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# EXTENSIONS OF THE MAXIMAL IDEAL SPACE OF A FUNCTION ALGEBRA

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**Let $A$ be a function algebra with its maximal ideal space $M_A$. Let $B$ be a function algebra such that $A \subset B \subset C(M_A)$. What can be said about $M_B$? We prove that $M_A = M_B$ if every point $x \in M_A$ has a fundamental neighborhood system $\{W\}$ such that the topological boundary $bW$ of each $W$ is contained in the Choquet boundary of $A$ or if $A$ is a normal function algebra. The first condition is satisfied if $M_A$ is a one dimensional topological space. Let $H(A)$ be the function algebra on $M_A$ generated by all functions which are locally approximable in $A$. We prove that $M_{H(A)} = M_A$ and then we try to generalize this result. If $f \in C(M_A)$ is such that $f$ is locally approximable in $A$ at every point where $f$ is different from zero then $M_A$ is the maximal ideal space of the function algebra generated by $A$ and $f$. We also look at closed subsets $F$ of $M_A$ such that $M_{H(F)} = F$ where $H(F)$ is the function algebra generated by restricting to $F$ all functions that are defined and locally approximable in $A$ in some neighborhood of $F$. These sets are called natural sets. We prove that there exists a smallest natural set $B(F)$ containing a closed set $F$ in $M_A$ and that the Silov boundary of $H(B(F))$ is contained in $F$. We also find conditions that guarantee that a closed set in $M_A$ is a natural set.**

If $X$ is a set and $f$ is a complex-valued function defined on $X$ then $|f|_V = \sup\{|f(x)| \mid x \in V\}$ for every $V \subset X$ and $f_V$ is the restriction of $f$ to $V$. If $V$ is a subset of a topological space $X$ then $bV$ is the topological boundary of $V$ in $X$. If $A$ is a function algebra we denote by $M_A$ its maximal ideal space, and $S_A$ its Shilov boundary. A point $x \in M_A$ is a strong boundary point in $A$ if $\{x\} = \cap P(f)$, where $P(f)$ are peak sets of $A$ in $M_A$. We shall use the wellknown fact that $S_A$ is the closure of the strong boundary points of $A$ in $M_A$. If $F$ is a closed set in $M_A$ then $\mathrm{Hull}_A(F) = \{x \in M_A \mid |f(x)| \leqq |f|_F$ for every $f \in A\}$. If $x \in \mathrm{Hull}_A(F)$ we say thet $F$ is a support of $x$. A minimal support of $x$ is a support $F$ of $x$ such that no proper closed subset of $F$ is a support of $x$. Now we have the principle of minimal supports. Let $F$ be a minimal support of $x$. Suppose $\{f_n\} \in A$ is such that $|f_n|_F \leqq K$ for some constant $K$ independent of $n$ and $\lim |f_n|_{W \cap F} = 0$, where $W$ is an open subset of $M_A$ such that $W \cap F$ is not empty. Then it follows that $\lim f_n(x) = 0$. If $F$ is a closed set in $M_A$ then $A_F$ is the function algebra on $F$ generated by functions $f \in C(F)$ such that $f = g$ on $F$ for some $g \in A$. Now $M_{A_F}$ can be identified with

