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A NOTE ON EBERLEIN'S THEOREM

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This paper is concerned with locally convex spaces which are closed, separable subspaces of their strong biduals. Let E be a space of this type. We first prove that, for an element of E'', weak* continuity on E' is equivalent to sequential weak* continuity on the convex, strongly bounded subsets of E'. We then prove Eberlein's theorem for spaces of this type; i.e., we prove that, for the weakly closed subsets of E, countable weak compactness coincides with weak compactness. Finally, we show that the separability hypothesis in our first theorem is necessary.

Our notation and terminology will be that of [1]. The letter E will always denote a locally convex, topological vector space over the field of real numbers. If we want to call attention to a specific, locally convex topology t on E, we will write $E[t]$. The dual of E will be denoted by E'. The weakest topology on E which renders each element of E' continuous will be denoted by $\sigma(E, E')$. We shall be working with the strong topology, $\beta(E', E)$, on E'. This is the topology of uniform convergence on the convex, $\sigma(E, E')$-bounded subsets of E. E'' will denote the dual of $E'[\beta(E', E)]$. We shall often identify E with its canonical image in E''. The topology induced on E by its strong bidual, $E''[\beta(E'', E')]$, will be denoted by $\beta^*(E, E')$. Recall that $\beta^*(E, E')$ is the topology of uniform convergence on the convex, $\beta(E', E)$-bounded subsets of E'.

DEFINITION. We shall say that E has property (S) if the following is true: An element w of E'' is in E if and only if $\lim wf_n = 0$, whenever $\{f_n\}$ is a $\beta(E', E)$-bounded sequence of points of E' which is $\sigma(E', E)$-convergent to zero.

THEOREM 1. *Suppose that $E[\beta^*(E, E')]$ is separable. Then E has property (S) if and only if E is a closed, linear subspace of $E''[\beta(E'', E')]$.*

Proof. We shall prove sufficiency first. Let w be in E'' and suppose that $\lim wf_n = 0$, whenever $\{f_n\}$ is a $\beta(E', E)$-bounded sequence of points of E' which is $\sigma(E', E)$-convergent to zero. Let B be a convex, $\beta(E', E)$-bounded subset of E' and let F be the dual of $E[\beta^*(E, E')]$. Clearly $E' \subset F$ and, by [1; Prop. 2, p. 65], B is relatively $\sigma(F, E)$-compact. Since E is $\beta^*(E, E')$-separable, the restriction of $\sigma(F, E)$ to B is metrizable. Hence $\sigma(E', E)$ is metrizable on every convex, $\beta(E' E)$-bounded subset of E'. This fact, together with our assumptions on w, implies that w is $\sigma(E', E)$-continuous on every convex, $\beta(E', E)$-bounded subset of E'. Thus, by [4; Th. 10, p. 97], w is in the completion of $E[\beta^*(E, E')]$. But w is in E'' and E is closed in $E''[\beta(E'', E')]$. It follows that w is in E.

Now assume that E has property (S). Let w be a point in the closure of E for $E'[\beta(E'', E')]$, and let $\{f_n\}$ be a $\beta(E', E)$-bounded sequence of points of E' which is $\sigma(E', E)$-convergent to zero. We may, for each fixed positive integer k, choose x_k in E such that: (a) $|wf_n - x_k f_n| \leqq 1/k$ for every n. The inequality

$$|wf_n - wf_m| \leqq |wf_n - x_k f_n| + |x_k f_n - x_k f_m| + |x_k f_m - wf_m|$$

shows that $\lim wf_n$ exists. But by (a), this limit is $\leqq 1/k$ for every k. Thus, E is closed in $E''[\beta(E'', E')]$.

THEOREM 2. *If E has property (S), then every weakly closed, countably weakly compact subset of E is weakly compact.*

Proof. Let M be a weakly closed, countably weakly compact subset of E. Let w be a point in the closure of M for $E''[\sigma(E'', E')]$ and let $\{f_n\}$ be a sequence of points of E' which is $\beta(E', E)$-bounded and $\sigma(E', E)$-convergent to zero. For each positive integer k we may choose x_k in M such that: $|x_k f_n - wf_n| \leqq 1/k$ for $n \leqq k$. Thus, for each fixed n, $\lim x_k f_n = wf_n$. Since M is countably weakly compact $\{x_k\}$ has a weak adherent point x_0 in M. It follows that $wf_n = x_0 f_n$ for every n. But then $\lim wf_n = 0$ and, since E has property (S), w is in E and hence in M.

Let B be a Banach space and let Q be a linear subspace of B'. Following Dixmier [2], we shall say that Q has positive characteristic if $\{x \text{ in } Q \mid \|x\| \leqq 1\}$ is weak* dense in some ball of B'. If Q has positive characteristic and is also norm closed in B', then it is easily seen that $\beta^*(B, Q)$ is equivalent to the norm topology of B. Thus, if B is separable, then Theorem 2 shows that compactness and countable compactness coincide for the closed subsets of $B[\sigma(B, Q)]$. This result was first obtained by I. Singer [6] who also showed that it is no longer true if B is nonseparable; see [7]. Hence, in Theorem 1, the separability of $E[\beta^*(E, E')]$ is necessary.

In the preceding application we made use of the following:

THEOREM 3. *If $E[\beta^*(E, E')]$ is both complete and separable, then E has property (S).*

Y. Komura [5] has shown that the strong bidual of a locally convex space need not be complete. Thus Theorem 3 is weaker than Theorem 1.

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