SEPARABLE CONJUGATE SPACES

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A Banach space B is reflexive if the natural isometric mapping of B into the second conjugate space B^{**} covers all of B^{**}. All conjugate spaces of a reflexive separable space B are separable. The nonreflexive space $l^{(1)}$ is separable and its first conjugate space is (m), which is non-separable. The space (c_0) is separable, its first conjugate space is $l^{(1)}$, and its second conjugate space is (m). An example is known of a non-reflexive Banach space whose conjugate spaces are all separable [4]. This space is pseudo-reflexive in the sense that its natural image in the second conjugate space has a finite-dimensional complement. The structure of such spaces has been studied carefully [2].

The main purpose of this paper is to show that the sequence started by $l^{(1)}$ and (c_0) can be extended to give a sequence $\{B_n\}$ of separable Banach spaces such that, for each n, the nth conjugate space of B_n is its first nonseparable conjugate space. The principal tool used is a theorem which states a sufficient condition on a space T for the existence of a space B with

$$B^{**} = \pi(B) \dotplus T ,$$

where $\pi(B)$ is the natural image of B in B^{**}. The following definition and notation will be used.

A *basis* for a Banach space B is a sequence $\{u^i\}$ such that, for each x of B, there is a unique sequence of numbers $\{a_i\}$ for which $\lim_{n\to\infty} \| x - \sum_1^n a_i u_i \| = 0$. A sequence $\{u_i\}$ is a basis for its closed linear span if and only if there is a number $\varepsilon > 0$ such that

$$\left\| \sum_1^{n+p} c_i x_i \right\| \geq \varepsilon \left\| \sum_1^n c_i x_i \right\|$$

for any numbers $\{c_i\}$ and positive integers n and p [1, page 111]. If ε can be $+1$, the basis is an *orthogonal basis*. It will be useful to classify bases as follows:

Type α. If $\{a_i\}$ is a sequence of numbers for which $\sup_n \| \sum_1^n a_i u_i \| < \infty$, then $\sum_1^\infty a_i u_i$ converges.

Type β. If f is a linear functional defined on B and $\| f \|_n$ is the norm of f on the closed linear span of $\{u_i \mid i \geq n\}$, then $\lim_{n\to\infty} \| f \|_n = 0$.

There are Banach spaces which have bases which are neither of type α nor of type β, while a basis is of both types if and only if the space

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is reflexive [3; Theorem 1].

The symbols C, (m), $l^{(1)}$, and (c_0) are used in the usual sense [1; pages 11, 12, 181]. The set of all $r+t$ with $r \in R$ and $t \in T$ is denoted by $R \dotplus T$. A space R is said to be *embedded* in a space S if R is mapped isomorphically and isometrically on a subspace of S; for $x \in R$, the image of x is indicated by $x^{(S)}$. In particular, $x^{(C)}$ is a continuous function defined on $[0, 1]$ and the value of $x^{(C)}$ at t is denoted by $x^{(C)}(t)$. If $w = (w_1, w_2, \cdots)$ is a sequence of numbers, then ${}^n w$ is the sequence obtained by replacing w_i by 0 if $i > n$. A *block* of w is a sequence ${}^n_m w$ obtained from w by replacing w_i by 0 if $i \le m$ or $i > n$. Two blocks ${}^{n_1}_{m_1} w$ and ${}^{n_2}_{m_2} w$ are said to *overlap* if the intervals $(m_1, n_1]$ and $(m_2, n_2]$ overlap.

LEMMA 1. *Let T be a Banach space with an orthogonal basis $\{u_i\}$. Then T can be embedded in (m) in such a way that:*

(i) *if $x = \sum_1^\infty a_i u_i$, then the first $2N$ coordinates of $x^{(m)}$ are zero if and only if $a_i = 0$ for $i \le N$;*

(ii) *if $\{a_i\}$ and $\{x_i^m\}$ are related by $x = \sum_1^\infty a_i u_i$ and $x^{(m)} = (x_1^m, x_2^m, \cdots)$, then $a_1, \cdots, a_N$ are each continuous functions of $x_1^m, \cdots, x_{2N}^m$ and $x_1^m, \cdots, x_{2N}^m$ are each continuous functions of $a_1, \cdots, a_N$;*

