TWO REMARKS ON FIBER HOMOTOPY TYPE

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Section 1 of this note considers the normal sphere bundle of a compact, connected, orientable manifold M^n (without boundary) differentiably imbedded in euclidean space R^{n+k}. (These hypotheses on M^n will be assumed throughout § 1.) It is shown that if k is sufficiently large then the normal sphere bundle has the fiber homotopy type of a product bundle if and only if there exists an S-map from S^n to M^n of degree one (i.e. for some p there exists a continuous map of degree one from S^{n+p} to the p-fold suspension of M^n). The proof is based on the fact that the Thom space of the normal bundle is dual in the sense of Spanier-Whitehead [8] to the disjoint union of M^n and a point.

Section 2 studies the tangent sphere bundle of a homotopy n-sphere. This has the fiber homotopy type of a product bundle if and only if n equals 1, 3 or 7. The proof is based on Adams' work [**1**].

If X is a space, S^kX will denote the k-fold suspension of X as in [8, **9**]. If X has a base point x_0, then S_0^kX will denote the k-fold reduced suspension and is the identification space S^kX/S^kx_0 obtained from S^kX by collapsing S^kx_0 to a point (to be used as base point for S_0^kX). There is a canonical homeomorphism $S_0^kX \approx S^k \times X/S^k \vee X$.

Two fiber bundles with the same fiber and with projections p_1: $E_1 \to B$, p_2: $E_2 \to B$ have the same *fiber homotopy type* [3, 4, **10**] if there exist fiber preserving maps f_i: $E_i \to E_{3-i}$ and fiber preserving[1] homotopies h_i: $E_i \times I \to E_i$ such that $h_i(x, 0) = f_{3-i}f_i(x)$, $h_i(x, 1) = x$.

Let ξ denote an oriented $(k-1)$-sphere bundle. The total space of ξ will be denoted by $\dot{E}$ and the total space of the associated k-disk bundle will be denoted by E. The *Thom space* $T(\xi)$ is the identification space $E/\dot{E}$ obtained from E by collapsing $\dot{E}$ to a single point (to be used as base point for $T(\xi)$). The following are easily verified:

(A) *If ξ_1, ξ_2 are $(k-1)$-sphere bundles of the same fiber homotopy type, then $T(\xi_1)$, $T(\xi_2)$ have the same homotopy type.*

(B) *If ξ is a product bundle, then $T(\xi)$ is homeomorphic to $S_0^k(B \cup p_0)$ (where $B \cup p_0$ is the disjoint union of B and a point, p_0, which is taken as the base point of $B \cup p_0$).*

1. The normal bundle. If X and Y are spaces we let $[X, Y]$ denote the set of homotopy classes of maps of X into Y and we let

Received May 19, 1959. The authors were supported by the Sloan Foundation and by the United States Air Force (Contract No. AF 49(638)-393 monitored by the Air Force Office of Scientific Research), respectively, during the period when this paper was in preparation.

[1] The phrase "fiber-preserving" means that $p_{3-i}f_i(x) = p_i(x)$ and $p_ih_i(x, t) = p_i(x)$.

$\{X, Y\}$ denote the set of S-maps of X into Y as in [8]. Thus, $\{X, Y\}$ is defined to be the direct limit of the sequence

$$[X, Y] \xrightarrow{S} [SX, SY] \xrightarrow{S} \cdots \xrightarrow{S} [S^pX, S^pY] \xrightarrow{S} \cdots$$

There is a natural map

$$\phi\colon [X, Y] \longrightarrow \{X, Y\}$$

which assigns to every homotopy class $[f] \varepsilon [X, Y]$ the S-map $\{f\}$ represented by any map of $[f]$. The following gives a sufficient condition for ϕ to be onto $\{X, Y\}$.