$\mathrm{Hull}_A(F)$. If $F$ is a closed set in $M_A$ such that $F = \mathrm{Hull}_A(F)$ we say that $F$ is an $A$-convex set. $A$ is a convex function algebra if every closed set in $M_A$ is $A$-convex. If $B$ is a function algebra on $M_A$ such that $A \subset B$ then the maximal ideal space $M_B$ contains $M_A$ and $S_B \subset M_A$. If $x \in M_B$ there exists a point $y(x) \in M_A$ such that $f(x) = f(y(x))$ for $f \in A$. If $V$ is a subset of $M_A$ we put $\{V\}_B = \{x \in M_B \mid y(x) \in V\}$. The set $\{V\}_B$ is called the fiber of $V$ in $M_B$. The correspondence between points $x$ in $M_A$ and the fibers $\{x\}_B$ is continuous in the following way: Let $W$ be an open neighborhood of $\{x\}_B$ in $M_B$ for some point $x \in M_A$. Then there exists a neighborhood $V$ of $x$ in $M_A$ such that $\{V\}_B \subset W$. If $W$ is an open set in $M_A$ then $H_0(W) = \{f \in C(W) \mid f$ is locally approximable in $A$ at every point in $W$, i.e., if $x \in W$ there exists a neighborhood $V \subset W$ of $x$ and $\{g_n\} \in A$ such that $\lim |g_n - f|_V = 0.\}$. We put $H_0(A) = H_0(M_A)$ and $H(A)$ is the function algebra generated by $H_0(A)$ on $M_A$. If $F$ is a closed set in $M_A$ then $H_0(F) = \{f \in C(F) \mid f = g$ on $F$ for some $g \in H_0(V)$, where $V$ is some neighborhood of $F\}$. We let $H(F)$ be the function algebra on $F$ generated by $H_0(F)$. We shall now discuss the results of this paper. The general problem which interests us here is the following: Let $A$ be a function algebra with its maximal ideal space $M_A$. Let $B$ be a function algebra such that $A \subset B \subset C(M_A)$. What can be said about $M_B$? In Lemma 1 we give the well-known construction which shows that $M_B$ in general is strictly larger than $M_A$. A point $x \in M_A$ is a stationary point if $\{x\}_B = \{x\}$ for every $B$ such that $A \subset B \subset C(M_A)$. $A$ is a resistent function algebra if $M_A$ consists of stationary points. In Theorem 2 we prove that $A$ is a resistent function algebra if every point $x \in M_A$ has a fundamental neighborhood system $\{W\}$ such that $\{bW\}$ consist of stationary points. We remark here that the Choquet boundary of $A$ is contained in the set of stationary points and that $A$ is resistent if $M_A = [0, 1]$. A function algebra $A$ on a compact set $X$ is regular if $A$ separates points from closed subsets of $X$. It is wellknown that if $X = M_A$ then $A$ is normal, i.e., $A$ separates disjoint closed sets. In Theorem 4 we prove that if $A$ is a regular function algebra on $X$ then $X$ consists of stationary points when we consider $X$ as a closed subset of $M_A$. We remark that if $A$ is a normal function algebra on $X$ then $X = M_A$. The rest of this paper is mostly devoted to a study of relations between $A$ and $H(A)$. We have never introduced the general concept of $A$-holomorphic functions as is done in [3]. We wish to point out that our methods come almost entirely from [3] and [4]. Our proof of Theorem 5 uses an argument which is essentially the same as in Lemma 3.1, p. 368, in [3]. We point out that Theorem 7 gives a proof of Rado's Theorem: Let $f \in C(F)$ where $F$ is a polynomially convex compact set in the complex plane. Assume that $f$ is analytic if $f$ is different from zero. Then it follows that $f$ is analytic in the interior of $F$ and hence $f \in P(F)$, i.e., $f$ can be uniformly approximated by polynomials on $F$. In Theorem 8 we prove that if $H(A)$ is a resistent function algebra then $A$ is a resistent function algebra. We also discuss the general problem of determining 'domains of holomorphy' in general function algebras. A closed set $F$ in $M_A$ is a natural set if $M_{H(F)} = F$. The main result about natural sets is contained in Theorem 10 which was essentially wellknown in [3]. Every closed subset $F$ of $M_A$ is contained in a smallest natural set $B(F)$, the barrier of $F$. We have also introduced the set $\hat{F} = \{y \in M_A \mid \{y\}_{H(F)} \cap M_{H(F)}$ is not empty$\}$. We know that $\hat{F} \subset B(F)$ and in general the inclusion is strict.[1] Theorem 12 is essentially wellknown in [5] but we believe our proof is different.

1. DEFINITION 1. A function algebra $A$ is resistent if $M_B = M_A$ for every function algebra $B$ such that $A \subset B \subset C(M_A)$.

LEMMA 1. *A resistent function algebra is convex.*

*Proof*. Let $A$ be a function algebra such that $\mathrm{Hull}_A(F) - F$ is not empty for some closed set $F$ in $M_A$. Let $B = \{g \in C(M_A) \mid g_F \in A_F\}$. Obviously $A \subset B \subset C(M_A)$ and now we prove that $M_A \neq M_B$. Let $x \in \mathrm{Hull}_A(F) - F$. If $g \in B$ we can find $\{f_n\} \in A$ such that $\lim |g - f_n|_F = 0$. Now we put $\hat{x}(g) = \lim f_n(x)$. It is easily seen that $\hat{x}$ is a well defined complex-valued homomorphism on $B$. Hence there exists a point $y \in M_B$ such that $\hat{x}(g) = g(y)$ for $g \in B$. In particular $f(x) = \hat{x}(f) = f(y)$ for $f \in A$. If $M_A = M_B$ it follows that $\hat{x}(g) = g(x)$ for $g \in B$. But now we choose $g \in B$ such that $g(x) = 1$ while $g = 0$ on $F$ and obtain a contradiction. Hence $M_A \neq M_B$ and the lemma follows.