(iii) *if $x^{(m)} = (x_1^m, x_2^m, \cdots)$, then $\| x^{(m)} \| = \limsup | x_i^m |$.*

Proof. Let T be embedded in the space C. Let $\{t_i\}$ be a sequence of numbers in the interval $[0, 1]$ for which the sequence $\{t_{2i-1}\}$, $i = 1, 2, \cdots$, is dense in $[0, 1]$ and, for each i, $u_i^{(C)}(t_{2i}) \neq 0$. If $x = \sum_1^\infty a_i u_i$, let $x^{(m)}$ be the sequence $(x_1^m, x_2^m, \cdots)$ for which

$$x_{2k-1}^m = \sum_1^k a_i u_i^{(C)}(t_{2k-1}),\ x_{2k}^m = \sum_1^k a_i u_i^{(C)}(t_{2k}) .$$

Then for any $t \in [0, 1]$,

$$\left| \sum_1^k a_i u_i^{(C)}(t) \right| \le \left\| \sum_1^k a_i u_i^{(C)} \right\| = \left\| \sum_1^k a_i u_i \right\| \le \left\| x \right\| .$$

Hence $\| x^{(m)} \| \le \| x \|$. But if $\varepsilon > 0$ and N is chosen so that $\| x - \sum_1^k a u_i \| < \varepsilon$ if $k > N$, then it follows from $\{t_{2k-1}\}$ being dense in $[0, 1]$ that

$$\| x^{(m)} \| \ge \sup_{k > N} \left| \sum_1^k a_i u_i^{(C)}(t_{2k-1}) \right| \ge \left\| x \right\| - \varepsilon .$$

Hence $\| x \| = \| x^{(m)} \|$ and T and its image in (m) are isometric. But if $x = \sum_{N+1}^\infty a_i u_i$, then $x_{2k-1}^m = x_{2k}^m = 0$ if $k \le N$. If $x_i^m = 0$ for $i \le 2N$, then the equations $x_{2k}^m = \sum_1^k a_i u_i^{(C)}(t_{2k}) = 0$, $k \le N$, successively imply $0 = a_1 = a_2 = \cdots = a_N$, since $u_k^{(C)}(t_{2k}) \neq 0$. The conclusion (ii) follows from this system of equations and the continuity of $\sum_1^N a_i u_i$ in $a_1, \cdots, a_N$, while (iii) follows from $\{t_{2k-1}\}$ being dense in $[0, 1]$.

LEMMA 2. *Let T be a Banach space with an orthogonal basis $\{u_i\}$ and let T be embedded in (m) as described in Lemma 1. Then the following are equivalent:*

(i) *the basis $\{u_i\}$ is of type α;*

(ii) *if $w \in (m)$, then $w = v + t$, with v an element of (m) which has all coordinates zero after the Mth $(M \geq 0)$ and t the image of an element of T, provided there is a sequence of elements $\{y_k\}$ of T for which $\sup \| y_k \| < \infty$ and*

$$\lim_{k \to \infty} y^m_{k,i} = w_i \text{ for } i > M ,$$

where $w = (w_1, w_2, \cdots)$ and $y_k^{(m)} = (y^m_{k,1}, y^m_{k,2}, \cdots)$.

Proof. Assume the basis $\{u_i\}$ is of type α and let $w = (w_i, w_2, \cdots)$ and $\{y_k\}$ satisfy the hypotheses of (ii). Since $\| y_k \|$ is bounded, there is a subsequence $\{z_k\}$ of $\{y_k\}$ such that

$$\lim_{k \to \infty} z^m_{k,i} = v_i$$

exists for $i \leq M$. Let $v = (w_1 - v_1, \cdots, w_M - v_M, 0, 0, \cdots)$. Also let $z_k = \sum_1^\infty a_i^k u_i$ for each k. It now follows from (ii) of Lemma 1 that $\lim_{k \to \infty} a_i^k = a_i$ exists for each i. Since the basis is orthogonal, $\| \sum_1^n a_i u_i \| \leq \sup \| z_k \|$. Since $\{u_i\}$ is a basis of type α, it then follows that $\sum_1^\infty a_i u_i$ is convergent. Also, $w - v = t$ is the (m)-image of $\sum_1^\infty a_i u_i$. This follows from the fact that the numbers a_i, $i \leq N$, continuously determine the first $2N$ coordinates of the (m)-image of $\sum_1^\infty a_i u_i$, while $z_k = \sum_1^\infty a_i^k u_i$, $\lim_{k \to \infty} a_i^k = a_i$, and $\lim_{k \to \infty} z^m_{k,i}$ exists and is the ith coordinate of $w - v$.