LEMMA 1. *Let Y be a k-connected CW-complex ($k \geq 1$) and let X be a finite CW-complex with*[2] *$H^q(X) = 0$ for $q > 2k+1$. Then $\phi([X, Y]) = \{X, Y\}$.*

Proof. It suffices to prove that under the hypotheses of the lemma the map S: $[X, Y] \to [SX, SY]$ is onto $[SX, SY]$ because then, for each $p \geq 0$, the map S: $[S^pX, S^pY] \to [S^{p+1}X, S^{p+1}Y]$ is onto $[S^{p+1}X, S^{p+1}Y]$ (because S^pY is $(p+k)$-connected and $H^q(S^pX) = 0$ for $q > 2k+p+1$ and $2(k+p)+1 \geq 2k+p+1$).

Choose base points $x_0 \varepsilon X$, $y_0 \varepsilon Y$ and let $[X, Y]'$ denote the set of homotopy classes of maps $(X, x_0) \to (Y, y_0)$. Since Y is simply-connected the natural map $[X, Y]' \to [X, Y]$ is a 1–1 correspondence. Since X, Y are CW-complexes the collapsing maps $SX \to S_0X$ and $SY \to S_0Y$ are homotopy equivalences (Theorem 12 of [11]) so there are 1–1 correspondences

$$[S_0X, S_0Y] \approx [S_0X, SY] \approx [SX, SY] .$$

Since S_0Y is simply connected, we also have a 1–1 correspondence $[S_0X, S_0Y]' \approx [S_0X, S_0Y]$. Hence, it suffices to show that $S_0([X, Y]') = [S_0X, S_0Y]'$.

Let ΩS_0Y denote the space of closed paths in S_0Y based at y_0. There is a canonical 1–1 correspondence $[S_0X, S_0Y]' \approx [X, \Omega S_0Y]'$ and a natural imbedding $Y \subset \Omega S_0Y$ such that the map S_0: $[X, Y]' \to [S_0X, S_0Y]'$ corresponds to the injection (see § 9 of [7])

$$[X, Y]' \longrightarrow [X, \Omega S_0Y]' .$$

Hence, it suffices to show this injection is onto or, equivalently, that the natural injection (without base point condition) $[X, Y] \to [X, \Omega S_0Y]$ is onto.

[2] When no coefficient group appears explicitly in the notation for a homology or cohomology group it is to be understood that the coefficient group is the group of integers. In dimension 0 the groups will be taken reduced.

Since Y is k-connected it follows from the suspension theorem (see § 7 of [9]) that

$$S_0 \colon \pi_i(Y) \longrightarrow \pi_{i+1}(S_0 Y)$$

is 1-1 if $i \leq 2k$ and is onto if $i \leq 2k+1$. Since S_0 corresponds to the injection map $\pi_i(Y) \to \pi_i(\Omega S_0 Y)$, this is equivalent to the statement that

$$\pi_i(\Omega S_0 Y,\ Y) = 0 \text{ for } i \leq 2k+1\ .$$

Since Y is simply-connected the groups $\pi_i(\Omega S_0 Y,\ Y)$ form a simple system for every i. Now the groups $H^i(X;\ \pi_i(\Omega S_0 Y,\ Y))$ vanish for every i because for $i \leq 2k+1$ the coefficient group vanishes while for $i > 2k+1$ the groups vanish because of the assumption on the cohomology of X. By Theorem 4.4.2 of [2] it follows that any map $X \to \Omega S_0 Y$ is homotopic to a map $X \to Y$, completing the proof.

REMARK. If in Lemma 1 we assume that $H^q(X) = 0$ for $q > 2k$, then a similar argument shows that ϕ is 1-1, however we shall not need this result.

Let $M^n \subset R^{n+k}$ be as in the introduction (i.e. M^n is a differentiably imbedded manifold which is compact, connected, orientable, and without boundary). The following result relates the normal bundle of M^n to M^n itself by means of duality.

