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CHARACTERIZING THE DISTRIBUTIONS OF THREE INDEPENDENT n-DIMENSIONAL RENDOM VARIABLES, X_1, X_2, X_3, HAVING ANALYTIC CHARACTERISTIC FUNCTIONS BY THE JOINT DISTRIBUTION OF $(X_1 + X_3,\ X_2 + X_3)$.

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Kotlarski characterized the distribution of three independent real random variables X_1, X_2, X_3 having nonvanishing characteristic functions by the joint distribution of the 2-dimensional vector $(X_1 + X_3, X_2 + X_3)$. In this paper, we shall give a generalization of Kotlarski's result for X_1, X_2, X_3 n-dimensional random variables having analytic characteristic functions which can meet the value zero.

In [3], Kotlarski shows that, for three independent random variables X_1, X_2, X_3, the distribution of $(X_1 + X_3, X_2 + X_3)$ determines the distributions of X_1, X_2 and X_3 up to a change of the location if the characteristic function of the pair $(X_1 + X_3,\ X_2 + X_3)$ does not vanish. Kotlarski also remarks that this result can be generalized in two ways. The statement remains true if the requirement that the pair $(X_1 + X_3,\ X_2 + X_3)$ has a nonvanishing characteristic function is replaced by the requirement that the random variables, X_1, X_2, X_3, possess analytic characteristic functions. The statement also remains true if X_1, X_2 and X_3 are n-dimensional real random vectors such that the pair $(X_1 + X_3,\ X_2 + X_3)$ has a nonvanishing characteristic function. In this paper, Kotlarski's result is generalized to the case where X_1, X_2, and X_3 are n-dimensional real random vectors possessing analytic characteristic functions.

1. Some notions and lemmas about analytic functions of several complex variables. Let R_n denote n-dimensional real Euclidean space, C_n denote n-dimensional complex Euclidean space, and let $f(t_1, \cdots, t_n)$ be defined on some domain D in C_n. The function f is said to be *analytic at the point* $(t_1^0, \cdots, t_n^0)$ in D if f can be represented by a convergent power series in some neighborhood of $(t_1^0, \cdots, t_n^0)$. The function f is said to be *analytic on the domain* D if it is analytic at every point in D. We now list several lemmas concerning analytic functions of several complex variables; for a discussion of these lemmas and further exposition on this theory, see [2].

LEMMA A. *If $f(t_1, \cdots, t_n)$ and $g(t_1, \cdots, t_n)$ are analytic at the* *point* $(t_1^0, \cdots, t_n^0)$, *and if* $f(t_1^0, \cdots, t_n^0) \neq 0$, *then the quotient* $\frac{g}{f}$ *is also analytic at* $(t_1^0, \cdots, t_n^0)$.

LEMMA B. (*Principle of analytic continuation*). *If* f *and* g *are analytic on some domain* D *in* C_n *and if* $f(t_1, \cdots, t_n) = g(t_1, \cdots, t_n)$ *at every point in some subdomain of* D, *then* $f(t_1, \cdots, t_n) = g(t_1, \cdots, t_n)$ *at all points of* D.

2. The main theorem and its proof.

THEOREM. *Let* X_1, X_2, X_3 *be three independent, real,* n*-dimensional random vectors, and let* $Z_1 = X_1 + X_3$, $Z_2 = X_2 + X_3$. *If the random vectors* X_k *possess characteristic functions* ϕ_k *which are analytic on domains* D_k, *with* $\bar{O} \in D_k$, $(k = 1, 2, 3)$, *then the distributions of* (Z_1, Z_2) *determines the distributions of* X_1, X_2 *and* X_3 *up to a change of the location.*

Proof. Let $t = (t_1, t_2, \cdots, t_n)$, $s = (s_1, s_2, \cdots, s_n)$ denote arbitrary points in C_n and $\bar{0} = (0, 0, \cdots, 0)$ denote the origin in C_n; let

$||t|| = \sqrt{|t_1|^2 + |t_2|^2 + \cdots + |t_n|^2}$ and let $t \cdot s = t_1 s_1 + t_2 s_2 + \cdots + t_n s_n$.