LEMMA 2. *Let $A$ be a convex function algebra and let*

$$A \subset B \subset C(M_A) .$$

*Then the fibers $\{x\}_B$ are connected in $M_B$ for every point $x \in M_A$.*

*Proof*. Suppose that some fiber $(x)_B$ is disconnected in $M_B$. Hence there exists a closed component $G$ of $\{x\}_B$ such that $G \subset M_B - M_A$. Now we can find a closed neighborhood $W$ of $G$ in $M_B$ such that $bW \cap \{x\}_B$ is empty and $W \subset M_B - M_A$. Let $F = \{y \in M_A \mid \{y\}_B \cap bW$ is not empty$\}$. Obviously $F$ is a closed subset of $M_A$ such that $x \notin F$. Let $y \in G$, then the local maximum principle shows that $|g(y)| \leqq |g|_{bW}$ for $g \in B$. It follows that $|f(x)| \leqq |f|_F$ for $f \in A$, hence $x \in \mathrm{Hull}_A(F)$,

[1] I am indebted to the referee for giving an example where $F \neq B(\hat{F})$.

a contradiction to the fact that $A$ is a convex function algebra.

THEOREM 1. *Let $V$ be a closed $A$-convex subset of $M_A$ such that $A_V$ is resistent. Let $f \in C(M_A)$ be such that $f = 0$ in $M_A - V$, then $M_{A(f)} = M_A$.*

*Proof.* Assume that $D = M_{A(f)} - M_A$ is not empty. Let $x \in D$ and choose a minimal support $F$ of $x$ such that $F \subset M_A$. Now $F \subset V$ is impossible since $A_V$ is a resistent function algebra. Because $f = 0$ in $M_A - V$ the principle of minimal supports shows that $f(x) = 0$. Choose $y \in M_A$ such that $g(x) = g(y)$ for $g \in A$. Since $y$ and $x$ are different points of $M_{A(f)}$ it follows that $f(y)$ must be different from zero, hence $y \in V$. We have now proved that $D \subset \{V\}_{A(f)}$. Now Lemma 1 shows that $A_V$ is a convex function algebra and Lemma 2 can be applied to show that $\{z\}_{A(f)}$ are connected in $M_{A(f)}$ for every $z \in V$. In particular $\{y\}_{A(f)}$ has no isolated points in $M_{A(f)}$. Since $D$ is an open subset of $M_{A(f)}$ we can find $x_1 \in D \cap \{y\}_{A(f)}$ such that $x_1 \neq x$. But now we get $f(x_1) = f(x) = 0$ and then $x$ and $x_1$ are not different points in $M_{A(f)}$, a contradiction.

DEFINITION 2. A point $x \in M_A$ is stationary if $\{x\}_B = \{x\}$ for every function algebra $B$ such that $A \subset B \subset C(M_A)$.

THEOREM 2. *Let $A$ be a function algebra such that every point $x \in M_A$ has a fundamental neighborhood system $\{W\}$ such that each $bW$ consists of stationary points, then $A$ is a resistent function algebra.*