LEMMA 2. *Let ξ be the normal $(k-1)$-sphere bundle of M^n in R^{n+k}. Then the Thom space $T(\xi)$ is weakly $(n+k+1)$-dual to the disjoint union $M^n \cup p_0$.*

Proof. Regard S^{n+k} as the one point compactification of R^{n+k}. Let E be a closed tubular neighborhood of M^n and assume E is contained in a large disk D^{n+k}. Then (D^{n+k}-interior E) is a deformation retract of $R^{n+k} - M^n = S^{n+k} - (M^n \cup (\text{point at infinity}))$. Using standard homotopy extension properties and the contractibility of D^{n+k} it follows that if $\dot{E}$ denotes the boundary of E then

$$T(\xi) = E/\dot{E} = D^{n+k}/(D^{n+k} - \text{interior } E)$$

has the homotopy type of the suspension $S(D^{n+k} - \text{interior } E)$. Since $(D^{n+k} - \text{interior } E)$ is an $(n+k)$-dual of $M^n \cup (\text{point at infinity})$, and the suspension of an $(n+k)$-dual is an $(n+k+1)$-dual, this completes the proof.

REMARK. Lemma 2 shows that the S-type of $T(\xi)$ depends only on that of M^n. If k is sufficiently large this implies that the homotopy type of $T(\xi)$ depends only on that of M^n. This suggests the conjecture that the fiber homotopy type of the normal bundle of any manifold $M^n \subset R^{n+k}$, k large, is completely determined by the homotopy type of M^n. A similar conjecture can be made for the tangent bundle.

THEOREM 1. *Let $M^n \subset R^{n+k}$ be as before and assume that $H_q(M^n) = 0$ for $q < r$ and that $k \geq \min(n - r + 2, 3)$. The following statements are equivalent*:

(1) *There is an S-map $\alpha \varepsilon \{S^n, M^n\}$ such that*

$$\alpha_*: H_n(S^n) \approx H_n(M_n) .$$

(2) *The normal sphere bundle of $M^n \subset R^{n+k}$ has the fiber homotopy type of a product bundle.*

(3) *The disjoint union $M^n \cup p_0$ is weakly $(n + k + 1)$-dual to $S_0^k(M^n \cup p_0)$.*

Proof. (1) $\Rightarrow$ (2). Let N denote the complement in S^{n+k} of an open tubular neighborhood of M^n. Then N is $(n + k)$-dual to M^n. The S-map α is $(n+k)$-dual to an S-map $\beta\varepsilon\{N, S^{k-1}\}$ such that $\beta^*: H^{k-1}(S^{k-1}) \approx H^{k-1}(N)$. Since $H^p(N) \approx H_{n+k-p-1}(M^n)$, we see that $H^p(N) = 0$ if $p > n + k - r - 1$. Since S^{k-1} is $(k - 2)$-connected, $k - 2 \geq 1$, and $k \geq n - r + 2$ (so $2(k-2)+1 \geq n+k-r-1$), it follows from Lemma 1 that there is a map f: $N \rightarrow S^{k-1}$ representing β. Then $f^*: H^{k-1}(S^{k-1}) \approx H^{k-1}(N)$. Let $\dot{E}$ be the boundary of N (so $\dot{E}$ is the normal $(k - 1)$-sphere bundle of M^n), and let F be a fiber of $\dot{E}$. Then the inclusion map $F \subset N$ induces an isomorphism $H^{k-1}(N) \approx H^{k-1}(F)$ (because by Corollaries III. 15 and I.5 of [10] or by Theorems 14 and 21 of [5] we have $H^{k-1}(\dot{E}) \approx H^{k-1}(M^n) + Z$ and the injection $H^{k-1}(N) \rightarrow H^{k-1}(\dot{E})$ maps isomorphically onto the second summand while the injection $H^{k-1}(\dot{E}) \rightarrow H^{k-1}(F)$ maps the second summand isomorphically.) Therefore, the map $f \mid \dot{E}$: $\dot{E} \rightarrow S^{k-1}$ has the property that its restriction to a fiber F induces an isomorphism of the cohomology of S^{k-1} onto that of F so is a homotopy equivalence of F with S^{k-1}. This implies (by Corollary 2 on p. 121 of [3]) that $\dot{E}$ has the same fiber homotopy type as a product bundle.