Now assume (ii) and let $\| \sum_1^n a_i u_i \|$ be a bounded function of n. Let $w = (w_1, w_2, \cdots)$ be the element of (m) whose first $2N$ coordinates are determined by $a_1, \cdots, a_N$. Take $M = 0$ and y_k to be the (m)-image of $\sum_1^k a_i u_i$. It then follows from (ii) that w is the (m)-image of some element of T, which can only be $\sum_1^\infty a_i u_i$.

THEOREM 1. *Let T be a Banach space which has an orthogonal basis of type α. Then there is a Banach space B which has a basis of type β and for which*

$$B^{**} = \pi(B) \dotplus T_1 ,$$

where $\pi(B)$ is the natural image of B in B^{**}, T and T_1 are isometric, and $\| r + t \| \geq \| t \|$ if $r \in \pi(B)$ and $t \in T_1$.

Proof. Let T_1 be the embedding of T in (m) as described in Lemma 1. The norm of (m) will be denoted by $\| \ \|$. For elements w of (m) which have only a finite number of nonzero coordinates, let

(1) $\theta(w) = \inf \| t \|$ for w a block of t, where t is either a member

of T_1 or has only one nonzero coordinate (note that $\theta(w)$ is defined only for elements w which are blocks of at least one $t \in T_1$ or which have only one nonzero coordinate);

(2) $h(w) = \{\inf \sum [\theta(b_i)]^2\}^{1/2}$, where $w = \sum b_i$, each b_i is a block of w, and no two blocks overlap.

(3) $|||\, x\, ||| = \inf \sum h(w_j)$ for $x = \sum w_j$.

In the above, all sums have a finite number of terms. The triangular inequality for $|||\ |||$ is a direct consequence of (3). Also, $|||\, x\, ||| \geq \|\, x\, \|$, since $\theta(w) \geq \|\, w\, \|$ and $h(w) \geq \|\, w\, \|$. Let B be the completion of the space of sequences with a finite number of nonzero coordinates, using the norm $|||\ |||$. The sequence of elements $\{u_i\}$ for which u_i has all coordinates 0 except the ith, which is 1, is an orthogonal basis for B. This means that $|||\sum_1^{n+p} a_i u_i ||| \geq ||| \sum_1^n a_i u_i |||$, which follows by noting that, if $\sum_1^{n+p} a_i u_i = \sum w_j$, then $\sum_1^n a_i u_i = \sum {}^n w_j$ and $h({}^n w_j) \leq h(w_j)$ for each j, where ${}^n w_j$ is obtained from w_j by replacing each coordinate after the nth by 0.

The basis $\{u_i\}$ is of type β. For suppose there is a linear functional f for which $\lim_{n\to\infty} |||\, f\, |||_n = K \neq 0$ and choose N so that $|||\, f\, |||_N \leq 7/6K$. Then there are two elements $x = \sum_{n_1}^{n_2} a_i u_i$, $y = \sum_{n_3}^{n_4} a_i u_i$, for which $N < n_1 \leq n_2 < n_3 \leq n_4$, $|||\, x\, ||| = |||\, y\, ||| = 1$, $f(x) > 7/8K$ and $f(y) > 7/8K$. Then

$$\frac{7}{4} K < f(x) + f(y) \leq \left(\frac{7}{6} K\right) |||\, x + y\, ||| \text{ and } |||\, x + y\, ||| > \frac{3}{2}\,.$$