*Proof.* Suppose that $B$ is a function algebra such that

$$A \subset B \subset C(M_A)$$

and assume that $D = M_B - M_A$ is not empty. Let $z \in D$ and choose $y \in M_A$ such that $f(z) = f(y)$ for $f \in A$. Choose an open neighborhood $V$ of $y$ in $M_A$ such that $bV$ consists of stationary points. Let $W$ be a closed $B$-convex neighborhood of $z$ in $M_B$ such that $W \subset D$. Now $\{V\}_B \cap W$ is open and closed in $W$. We apply Shilov's Idempotent Theorem to the function algebra $B_W$. Hence we find $\{f_n\} \in B$ such that $\lim |f_n - 1|_{W \cap \{V\}_B} = 0$ while $\lim |f_n|_{W - \{V\}_B} = 0$. Choose a minimal support $F$ of $z$ such that $F \subset bW$. It follows from the principle of minimal supports that $F \subset bW \cap \{V\}_B$. Now we let $V$ shrink to $y$ in $M_A$ and it follows that $z \in \mathrm{Hull}_B(bW \cap \{y\}_B)$. This holds for every $z \in D \cap \{y\}_B$ when $W$ is a closed $B$-convex neighborhood of $z$ such that $W \subset D$. Now we choose a strong boundary point $x \in D \cap \{y\}_B$ of the function algebra $B_{\{y\}_B}$ to obtain a contradiction.

DEFINITION 4. A point $x \in M_A$ is locally regular if there exists a neighborhood $V$ of $x$ such that to every $y \in V - \{x\}$ there exists $f \in A$ with $f = 0$ in a neighborhood of $y$ and $f(x) = 1$.

THEOREM 3. *A locally regular point is a stationary point.*

*Proof.* Let $x \in M_A$ be a locally regular point. Let $B$ be a function algebra such that $A \subset B \subset C(M_A)$. Let $D = M_B - M_A$ and assume that $\{x\}_B \cap D$ is not empty. Let $V$ be an open neighborhood of $x$ in $M_A$ such that to every $y \in V - \{x\}$ there exists $f \in A$ with $f = 0$ in a neighborhood of $y$ and $f(x) = 1$. Let $z \in \{x\}_B \cap D$ and choose a closed neighborhood $W$ of $z$ in $M_B$ such that $W \subset D \cap \{V\}_B$. Let $F$ be a minimal support of $z$ such that $F \subset bW$. It follows now that $F \subset \{x\}_B$ holds. Hence $z \in \mathrm{Hull}_B(bW \cap \{x\}_B)$ and we obtain a contradiction if we choose a suitable point $z \in D \cap \{x\}_B$. Hence $\{x\}_B \cap D$ must be empty and it follows that $x$ is a stationary point.

THEOREM 4. *Let $A$ be a regular function algebra on a compact set $X$. Then every point $x \in X \cap M_A$ is a stationary point.*

*Proof.* Let $x \in X \cap M_A$ and put $R(x) = \{y \in M_A \mid \text{there exists } g \in A$ with $g = 0$ in a neighborhood of $y$ and $g(x) = 1\}$. We shall now prove that $R(x) = M_A - \{x\}$ and then it follows from Theorem 3 that $x$ is a stationary point. Let $y \in M_A - \{x\}$ and choose $g \in A$ such that $g(y) = 1$ and $g(x) = 0$. Let $V = \{z \in M_A \mid |g(z)| > 1/2\}$ and let $W = \{z \in X \mid |g(z)| \leqq 1/2\}$. We choose $f \in A$ such that $f = 0$ on $X - W$ and $f(x) = 1$. If $z \in V$ we can choose a minimal support $F$ of $z$ such that $F \subset X$. Obviously $F \cap (X - W)$ is not empty and the principle of minimal supports implies that $f(z) = 0$. Hence $f = 0$ on $V$ and $f(x) = 1$, i.e., $y \in R(x)$.

THEOREM 5. *Let $F$ be a closed subset of $M_A$ and let $f \in CM_A$ be such that $f$ is locally approximable in $A$ at every point in $M_A - F$. Then $M_{A(f)} - M_A \subset \{\mathrm{Hull}_A(F)\}_{A(f)}$.*

*Proof.* Let $D = M_{A(f)} - M_A$. Let $K = \mathrm{Hull}_{A(f)}(bD)$ and let $C = A(f)_K$. We have $D \subset K = M_C$ and $bD$ contains the Shilov boundary of $C$. Let $x \in bD$ be a strong boundary point of $C$. Assume that $x \in M_A - F$. Choose a closed neighborhood $V$ of $x$ in $M_A$ such that there exists $\{g_n\} \in A$ with $\lim |g_n - f|_V = 0$. Now we choose $h \in C$ such that $h(x) = |h|_K = 1$ and $\{x \in K \mid |h(x)| \geqq 1/2\} \subset \{V\}_{A(f)}$. Let

$$D_1 = \{x \in D \mid |h(x)| > 1/2\} .$$

The topological boundary $bD_1$ of $D_1$ in $K$ is obviously contained in the set $T = \{x \in bD \mid |h(x)| \geqq 1/2\} \cup \{x \in K \mid |h(x)| = 1/2\}$. Choose a point