(2) $\Rightarrow$ (3). By Lemma 2, $T(\xi)$ is weakly $(n + k + 1)$-dual to $M^n \cup p_0$. If ξ is of the same fiber homotopy type as a product bundle, it follows from (A), (B) that $T(\xi)$ is of the same homotopy type as $S_0^k(M^n \cup p_0)$. Combining these two statements gives the result.

(3) $\Rightarrow$ (1) assume $M^n \cup p_0$ is weakly $(n + k + 1)$-dual to $S_0^k(M^n \cup p_0)$. The map $M^n \cup p_0 \rightarrow S^0$ collapsing each component of $M^n \cup p_0$ to a single point represents an S-map β: $S_0^k(M^n \cup p_0) \rightarrow S_0^k(S^0) = S^k$ such that $\beta^*: H^k(S^k) \approx H^k(S_0^k(M^n \cup p_0))$. By duality there is an S-map $\alpha\varepsilon\{S^n, M^n \cup p_0\}$ such that $\alpha_*: H_n(S^n) \approx H_n(M^n \cup p_0) \approx H_n(M^n)$. Since.

$$\{S^n,\ M^n \cup p_0\} \approx \{S^n,\ M^n\} + \{S^n,\ S^0\}\ ,$$

the result is proved.

As a corollary we obtain the following result proved by Massey [4].

COROLLARY. *Let M^n be a homology sphere. Then the normal bundle of M^n in R^{n+k} has the same fiber homotopy type as a product bundle.*

Proof. Since $r = n$, the case $k \geq 3$ follows from the theorem. For the cases $k = 1, 2$ it is well known that the normal bundle is, in fact, trivial.

REMARK. Puppe [6] calls a manifold "sphere-like" if the unstable group $\pi_{n+1}(SM^n)$ contains an element of degree one. (The group $\pi_n(M^n)$ can contain an element of degree one if and only if M^n is a homotopy sphere.) Theorem 1 shows that the normal sphere bundle of a sphere-like manifold $M^n \subset R^{n+k}$ has the fiber homotopy type of a product bundle provided k is sufficiently large. An example of a manifold with trivial normal bundle which is not sphere-like is provided by the real projective 3-space.

2. The tangent bundle. Let M^n be as above (i.e. compact, connected, orientable, and without boundary), but let E denote a closed tubular neighborhood of the diagonal in $M^n \times M^n$. If the tangent bundle has the fiber homotopy type of a product bundle, then there exists a map $\dot{E} \rightarrow S^{n-1}$ (where $\dot{E}$ is the boundary of E) having degree one on each fiber. This gives rise to a map $(E,\ \dot{E}) \rightarrow (D^n,\ S^{n-1}) \rightarrow (S^n, \text{point})$ of degree one and, hence, to a map

$$M^n \times M^n \longrightarrow M^n \times M^n/(M^n \times M^n\text{-interior } E) = E/\dot{E} \longrightarrow S^n$$

which has degree (1, 1) (the degree is (1, 1) because a generator of $H^n(S^n)$ maps, under the homomorphism induced by the above composite, into a cohomology class of $M^n \times M^n$ dual under Poincaré duality to the diagonal class of $H_n(M^n \times M^n)$).

THEOREM 2. *Suppose that M^n has the homotopy type of an n-sphere. Then the tangent bundle has the fiber homotopy type of a product bundle if and only if n equals* 1, 3 *or* 7 (*and in this case the tangent bundle is a product bundle*).

Proof. If a map $S^n \times S^n \rightarrow S^n$ of degree (1, 1) exists, then according to Adams n must be equal to 1, 3 or 7 (see Theorem la of [1]).

Conversely, if n equals 1, 3 or 7 then $\pi_{n-1}(SO(n)) = 0$. Using obstruction theory it follows that any homotopy n-sphere is parallelizable. This completes the proof.

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