Since θ and h are both monotone decreasing as a block has coordinates at the ends replaced by zeros, there exists $\{x_j\}$ and $\{y_j\}$ such that $x = \sum x_j$, $y = \sum y_j$, $\sum h(x_j) < |||\, x\, ||| + \varepsilon$, and $\sum h(y_j) < |||\, y\, ||| + \varepsilon$, where each x_j has zero coordinates outside the index interval $[n_1, n_2]$ and each y_j has zero coordinates outside the index interval $[n_3, n_4]$. Now replace the sets $\{x_j\}$ and $\{y_j\}$ by $\{\bar{x}_j\}$ and $\{\bar{y}_j\}$ defined as follows: if $h(x_p)$ is the smallest of all the numbers $h(x_j)$ and $h(y_j)$, then let $\bar{x}_1 = x_p$ and $\bar{y}_1 = [h(x_p)/h(y_r)]y_r$ (for some r) and replace y_r by $[1 - h(x_p)/h(y_r)]y_r$. The analogous process is used if h takes on its minimum at one of the y_j's. This process creates two new elements and eliminates one old one at each step, until all of the x_j's or all of the y_j's are eliminated. If only x_j's remain, say x_{p_j}'s, then $\sum h(x_{p_j}) < \varepsilon$, and similarly $\sum h(y_{p_j}) < \varepsilon$ if only y_j's remain. Also

$$\sum h(\bar{x}_j) - \varepsilon = \sum h(\bar{y}_j) - \varepsilon < |||\, x\, ||| = |||\, y\, ||| = 1$$

and $h(\bar{x}_j) = h(\bar{y}_j)$ for each j. For each j, there are nonoverlapping blocks $\{\bar{x}_{ji}\}$ and $\{\bar{y}_{ji}\}$ such that

$$h(\bar{x}_j) = h(\bar{y}_j) = \{\textstyle\sum_i [\theta(\bar{x}_{ji})]^2\}^{1/2} = \{\textstyle\sum_i [\theta(\bar{y}_{ji})]^2\}^{1/2}\,.$$

Then

$$h(\bar{x}_j + \bar{y}_j) \leq \{\sum_i [\theta \bar{x}_{ji})]^2 + \sum_i [\theta(\bar{y}_{ji})]^2\}^{1/2} = \sqrt{2}\, h(\bar{x}_j) .$$

Hence

$$|||x + y||| \leq \sum h(\bar{x}_j + \bar{y}_j) + \varepsilon \leq \sqrt{2} \sum h(\bar{x}_j) + \varepsilon \leq \sqrt{2} + \varepsilon .$$

Since $|||x + y||| > 3/2$, this is contradictory if $\sqrt{2} + \varepsilon < 3/2$. It has therefore been shown that $\{u_i\}$ is a basis of type β.

Since $\{u_i\}$ is an orthogonal basis of type β for B, it follows that B^{**} consists of all sequences $F = (F_1, F_2, \cdots)$ for which

$$|||F||| = \lim_{n\to\infty} |||(F_1, \cdots, F_n, 0, 0, \cdots)|||$$

exists [4; page 174]. Note first that if $t = (t_1, \cdots) \in T_1$, then

$$|||(t_1, \cdots, t_n, 0, 0, \cdots)||| = ||(t_1, \cdots, t_n, 0, 0, \cdots)||$$

and $\lim_{n\to\infty} |||(t_1, \cdots, t_n, 0, 0, \cdots)||| = |||t||| = ||t||$. Thus $T_1 \subset B^{**}$. Also, the natural mapping of B into B^{**} is merely the mapping of a sequence in B onto the identical sequence in B^{**}. It then follows that $|||r + t||| \geq |||t|||$ if $r \in \pi(B)$ and $t \in T_1$, since r can be approximated by a sequence with a finite number of nonzero coordinates but (Lemma 1) $||t|| = \limsup |t_i|$.

Now suppose that $F = (F_1, F_2, \cdots)$ is a sequence for which $\lim_{n\to\infty}|||^nF|||$ exists; i.e., $F \in B^{**}$. It will be shown that there is an element v of $\pi(B) + T_1$ for which $|||F - v||| \leq 15/16\, |||F|||$. Successive application of this would then establish that $F \in \pi(B) + T_1$. For each n, there are ${}^nw_j$ and blocks $b^n_{j,i}$, which are either blocks of elements of T_1 or have only one nonzero coordinate, such that

$$|||^nF||| = \sum_j h({}^nw_j),\ {}^nF = \sum_j {}^nw_j,\ \text{and}\ h({}^nw_j) = \{\sum_i [\theta(b^n_{j,i})]^2\}^{1/2} ,$$

where each ${}^nw_j$ and each $b^n_{j,i}$ have all coordinates zero after the nth. This follows by a limit argument, using the facts (1) that there are only a finite number K_n of ways of choosing division points for nonoverlapping blocks from the integers $1, 2, \cdots, n$ and (2) that it follows from Lemma 1 and the orthogonality of the basis for T that $\theta(b^{2N}_{j,i})$, for a block $b^{2N}_{j,i}$ which has zero coordinates beyond the $2N$th coordinate, can be evaluated by using only members of the span of the first N basis elements of T.