$x_1 \in D_1$. Now the local maximum principle shows that we can find a minimal support $F$ of $x_1$ in $C$ such that $F \subset T$. Since $|h(x_1)| > 1/2$ it follows that $F \cap bD$ contains an open subset of $F$. Since $F \subset T \subset \{V\}_{A(f)}$ we have $|g|_F \leqq |g|_V$ for $g \in A$. Now $\lim |g_n - f|_{F \cap bD} \leqq \lim |g_n - f|_V = 0$ and the principle of minimal supports shows that $\lim g_n(x_1) = f(x_1)$ holds. Now we also have $x_1 \in \{y_1\}_{A(f)}$ for some point $y_1 \in V$. Hence $f(y_1) = \lim g_n(y_1) = g_n(x_1) = f(x_1)$ and then $x_1$ and $y_1$ cannot be different points in $M_{A(f)}$, a contradiction. We have now proved that every strong boundary point of $C$ must belong to $F$. It follows that $S_C \subset F$ and hence $M_{A(f)} - M_A \subset \text{Hull}_{A(f)}(F)$. This implies that $M_{A(f)} - M_A \subset \{\text{Hull}_A(F)_{A(f)}$.

LEMMA 3. *Let $A$ be a function algebra on a compact set $X$. Let $F$ be a closed subset of $X$. Then there exists a point $x \in F$ such that if $m$ is a representing measure of $x$ in $A$ with $m(F) = 1$ then $m = e_x$, i.e., $m$ is the unit point mass at $x$.*

*Proof*. Choose a strong boundary point $x \in F$ of the function algebra $A_F$.

THEOREM 6. *Let $A \subset B \subset C(M_A)$. Let $f \in B$ be such that $f \in H_0(A)$. Then $f$ is constant on each fiber $\{x\}_B$ for $x \in M_A$.*

*Proof*. If $x \in M_B$ we denote by $y(x)$ the point in $M_A$ such that $x \in \{y(x)\}_B$. Let $d(x) = |f(x) - f(y(x))|$ and assume that $d(x)$ is different from zero. Let $F = \{x \in M_B \mid d(z) = \|d\| = \sup d(z)\}$. Obviously $F$ is a closed subset of $M_B$ and $F \cap M_A$ is empty. Let $x \in F$ and choose an open neighborhood $V$ of $y(x)$ in $M_A$ such that there exists $\{g_n\} \in A$ with $\lim |g_n - f|_V = 0$. Choose now a closed neighborhood $W$ of $x$ in $M_B$ such that $W \subset \{V\}_B \cap (M_B - M_A)$. Let $T$ be a minimal support of $x$ such that $T \subset bW$. Now we can find a positive measure on $T$ such that $g(x) = \int g dm$ from $g \in B$. It follows that $|f(x) - g_n(y(x))| = |f(x) - g_n(x)| \leqq \int |f - g_n| dm$ for every $n$. Hence we also get

$$|f(x) - f(y(x))| \leqq \int |f(z) - f(y(z))| dm(z) .$$

It follows that $|f(z) - f(y(z))| = \|d\|$ for every $z \in T$, hence $T \subset F$. We have now proved that $x \in \text{Hull}_B(bW \cap F)$ for every $x \in F$ and every closed neighborhood $W$ of $x$ such that $W \subset (M_B - M_A)$. Now we derive a contradiction from Lemma 3.

THEOREM 7. *Let $f \in C(M_A)$ and suppose that $f$ is locally approximable in $A$ at every point where $f$ is different zero. Then $M_{A(f)} = M_A$ and $\text{Hull}_A(F) = \text{Hull}_{A(f)}(F)$ for every closed subset $F$ of $M_A$.*