Let $\phi_k = Ee^{it \cdot X_k}$, the characteristic function of X_k, be defined on the domain $D_k \in C_n$, $(k = 1, 2, 3)$. Then, letting $\phi(t, s)$ denote the characteristic function of the distribution of the pair (Z_1, Z_2), we have

$$\begin{aligned} \phi(t, s) &= Ee^{i(t \cdot Z_1 + s \cdot Z_2)} \\ &= Ee^{i(t \cdot X_1 + s \cdot X_2 + (t+s) \cdot X_3)} \\ &= Ee^{it \cdot X_1} \; Ee^{is \cdot X_2} \; Ee^{i(t+s) \cdot X_3} \\ &= \phi_1(t) \; \phi_2(s) \; \phi_3(t + s) \end{aligned}$$

where this function is defined on the domain

$$D = \{(t, s) : \; t \in D_1, s \in D_2, (t + s) \in D_3\} \in C_{2n} .$$

Let U_1, U_2, U_3 be three other independent, real, n-dimensional random vectors possessing characteristic functions ψ_1, ψ_2, ψ_3 which are analytic on domains D_1^*, D_2^*, D_3^*. Let $V_1 = U_1 + U_3$, $V_2 = U_2 + U_3$ and let $\psi(t, s) = Ee^{i(t \cdot V_1 + s \cdot V_2)}$. Calculations analogous to those above yield

$$\psi(t, s) = \psi_1(t) \; \psi_2(s) \; \psi_3(t + s)$$

on

$$D^* = \{(t, s) : \; t \in D_1^*, \; s \in D_2^*, \; (t + s) \in D_3^*\} \in C_{2n} .$$

Suppose that the pairs (Z_1, Z_2) and (V_1, V_2) have the same distribution; we shall show that the distributions of X_k and U_k, $(k = 1, 2, 3)$ are equal up to a shift. If the pairs (Z_1, Z_2) and (V_1, V_2) have the same distribution, their characteristic functions are equal so that $D = D^*$ and

$$\psi_1(t)\ \psi_2(s)\ \psi_3(t+s) = \phi_1(t)\ \phi_2(s)\ \phi_3(t+s)\ . \tag{1}$$

Since each of the functions in equation (1) is analytic and equal to 1 at $\bar{0}$, there exists a domain $D^{**} \in C_{2n}$ of the form

$$\{(t, s):\ \sqrt{\|t\|^2 + \|s\|^2} < \alpha,\ \alpha > 0\}$$

such that, on D^{**}, $|\phi_1(t)| > 1/2$, $|\phi_2(s)| > 1/2$, $|\phi_3(t+s)| > 1/2$ and similar conditions hold for ψ_1, ψ_2, ψ_3. Then on D^{**} equation (1) can be rewritten

$$\frac{\psi_1(t)}{\phi_1(t)}\ \frac{\psi_2(s)}{\phi_2(s)} = \frac{\phi_3(t+s)}{\psi_3(t+s)}\ . \tag{2}$$

Letting $\chi_1(t) = \psi_1(t)/\phi_1(t)$, $\chi_2(t) = \psi_2(t)/\phi_2(t)$, $\chi_3(t) = \phi_3(t)/\psi_3(t)$, Lemma A asserts that each χ_k, $(k = 1, 2, 3)$, is analytic for $\|t\| < \alpha$. Then on D^{**} equation (2) becomes

$$\chi_1(t)\ \chi_2(s) = \chi_3(t+s)\ . \tag{3}$$

For $s = \bar{0}$, this equation reduces to $\chi_1(t) = \chi_3(t)$; similarly, setting $t = \bar{0}$ yields $\chi_2(s) = \chi_3(s)$ so that, on D^{**},

$$\chi_3(t)\ \chi_3(s) = \chi_3(t+s)\ . \tag{4}$$

In [1], it is shown that the only nonzero analytic solutions of (4) are the exponential functions, $e^{c\cdot t}$ where $c \in C_n$.

Therefore, for $\|t\| < \alpha$, $\psi_3(t) = e^{-c\cdot t}\ \phi_3(t)$; since ψ_3 and ϕ_3 are analytic on D_3, Lemma B asserts that $\psi_3(t) = e^{-c\cdot t}\ \phi_3(t)$ for all $t \in D_3$. Since $\chi_3(t) = \chi_1(t)$ for $\|t\| < \alpha$, $\chi_1(t) = e^{c\cdot t}$ so that $\psi_1(t) = e^{c\cdot t}\ \phi_1(t)$ for $\|t\| < \alpha$. Again, Lemma B asserts that $\psi_1(t) = e^{c\cdot t}\ \phi_1(t)$ for all $t \in D_1$. A similar argument yields $\psi_2(t) = e^{c\cdot t}\ \phi_2(t)$ for all $t \in D_2$.