If $m < n$ and ${}^mw^n_j$ is obtained from ${}^nw_j$ by replacing coordinates after the mth by zeros, then

$$|||^mF||| \leq \sum_j h({}^mw^n_j) \leq |||^nF||| \leq |||F||| .$$

If ${}^mw^n_{j_1}$ and ${}^mw^n_{j_2}$ are of the "same type" in the sense that they are divided into blocks by using the same division points, then it follows by using these same division points for ${}^mw^n_{j_1} + {}^mw^n_{j_2}$ that

$$h({}^m w^n_{j_1} + {}^m w^n_{j_2}) \leq h({}^m w^n_{j_1}) + h({}^m w^n_{j_2}) .$$

For each $n > m$, let ${}^m\hat{w}^n_j$ be the sum of all ${}^m w^n_{j_i}$ of the "same type" as ${}^m\hat{w}^n_j$. A limit argument gives a sequence of integers $\{n_i\}$ such that $\lim {}^m\hat{w}^{n_i}_j = {}^m\overline{w}_j$ exists for each "type". If $m < n$, then there exist $\bar{b}^n_{j,i}$ such that

$$|||\,{}^mF\,||| \leq \textstyle\sum_j h({}^m\overline{w}_j) \leq \sum_k h({}^n\overline{w}_k) \leq |||\,F\,||| ,$$
$$h({}^m\overline{w}_j) = \{\textstyle\sum_i [\theta(\bar{b}^m_{j,i})]^2\}^{1/2},\ {}^mF = \sum {}^m\overline{w}_j ,$$

and ${}^m\overline{w}_j$ is equal to the sum of all ${}^m\overline{w}^n_j$ which are of the same type as ${}^m\overline{w}_j$ and are obtained from ${}^n\overline{w}_j$ by replacing all coordinates after the mth by zeros. The points used to divide ${}^m\overline{w}_j$ into the blocks $\bar{b}^m_{j,i}$ will be called the *division points* of ${}^m\overline{w}_j$.

Choose M so that $|||\,{}^mF\,||| > 15/16\,|||\,F\,|||$. Note that if ${}^m\overline{w}_j$ is of a particular type and $n > m$, then ${}^m\overline{w}_j$ is the sum of one or more elements obtained from the ${}^n\overline{w}_k$'s by replacing coordinates after the mth by zeros. For $n > m \geq M$, let ${}^nt$ be the sum of all ${}^n\overline{w}_k$'s which have no division points between M and n and let ${}^mt^n$ be obtained from ${}^nt$ by replacing coordinates after the mth by zeros. Let $\{n_i\}$ be chosen so that

$$\lim_{i\to\infty} {}^mt^{n_i} = {}^m\bar{t}$$

exists for each $m \geq M$. Let $\bar{t}$ be defined so as to have the same first m coordinates as ${}^m\bar{t}$. Then any finite block of $\bar{t}$ whose first M coordinates are zero is also approximately a block of an element of T_1 and these elements of T_1 are of bounded norm. It then follows from Lemma 2 that there is an element v_0, with a finite number of nonzero coordinates, such that $v_0 + \bar{t} \in T_1$. Thus

$$\bar{t} \in \pi(B) \dotplus T_1 .$$

First assume that $|||\,\bar{t}\,||| > 1/8\,|||\,F\,|||$ and choose N so that

$$|||\,{}^n\bar{t}\,||| > 1/8\,|||\,F\,||| \text{ if } n > N .$$

For $n > N$, choose $p > n$ so that

$$|||\,{}^n\bar{t} - {}^nt^p\,||| < \frac{1}{32}\,|||\,F\,||| .$$

Since $|||\,{}^nF\,||| \leq \sum_j h({}^n\overline{w}_j)$, discarding all ${}^n\overline{w}^p_j$ without division points between M and p gives

$$\begin{aligned} |||\,{}^nF - {}^nt^p\,||| &\leq \textstyle\sum h({}^n\overline{w}_j) - |||\,{}^nt^p\,||| \\ &\leq |||\,F\,||| - |||\,{}^nt^p\,||| . \end{aligned}$$