*Proof.* Let $F$ be a closed subset of $M_A$ such that $F = \text{Hull}_{A(f)}(F)$. Let us put $G = \text{Hull}_A(F)$ and assume that $D = G - F$ is not empty. Let $C = A(f)_G$. We see that the Shilov boundary $S_C$ of $C$ meets $D$. Hence we can find $x \in D$ such that $x$ is a strong boundary point of $C$. Let us assume that $f(x) \neq 0$. Choose a closed neighborhood $V \subset (M_A - F)$ of $x$ in $M_A$ such that there exist $\{g_n\} \in A$ with $\lim |g_n - f|_V = 0$. Now we choose $h \in C$ such that if $P(h) = \{x \in G \mid h(x) = |h|_G\}$ then $x \in P(h)$ and $P(h) \subset V$ with $P(h) \cap bV$ empty. Since $h \in C$ we can find $\{h_n\} \in A$ with $\lim |h_n - h|_{V \cap G} = 0$. Now the local maximum principle shows that $|g(x)| \leqq |g|_{bV \cap G}$ for $g \in A$. It follows that $|h(x) = \lim |h_n(x)| \leqq \lim |h_n|_{bV \cap G} = |h|_{bV \cap G}$, contradiction to the fact that $P(h) \cap bV$ is empty. Hence we have proved that if $x \in D$ is a strong boundary point of $C$ then $f(x) = 0$. If $x \in D$ we can choose a minimal support $T$ of $x$ such that $T \subset S_C$. Since $F = \text{Hull}_{A(f)}(F)$ it follows that $T \cap D$ is not empty. Since $f = 0$ on $S_C \cap D$ it follows from the principle of minimal supports that $f(x) = 0$. Hence we have proved that $f = 0$ on $D$. But then $A(f)_D = A_D$ and it follows easily that $D$ cannot contain any strong boundary point of $C$. Hence $S_C \subset F$ which shows that $D$ must be empty. We have now proved that $\text{Hull}_A(F) = \text{Hull}_{A(f)}(F)$ for every closed subset $F$ of $M_A$. In particular we see that $Z(f) = \{x \in M_A \mid f(x) = 0\}$ is an $A$-convex set and using Theorem 5 it follows easily that $M_A = M_{A(f)}$.

COROLLARY 1. *$M_A = M_{H(A)}$ and $\text{Hull}_A(F) = \text{Hull}_{H(A)}(F)$ for every closed subset $F$ of $M_A$.*

THEOREM 8. *If $H(A)$ is a resistent function algebra then $A$ is a resistent function algebra.*

*Proof.* If $A$ is not a resistent function algebra we can find $g_1 \cdots g_k \in C(M_A)$ such that $g_1 \cdots g_k$ have no common zero on $M_A$ while $g_i(z) = \cdots = g_k(z) = 0$ for some point $z \in M_{A(g_1 \cdots g_k)}$. Because $H(A)$ is resistent we can find $h_1 \cdots h_k$, where each $h_i$ is a polynomial in $g_1 \cdots g_k$ with coefficients in $H_0(A)$, such that $|h_1 g_1 + \cdots + h_k g_k - 1|_{M_A} < 1/2$. Let $h_i = \sum f_{iv} g^v$, where $v$ runs over a finite set of multi-indices $(v_1 \cdots v_k)$ and $g^v = g_1^{v_1} \cdots g_k^{v_k}$. Each $f_{iv} \in H_0(A)$ and we define $f_{iv}$ on $M_{A(g_1 \cdots g_k)}$ by letting $f_{iv}$ be constant on each fiber of $M_{A(g_1 \cdots g_k)}$ over points of $M_A$. Each $g^v$ is defined on $M_{A(g_1 \cdots g_k)}$ in the usual way. In this way we can extend each $h_i$ to $M_{A(g_1 \cdots g_k)}$. Call these extensions $H_1 \cdots H_k$. It is easily seen that $H = H_1 g_1 + \cdots + H_k g_k$ is locally approximable in $A(g_1 \cdots g_k)$ on $M_{A(g_1 \cdots g_k)}$. Now $H(z) = 0$ while $|H - 1|_{M_A} < 1/2$ and since $M_A$ contains the Shilov boundary of $A(g_1 \cdots g_k)$ we derive a contradiction from Corollary 1.

