K(s^k)$ of residue classes modulo s^k and at most s^k other numbers. Hence by Lemma 6A

$$\sum_{N_1 < m \leqq N_2} \Pi(s, t ; vr^m) \leqq w\, \alpha_{12} v_p s^{(1-\alpha_{13})k} + s^k \leqq \alpha_{17}(N_2 - N_1)^{1-\alpha_{18}} v_p .$$

B. If $p \nmid r$, then we proceed as in part A. We first take the sum over m and use Lemma 6B.

If $p/$r, then our argument is as follows. Consider, for example, the part of the sum with $n \leqq m$. Changing the notation in n, m, we see that this part of the sum (12) equals

$$\sum_{n=0}^{N_2-N_1-1} \sum_{m=N_1+1}^{N_2-n} \Pi(s, t ; (ur^n + v) r^m) .$$

Except for possibly one exceptional n we have $(ur^n)_p \neq v_p$ and therefore $(ur^n + v)_p \leqq v_p \leqq \max(u_p, v_p)$. If n is not exceptional, then the already proved Lemma 7A can be applied to the inner sum and we obtain the bound

$$\alpha_{17}(N_2 - N_1 - n)^{1-\alpha_{18}} \max(u_p, v_p) .$$

Taking the sum over n we obtain (12).

7. A fundamental lemma. Generalizing $M_N(\xi, r, I)$ we write ${}_{N_1}M_{N_2}(\xi, r, I)$ for the number of indices i in $N_1 < i \leqq N_2$ such that $\{r^i\xi\}$ lies in I. We put

$$ {}_{N_1}R_{N_2}(\xi, r, I) = {}_{N_1}M_{N_2}(\xi, r, I) - (N_2 - N_1)|I| . $$

Fundamental lemma. *Assume $r \not\sim s$. Then*

$$ \int_0^1 {}_{N_1}R^2_{N_2}(T_{s,t}\xi, r, I)d\xi \leqq \alpha_{21}(N_2 - N_1)^{2-\alpha_{22}} . $$

Proof. It is enough to prove this for intervals of the type $I = [0, \gamma)$. Then

$$ {}_{N_1}M_{N_2}(\xi, r, I) = \sum_{N_1 < n \leqq N_2} \{r^n\xi, \gamma\} $$

and

$$ \int_0^1 {}_{N_1}M_{N_2}(T_{s,t}\xi, r, I)d\xi = \sum_{N_1 < n \leqq N_2} \mu(\gamma, r^n) \tag{13} $$

$$ \int_0^1 {}_{N_1}M^2_{N_2}(T_{s,t}\xi, r, I)\, d\xi = \sum_{N_1 < n, m \leqq N_2} \mu(\gamma, r^n, r^m) . \tag{14} $$

Now we combine (8) and Lemma 7. We obtain, together with (6),

$$ \begin{aligned} \sum_{N_1 < n, m \leqq N_2} \mu(\gamma, r^n, r^m) &\leqq (\gamma + \eta)^2(N_2 - N_1)^2 \\ &+ 2(\gamma + \eta) \sum_{v \neq 0} \frac{v_p}{\eta v^2} \alpha_{17}(N_2 - N_1)^{2-\alpha_{18}} \\ &+ \sum_{u \neq 0} \sum_{v \neq 0} \frac{\max(u_p, v_p)}{\eta u^2 \eta v^2} \alpha_{19}(N_2 - N_1)^{2-\alpha_{20}} . \end{aligned} $$

Since the sums

$$ \sum_{v \neq 0} \frac{v_p}{v^2} , \qquad \sum_{u \neq 0} \sum_{v \neq 0} \frac{\max(u_p, v_p)}{u^2 v^2} $$

are convergent, and since η was arbitrary, we have

$$ \sum_{N_1 < n, m \leqq N_2} \mu(\gamma, r^n, r^m) - (N_2 - N_1)^2\gamma^2 \leqq \alpha_{23}(N_2 - N_1)^{2-\alpha_{24}} . $$

In the same fashion we can prove

$$ \left| \sum_{N_1 < n, m \leqq N_2} \mu(\gamma, r^n, r^m) - (N_2 - N_1)^2\gamma^2 \right| \leqq \alpha_{23}(N_2 - N_1)^{1-\alpha_{24}} $$

$$ \left| \sum_{N_1 < n \leqq N_2} \mu(\gamma, r^n) - (N_2 - N_1)\gamma \right| \leqq \alpha_{25}(N_2 - N_1)^{1-\alpha_{26}} . $$

These two inequalities, together with (13) and (14), give the Fundamental Lemma.

8. Proof of the theorems. Once the Fundamental Lemma is shown, we can prove Theorem 2 by the standard method developed in [2].

By J_B, $B > 0$, we denote the set of intervals $[\beta, \gamma)$, $0 \leqq \beta < \gamma < 1$ of the type $\beta = a2^{-b}$, $\gamma = (a+1)2^{-b}$, where $0 \leqq b \leqq \alpha_{22}B/2$. By P_B we denote the set of all pairs of integers N_1, N_2 having $0 \leqq N_1 < N_2 \leqq 2^B$ of the type $N_1 = a2^b$, $N_2 = (a+1)2^b$ for integers a and $b \geqq 0$.