Hence $|||\,{}^n F - {}^n\bar{t}\,||| < |||\,F\,||| - |||\,{}^n\bar{t}\,||| + 1/16\,|||\,F\,||| < 15/16\,|||\,F\,|||$. Since n was an arbitrary integer with $n > N$, it follows that

$$|||\,F - \bar{t}\,||| \le \frac{15}{16}\,|||\,F\,||| \;.$$

Now assume that $|||\,\bar{t}\,||| \le 1/8\,|||\,F\,|||$. Then $|||\,{}^n\bar{t}\,||| \le 1/8\,|||\,F\,|||$ for all n. Choose q so that

$$|||\,{}^M\bar{t} - {}^M t^q\,||| < \frac{1}{16}\,|||\,F\,||| \;.$$

For each ${}^q\bar{w}_j$ which has a division point between M and q, let u_j^q be obtained from ${}^q\bar{w}_j$ by replacing all coordinates after the last such division point by zeros. Let

$$u = \textstyle\sum_j u_j^q \;.$$

Choose $n > q$. Then ${}^n F = \sum {}^n\bar{w}_j$ and

$$\begin{aligned} |||\,{}^M F\,||| &\le \textstyle\sum h({}^M\bar{w}_j^n) \le \sum h(u_j^q) + |||\,{}^M t^q\,||| \\ &< \textstyle\sum h(u_j^q) + \dfrac{3}{16}\,|||\,F\,||| \;. \end{aligned}$$

Since $|||\,{}^M F\,||| > 15/16\,|||\,F\,|||$, we have $\sum h(u_j^q) > 3/4\,|||\,F\,|||$. Now consider $F - u$. Since $|||\,{}^n F\,||| \le \sum h({}^n\bar{w}_j)$, where $h({}^n\bar{w}_j) = \{\sum_i [\theta(\bar{b}_{j,i}^n)]^2\}^{1/2}$, we have

$$\begin{aligned} {}^n(F - u) &= \textstyle\sum {}^n\bar{w}_j - \sum u_j^q = \sum {}^n\tilde{w}_j \;, \\ |||\,{}^n(F - u)\,||| &\le \textstyle\sum h({}^n\tilde{w}_j) \;, \end{aligned}$$

where ${}^n\tilde{w}_j$ is obtained from ${}^n\bar{w}_j$ by replacing all coordinates before the last division point between M and q by zeros (if there is no such point, then ${}^n\tilde{w}_j = {}^n\bar{w}_j$). The following trivial facts will be used: If A and B are nonnegative and

$$\text{if } \sqrt{3}\,A < B, \text{ then } \sqrt{A^2 + B^2} > 2A\,;$$

$$\text{if } \sqrt{3}\,A \ge B, \text{ then } B < \sqrt{A^2 + B^2} - \frac{1}{4}A \;.$$

Each ${}^n\bar{w}_j$ which has a division point between M and q makes a contribution to some u_j^q. For such an ${}^n\bar{w}_j$, let

$$h({}^n\bar{w}_j) = [\textstyle\sum_r (A_r)^2 + \sum_s (B_s)^2]^{1/2} \;,$$

where the A_r's and B_r's are, respectively, the values of $\theta(\bar{b}_{j,i}^n)$ for $\bar{b}_{j,i}^n$ a block of some u_j^q and $\bar{b}_{j,i}^n$ *not* a block of any u_j^q. Then

$$h(u_j^q) \leq \sum [\sum_r (A_r)^2]^{1/2} ,$$

where the sum is over all ${}^n\overline{w}_j$ which make a contribution to u_j^q. Let $\sum_r (A_r)^2$ be of class (1) or of class (2) according as

$$\sqrt{3}\,[\sum (A_r)^2]^{1/2} < [\sum (B_s)^2]^{1/2} \text{ or } \sqrt{3}\,[\sum (A_r)^2]^{1/2} \geq [\sum (B_s)^2]^{1/2} .$$