THEOREM 9. *Let $f \in C(M_A)$ be such that $f^n + a_1 f^{n-1} + \cdots + a_n = 0$*

*on $M_A$ where $a_1 \cdots a_n \in A$, then $M_A = M_{A(f)}$.*

*Proof.* Let $g = nf^{n-1} + (n-1)a_1 f^{n-2} + \cdots + a_{n-1}$. It is well known that $f$ is locally approximable in $A$ at every point $x \in M_A$ where $g(x)$ is different from zero. (See [1], Th. 3.2.5, p. 71.) It follows that $g$ is locally approximable in $A$ at every point where $g$ is different from zero. Now Theorem 7 shows that $Z(g)$ is $A$-convex and then Theorem 5 shows that $M_{A(f)} - M_A \subset \{Z(g)\}_{A(f)}$. Let us put $B = A_{Z(g)}$, then $M_B = Z(g)$ and the restriction of $f$ to $M_B$ satisfies the equation $nf^{n-1} + (n-1)b_1 f^{n-2} + \cdots + b_{n-1} = 0$ where $b_i \in B$ are the restrictions of $a_i$ to $Z(g)$. Since $M_{A(f)} - M_A \subset \{Z(g)\}_{A(f)}$ we see that $M_{B(f)} - M_B$ is not empty if $M_{A(f)} - M_A$ is not empty. Hence we can use induction over $n$ to prove that $M_{A(f)} = M_A$.

Let $A$ be a function algebra. If $F$ is a closed subset of $M_A$ we have defined the function algebra $H(F)$. We are now interested in the maximal ideal space of $H(F)$.

DEFINITION. If $F$ is a closed subset of $M_A$ we put $\hat{F} = \{y \in M_A \mid \{y\}_{H(F)} \cap M_{H(F)}$ is not empty$\}$.

DEFINITION. A natural set in $M_A$ is a closed subset $F$ of $M_A$ such that $F = M_{H(F)}$.

LEMMA 4. *$(\cap F_a)\hat{} \subset \cap \hat{F}_a$ for every family $\{F_a\}$ of closed subsets of $M_A$.*

*Proof.* Let $y \in M_A$ be such that $y \in (\cap F_a)\hat{}$. Hence there exists a complex-valued homomorphism $C$ of $H(\cap F_a)$ such that $C(g) = g(y)$ for $g \in A$. If $f \in H(F_a)$ the restriction of $f$ to $\cap F_a$ obviously gives an element of $H(\cap F_a)$. Hence $C$ can be restricted to $H(F_a)$ and we obtain a complex-valued homomorphism of $H(F_a)$ such that $C(g) = g(y)$ for $g \in A$.

THEOREM 10. *Let $F$ be a closed subset of $M_A$ such that $F = \hat{F}$, then $M_{H(F)} = F$.*

*Proof.* Let $f \in H_0(F)$ and define $d(x) = |f(x) - f(y(x))|$ on $M_{H(F)}$ where $y(x)$ is the point in $F$ such that $g(x) = g(y(x))$ for $g \in A$. Assume that $d$ is not identical zero. Let $D = \{x \in M_{H(F)} \mid d(x) > 0\}$. Obviously $D \cap F$ is empty and hence $D$ lies off the Shilov boundary of $H(F)$. Hence $D \subset K = \mathrm{Hull}_{H(F)}(bD)$. Let us put $C = H(F)_K$ and choose $x \in bD$ such that $x$ is a strong boundary point of $C$. Choose a closed neighborhood $V$ of $y(x)$ in $M_A$ such that there exists $\{g_n\} \in A$ with $\lim |g_n - f|_{V \cap F} = 0$. Now we choose $h \in C$ such that $h(x) = |h|_K = 1$ and $\{x \in K \mid |h(x)| \geqq 1/2\} \subset \{V \cap F\}_{H(F)}$. Now we obtain a contradiction using the same argument as in the final part of Theorem 5. Hence we have proved that if $f \in H_0(F)$ then $f$ is constant on each fiber $\{x\}_{H(F)}$ when $x \in F$. Since $H_0(F)$ is a dense subalgebra of $H(F)$ it follows that $F = M_{H(F)}$.

COROLLARY 2. *If* $\{F_a\}$ *is a family of natural set of* $M_A$ *then* $\cap F_a$ *is a natural set.*

*Proof.* Lemma 4 shows that $(\cap F_a)^\wedge \subset \cap \hat{F}_a = \cap F_a$ and then Theorem 10 implies that $\cap F_a$ is a natural set.

DEFINITION. If $F$ is a closed subset of $M_A$ then $B(F)$ is the intersection of all natural sets containing $F$. $B(F)$ is called the barrier of $F$.

Corollary 2 shows that $B(F)$ is the smallest natural set containing a closed subset $F$ of $M_A$.