LEMMA 8. *Assume $r \not\sim s$. Then*

$$\sum_{(N_1, N_2) \in P_B} \sum_{I \in J_B} \int_0^1 {}_{N_1}R^2_{N_2}(T_{s,t}\xi, r, I) d\xi \leqq \alpha_{27} 2^{2B(1-\alpha_{28})} .$$

Proof. Because of the Fundamental Lemma the left hand side is not greater than

$$\alpha_{21} 2^{\alpha_{22}B/2+1} \Sigma ,$$

where $2^{\alpha_{22}B/2+1}$ is an upper bound for the number of intervals in J_B and

$$\Sigma = \sum_{(N_1, N_2) \in P_B} (N_2 - N_1)^{2-\alpha_{22}} . \tag{15}$$

In (15) each value of $N_2 - N_1 = 2^b$ occurs 2^{B-b} times, so that

$$\Sigma = \sum_{b=0}^{B} 2^{B-b+b(2-\alpha_{22})} \leqq \alpha_{29} 2^{2B(1-\alpha_{22}/2)} .$$

Hence Lemma 8 is true with $\alpha_{28} = \alpha_{22}/4$.

LEMMA 9. *For large B there exists a set E_B of measure not greater than $2^{-\alpha_{30}B}$ such that*

$$R_N(T_{s,t}\xi, r, I) \leqq 2^{B(1-\alpha_{31})} \tag{16}$$

for all I, $N \leqq 2^B$ and all ξ in $[0, 1)$ but not in E_B.

Proof. We define E_B to be the set consisting of all ξ in $[0, 1)$ for which it is not true that

$$\sum_{(N_1, N_2) \in P_B} \sum_{I \in J_B} {}_{N_1}R^2_{N_2}(T_{s,t}\xi, r, I) \leqq 2^{2B(1-\alpha_{28}/2)} . \tag{17}$$

Lemma 8 assures that the measure of E_B does not exceed

$$\alpha_{27} 2^{-2B\alpha_{28}/2} < 2^{-\alpha_{30}B}$$

for large B. We have to show that (16) is a consequence of (17).

We first assume I to be of the type $I = [0, \gamma)$, $\gamma = a2^{-b}$, where $0 \leqq b \leqq \alpha_{22}B/2$. Then the interval $[0, \gamma)$, is the sum of at most $b < B$ intervals I, $I \in J_B$, as may be seen by expressing a in the binary scale.

Expressing N in the binary scale we see that the interval $[0, N)$ can be expressed as a union of at most B intervals $[N_1, N_2)$, where the pair $N_1, N_2 \in P_B$. Hence we can write $R_N(T_{s,t}\xi, r, I)$ as a sum of ${}_{N_1}R_{N_2}(T_{s,t}\xi, r, I)$ over at most B^2 sets N_1, N_2, I, where $N_1, N_2 \in P_B$, $I \in J_B$:

$$R_N(T_{s,t}\xi, r, I) = \Sigma_{N_1}R_{N_2}(T_{s,t}\xi, r, I) .$$

Hence by (17) and Cauchy's inequality,

$$R_N^2(T_{s,t}\xi, r, I) \leqq B^2 2^{2B(1-\alpha_{28}/2)} < 2^{2B(1-\alpha_{32})}$$

for large B.

Next, let $I = [0, \gamma)$ be of the type $a2^{-b} \leqq \gamma \leqq (a+1)2^{-b}$, where $\alpha_{22}B/4 < b \leqq \alpha_{22}B/2$. Then

$$\begin{aligned} &| R_N(T_{s,t}\xi, r, [0, \gamma)) | = | M_N(T_{s,t}\xi, r, [0, \gamma)) - \gamma N | \\ &\leqq | R_N(T_{s,t}\xi, r, [0, (a+1)2^{-b})) | + | R_N(T_{s,t}\xi, r, [0, a2^{-b})) | + 2^{-b}N \\ &\leqq 2 \cdot 2^{B(1-\alpha_{32})} + 2^{(1-\alpha_{22}/4)B} < 2^{B(1-\alpha_{33})} . \end{aligned}$$

The Lemma now follows from

$$| R_N(\ , \ , [\beta, \gamma)) | \leqq | R_N(\ , \ , [0, \beta)) | + | R_N(\ , \ , [0, \gamma)) | .$$

Proof of Theorem 2. Since $\Sigma 2^{-\alpha_{30}B}$ is convergent, there exists for almost all ξ a $B_0 = B_0(\xi)$ such that $\xi \notin E_B$ for $B \geqq B_0$. If $N \geqq 2^{B_0}$, then we can find a $B \geqq B_0$ satisfying $2^{B-1} < N \leqq 2^B$ and Lemma 9 yields

$$R_N(T_{s,t}\xi, r, I) < 2^{B(1-\alpha_{31})} < 2N^{1-\alpha_{31}} < N^{1-\alpha_1}$$

for large enough N.

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