Since $\sum h(u_j^q) > 3/4 \,|||F|||$, the sum of all terms of class (1) is not larger than $1/2\,|||F|||$ (otherwise we would have $\sum h({}^n\overline{w}_j) > |||F|||$) and the sum of all terms of class (2) is greater than $1/4\,|||F|||$. But for a term of class (2),

$$[\sum (B_s)^2]^{1/2} < h({}^n\overline{w}_j) - \frac{1}{4}[\sum (A_r)^2]^{1/2} .$$

Adding these inequalities for each ${}^n\overline{w}_j$ and discarding each $\sum (A_r)^2$ which is of class (1) gives

$$\sum h({}^n\tilde{w}_j) < \sum h({}^n\overline{w}_j) - \frac{1}{16}\,|||F||| \text{ and } |||{}^n(F-u)||| < \frac{15}{16}\,|||F||| .$$

Since n was an arbitrary integer with $n > q$, it follows that

$$|||F - u||| \leq \frac{15}{16}\,|||F||| .$$

The importance of the assumption in Theorem 1 that T_1 have a basis of type α is made clear by the fact that the theorem breaks down if T_1 has a subspace isomorphic with (c_0). In fact, in this case there can not be a separable space B with

$$B^{**} = \pi(B) \dotplus T_1$$

and T_1 separable, whether or not B and T_1 have bases. This follows from the fact that if a conjugate space R^* contains a subspace isomorphic with (c_0), then R^* contains a subspace isomorphic with (m) and is not separable. To establish this fact, suppose that $\{F_n\}$ are continuous linear functionals defined on some Banach space B and that the closed linear span of $\{F_n\}$ is isomorphic with (c_0), the correspondence being

$$\sum_1^\infty a_i F_i \leftrightarrow (a_1, a_2, \cdots) .$$

For any bounded sequence $w = (w_1, w_2, \cdots)$, define F_w by

$$F_w(f) = \lim_{n\to\infty}\Big(\sum_1^n w_i F_i\Big)(f) ,$$

for each f of B. This limit exists, since if it did not there would exist

$\varepsilon > 0$ and $G_1 = \sum_1^{n_1} w_i F_i$, $G_2 = \sum_{n_2}^{n_3} w_i F_i, \cdots$, with $1 \leq n_1 < n_2 \leq n_3 < n_4 \leq \cdots$, such that $G_i(f) > \varepsilon$. Then correct choice of signs would give

$$\sum_1^n \pm G_i(f) > n\varepsilon ,$$

which contradicts the boundedness of $\| \sum^n \pm G_i \|$. Clearly the correspondence with (c_0) is thus extended to a bicontinuous correspondence with (m).

THEOREM 2. *For any positive integer n, there is a Banach space B_n such that the nth conjugate space of B_n is the first nonseparable conjugate space of B_n.*

Proof. Let $B_1 = l^{(1)}$ and $B_2 = (c_0)$. Then B_1 has a basis of type α and B_2 has a basis of type β. In the following, the notation $R \dotplus S$ is used only if $\| r + s \| \geq \| s \|$ whenever $r \in R$ and $s \in S$. It follows from Theorem 1 that there is a separable Banach space B_3 with a basis of type β for which

$$B_3^{**} = B_3 \dotplus l^{(1)} = B_3 \dotplus B_2^*$$

Then B_3^{***} is nonseparable and B_3^* has a basis of type α [3, Theorem 3]. Now suppose that, for $k \leq n$, B_k has been found for which

$$B_k^{**} = B_k \dotplus B_{k-1}^*$$

if $k \geq 3$, B_k has a basis of type β if $k \geq 2$, and the kth conjugate space of B_k is the first nonseparable conjugate space of B_k. Then B_n^* has a basis of type α and it follows from Theorem 1 that there exists a separable space B_{n+1} which has a basis of type β and for which

$$B_{n+1}^{**} = B_{n+1} \dotplus B_n^* .$$

Then $B_{n+1}^{***} = B_{n+1}^* \dotplus B_n \dotplus B_{n-1}^*$. The $(n-2)$nd conjugate space of B_{n-1}^* is the first nonseparable conjugate space of B_{n-1}^*, while the $(n-2)$nd conjugate space of B_n is separable. Hence the $(n+1)$st conjugate space of B_{n+1} is the first nonseparable conjugate space of B_{n+1}.

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