LEMMA 5. *Let* $F$ *be a natural set. Let* $f \in H(F)$ *and let* $F_1 = \{x \in F \,||\, f(x)| \leqq 1\}$. *Then* $F_1$ *is a natural set.*

*Proof.* Let $z \in M_{H(F_1)}$. If $g \in H(F)$ the restriction of $g$ to $F_1$ gives an element of $H(F_1)$. It follows that $g(z) = g(y)$ for some point $y \in M_{H(F)}$ when $g \in H(F)$. In particular $f(z) = f(y)$ and since $|f(z)| \leqq |f|F_1$ it follows that $y \in F_1$. Hence we have proved that $F_1 = \hat{F}_1$ and now Theorem 10 implies that $F_1$ is a natural set.

THEOREM 11. *Let* $F$ *be a closed subset of* $M_A$. *Let* $S(F)$ *be the Shilov boundary of* $H(B(F))$. *Then* $S(F) \subset F$.

*Proof.* Assume that $S(F)$ meets $B(F) - F$. Hence we can find $x \in B(F) - F$ such that $x$ is a strong boundary point of $H(B(F))$. Now we can choose $f \in H(B(F))$ such that $F_1 = \{x \in B(F) \,||\, f(x)| \leqq 1\}$ contains $F$ and omits the point $x$.

Lemma 5 shows that $F_1$ is a natural set, a contradiction to the fact that $B(F)$ is the smallest natural set containing $F$.

We finally give some examples of natural subsets of $M_A$.

DEFINITION. An $A$-analytic polyhedron $P$ is a closed set in $M_A$ of the form $P = \{x \in V \,||\, f_a(x)| \leqq 1$ where $V$ is an open neighborhood of $P$ and $\{f_a\}$ is a family in $H_0(V)\}$.

THEOREM 12. *An* $A$*-analytic polyhedron is a natural set.*

*Proof.* Let $U$ be an open neighborhood of $P$ and $W$ a closed set containing $U$ such that $W \subset V$. Now we can find finitely many $\{f_a\}$, say $f_1 \cdots f_k$ such that $P_1 = \{x \in W \| f_i(x) | \leqq 1,\ i = 1 \cdots k\}$ is contained in $U$. Now we can prove that $P_1$ is a natural set using the same argument as in the final part of Theorem 5. Finally we let $U$ shrink to $P$ and obtain natural sets $\{P_U\}$ such that $P = \cap P_U$. Now Corollary 2 shows that $P$ is a natural set.

DEFINITION. If $F$ is a closed subset of $M_A$ we put $R_0(F) = \{h \in C(F) | h = f/g$ where $f, g \in A$ and $g$ has no zero on $F\}$.

We let $R(F)$ be the function algebra on $F$ generated by $R_0(F)$.

DEFINITION. If $F$ is a closed subset of $M_A$ we put $\mathrm{Hull}_R(F) = \{x \in M_A | g(x) \in g(F)$ for $g \in A\}$.

THEOREM 13. *$M_{R(F)} = \mathrm{Hull}_R(F)$ for every closed set $F$ in $M_A$ and if $M_{R(F)} = F$ then $F$ is a natural set.*

*Proof.* If $y \in M_{R(F)}$ we choose $x \in M_A$ such that $g(y) = g(x)$ for $g \in A$. It is easily seen that $x \in \mathrm{Hull}_R(F)$ and that $(f/g)(y) = f(x)/g(x)$ when $f/g \in R_0(F)$. Since $R_0(F)$ is dense in $R(F)$ it follows that $y$ is uniquely determined by $x$. Conversely if we choose $x \in \mathrm{Hull}_R(F)$ then the mapping $X; f/g \rightarrow f(x)/g(x)$ is well defined on $R_0(F)$. We have $|f(x)/g(x)| \leqq |f/g|_F$ for if $f(x) = g(x)$ while $|f/g|_F < 1$ we see that $(g - f)$ is different from zero on $F$ and hence $(g - f)(x) \in (g - f)(F)$ is different from zero, a contradiction. Hence we can extend $X$ to $R(F)$ and we obtain a complex-valued homomorphism on $R(F)$ such that $g$ is mapped into $g(x)$ when $g \in A$. This proves that $M_{R(F)} = \mathrm{Hull}_R(F)$. If $M_{R(F)} = F$ then Corollary 1 can be applied to prove that $F$ is a natural set.

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