Since $\phi(-t) = \overline{\phi(t)}$, the conjugate of $\phi(t)$, for any characteristic function ϕ and any $t \in R_n$, it follows that $c = ib$ where $b \in R_n$. Therefore, $\psi_1(t) = e^{ib\cdot t}\ \phi_1(t)$, $\psi_2(t) = e^{ib\cdot t}\ \phi_2(t)$, $\psi_3(t) = e^{-ib\cdot t}\ \phi_3(t)$. From this it follows that the distributions of X_k are equal to those of U_k, $(k = 1, 2, 3)$, up to a change of the location, and the proof is complete.

3. Applications of the theorem. The following two examples show how the theorem can be applied to random vectors X_1, X_2, X_3, of the same dimension, which possess analytic characteristic functions and for which the characteristic function of $(X_1 + X_3, X_2 + X_3)$ assumes the value zero.

Let $X = (X_1, \cdots, X_n)$ denote a random vector; then X has multinomial distribution, $Mu(r; P_1, \cdots, P_n)$, of order r with parameters $P_1, \cdots P_n, 0 \leqq P_j, P_1 + P_2 + \cdots + P_n \leqq 1$, if, for every set of integers

$$\{k_j\colon j = 1, 2, \cdots, n, k_j \geqq 0, \sum_1^n k_i \leqq r\} ,$$

$$P(X_1 = k_1, \cdots, X_n = k_n) = \frac{r!\, P_1^{k_1} \cdots P_n^{k_n}\, P_0^{r-k_1 \cdots -k_n}}{k_1!\, k_2! \cdots k_n! (r - k_1 - \cdots - k_n)!}$$

where $P_0 = 1 - P_1 - P_2 - \cdots - P_n$. The characteristic function of X, $\phi(t_1, \cdots, t_n) = (P_0 + P_1 e^{it_1} + \cdots + P_n e^{it_n})^r$, is clearly an analytic function on C_n. Notice that, for the choice of parameters $P_1 = P_2 = \cdots = P_n = 1/2n$, $P_0 = 1/2$, ϕ has zeros at the points $((2m_1 + 1)\pi, (2m_2 + 1)\pi, \cdots, (2m_n + 1)\pi)$, where $m_1, m_2, \cdots, m_n$ are integers. Let $Mu^*(r_1, r_2, r_3; P_1, P_2, \cdots, P_n)$ denote the joint distribution of the pair (Z_1, Z_2) where $Z_1 = X_1 + X_3, Z_2 = X_2 + X_3$ and each X_k, $(k = 1, 2, 3)$ has distribution $Mu(r_k; P_1, \cdots, P_n)$. With these definitions, the above theorem asserts the following result.

COROLLARY 1. *Let X_1, X_2, X_3 be three independent, n-dimensional, random vectors and let $Z_1 = X_1 + X_3$, $Z_2 = X_2 + X_3$. If the pair (Z_1, Z_2) has distribution $Mu^*(r_1, r_2, r_3; P_1, \cdots, P_n)$, then, except for perhaps a change of location, the distribution of X_k is Mu $(r_k; P_1, \cdots, P_n)$, $(k = 1, 2, 3)$.*

As another application of the above theorem, let X be a 2 dimensional real random vector and let us say that X has distribution $U(a)$, $a > 0$, if its distribution has density function

$$f(x, y) = \begin{cases} \dfrac{1}{2a^2} \text{ for } |x| + |y| \leqq a \\ 0 \text{ for } |x| + |y| > a \end{cases} .$$

If X has distribution $U(a)$, its characteristic function

$$\phi_X(t_1, t_2) = \frac{\sin\left[(t_1 + t_2)\dfrac{a}{2}\right] \sin\left[(t_1 - t_2)\dfrac{a}{2}\right]}{a^2\left(\dfrac{t_1 + t_2}{2}\right)\left(\dfrac{t_1 - t_2}{2}\right)} ,$$

is an analytic function defined on C_2 with zeros at the points (t_1, t_2) where $(t_1 \pm t_2) = 2\pi/a\ m$, $m = \pm 1, \pm 2, \cdots$. Let $U^*(a_1, a_2, a_3)$ denote the joint distribution of the pair (Z_1, Z_2) where $Z_1 = X_1 + X_3$ and

$Z_2 = X_2 = X_3$ and each X_k has distribution $U(a_k)$, $(k = 1, 2, 3)$. With these definitions, the above theorem asserts the following result.

COROLLARY 2. *Let* X_1, X_2, X_3 *be three independent* 2-*dimensional random vectors and let* $Z_1 = X_1 + X_3, Z_2 = X_2 + X_3$. *If the pair* (Z_1, Z_2) *has distribution* $U^*(a_1, a_2, a_3)$, *then, except for perhaps a change of location, the distribution of* X_k *is* $U(a_k)$, $(k = 1, 2, 3)$.

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