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A NECESSARY AND SUFFICIENT CONDITION FOR THE EMBEDDING OF A LINDELOF SPACE IN A HAUSDORFF $\mathscr{K}_\sigma$ SPACE

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It is known that complete regularity characterizes the Hausdorff topological spaces which are embeddable in a compact Hausdorff space. The theory of $\mathscr{K}$-analytic and $\mathscr{K}$-Borelian sets leads naturally to the search for an analogous criterion for the embedding of a Hausdoff space in a Hausdorff $\mathscr{K}_\sigma$ space. (A Hausdorff $\mathscr{K}_\sigma$ space is a Hausdorff space which is equal to a countable union of its compact subsets.) We shall give an answer to this problem for Lindelof spaces.

Strong regularity and strong normality of a closed subspace with respect to a given Hausdorff space are defined. It is shown that a Hausdorff Lindelof space is embeddable in a Hausdorff $\mathscr{K}_\sigma$ if and only if X is equal to a union of an increasing sequence of its strongly regular closed subspaces. An example is given of a nonregular space which is equal to a union of an increasing sequence of its strongly normal subspaces.

One might think that if a Hausdorff space were equal to a union of an increasing sequence of its closed completely regular subspaces, it would be embeddable in a Hausdorff $\mathscr{K}_\sigma$. However, in [3] an example of a Hausdorff space which is equal to a union of an increasing sequence of its closed normal subspaces and which is not embeddable in a Hausdorff $\mathscr{K}_\sigma$ is given.

In 1959 in [1], Professor G. Choquet proved that a $\mathscr{K}$-analytic space is embeddable in a space in which it is $\mathscr{K}$-Souslin if and only if it is embeddable in a Hausdorff $\mathscr{K}_\sigma$ space. Since all $\mathscr{K}$-analytic spaces are Lindelof, it is desirable to characterize Lindelof spaces which are embeddable in Hausdorff $\mathscr{K}_\sigma$ spaces.

2. Preliminaries.

We will need the following definitions.

DEFINITION 2.1. Let Y be a closed subspace of a Hausdorff topological space X. Y is said to be *strongly regular with respect to* X if for all subspaces A closed in Y and for all $x \in (Y \backslash A)$, there exist $0, P$ open *in* X such that:

$$A \subset 0; x \in P; 0 \cap P = \varnothing .$$

Clearly such a subspace Y is a regular topological space in the subspace topology. The converse is false, because there exist closed and regular subspaces Y of a Hausdorff space X that are not strongly regular with respect to X. For example, let $[0, 1]$ have the following topology $\mathscr{T}$:

$$0 \in \mathscr{T} \text{ if and only if } 0 = 0' \cup (0'' \cap Q)$$

where $0', 0''$ are open in the usual topology and Q is the set of rationals in $[0, 1]$.

Denote by X the Hausdorff topological space $([0, 1], \mathscr{T})$. Let I be the set of irrationals in $[0, 1]$; and let $Y = I \cup \{q\}$, where q is a rational in $[0, 1]$. Now Y is a closed and regular subspace of X, but Y is not strongly regular with respect to X because every open set in X containing I is everywhere dense in X.

DEFINITION 2.2. Let Y be a closed subspace of a Hausdorff topological space X. Y is *strongly normal with respect to* X if for every two closed subspaces A, B of Y, there exist open sets 0, P *in* X such that:

$$A \subset 0; B \subset P; \text{ and } 0 \cap P = \varnothing .$$

Obviously, such a Y is a normal topological space in the subspace topology. But note in the example given above, Y is a closed and normal subspace of X, but Y is not strongly normal with respect to X.

Now several lemmas concerning these properties will be given.

LEMMA 2.1. *Let Y be a closed subspace of a Hausdorff space X. Then the following conditions are equivalent.*

(1) *Y is strongly regular with respect to X.*

(2) *For each $y \in Y$ and for each 0 open in X and containing y, there exists an open set P in X such that:*

$$y \in P \subset \bar{P}^x \subset 0 ,$$

where $\bar{P}^x =$ the closure of P in X.

(3) *For each closed $A \subset Y$ and each $y \in (Y \backslash A)$, there exists an open 0 in X such that:*

$$y \in 0 \quad \text{and} \quad \bar{0}^x \cap A = \varnothing .$$

Proof. (The same proof as used to prove regularity in the classical sense.)

LEMMA 2.2. *Let Y be a closed subspace of a Hausdorff Lindelof space X. If Y is strongly regular with respect to X, Y is strongly normal with respect to X.*

Proof. Let A, B be two closed disjoint subspaces of Y. After Lemma 2.1 we have:

(1) for each $x \in A$, there exists an open subspace of X, 0_x containing x, such that:

$$\bar{0}_x^X \cap B = \varnothing; \ (\bar{0}_x^X = \text{closure of } 0_x \text{ in } X);$$

(2) for each $y \in B$, there exists an open subspace of X, P_y containing y, such that:

$$\bar{P}_y^X \cap A = \varnothing .$$

Since X is Lindelof, the open cover of X consisting of

$$\{\{0_x\}_x \in A, \{P_y\}_y \in B \quad \text{and} \quad X \backslash (A \cup B)\}$$

has a countable subcover. That is, there exist sequences $\{0_n\}_{n=1}^{\infty}$, $\{P_n\}_{n=1}^{\infty}$ of open subsets of X such that:

$A \subset \bigcup_{n=1}^{\infty} 0_n$; $B \subset \bigcup_{n=1}^{\infty} P_n$; with $\bar{0}_n^X \cap B = \bar{P}_n^X \cap A = \varnothing$ for each n.

Define $0_1' = 0_1$ and $P_1' = P_1$; and by induction for each n

$$0_n' = 0_n \backslash \bigcup_{j=1}^{n} \bar{P}_j^X, \ P_n' = P_n \backslash \bigcup_{j=1}^{n} \bar{0}_j^X .$$

Now $0_n' \cap P_j = \varnothing$ for all $j \leqq n$ implies that $0_n' \cap P_j' = \varnothing$ for all $j \leqq n$. Similarly $0_j \cap P_n' = \varnothing$ for all $j \leqq n$ implies that $0_j' \cap P_n' = \varnothing$ for all $j \leqq n$. Thus, $0_j' \cap P_n' = \varnothing$ for all j and n. Then

$$\left(\bigcup_{j=1}^{\infty} 0_j'\right) \cap \left(\bigcup_{n=1}^{\infty} P_n'\right) = \varnothing .$$

For each n, $\bar{0}_n^X \cap B = \bar{P}_n^X \cap A = \varnothing$. Therefore, $A \subset \bigcup_{j=1}^{\infty} 0_j'$ and $B \subset \bigcup_{n=1}^{\infty} P_n'$ and these unions are disjoint. Thus, Y is strongly normal with respect to X.

3. Embedding in Hausdorff $\mathscr{K}_\sigma$ spaces.

THEOREM 3.1. *Let X be a Hausdorff Lindelof space. Then a necessary and sufficient condition for X to be embeddable in a Hausdorff $\mathscr{K}_\sigma$ space is the following: there exists a sequence $\{X_n\}_{n=1}^{\infty}$ of subspaces of X such that:*

(1) $X = \bigcup_{n=1}^{\infty} X_n$, *and for each* n

(2) $X_n \subset X_{n+1}$

(3) X_n *is closed and strongly regular with respect to* X.

Proof. *Necessity*. By hypothesis, X is embeddable in a Hausdorff $E = \bigcup_{n=1}^{\infty} K_n$, where K_n is a compact subspace of E. Without loss of generality, we can suppose that $K_n \subset K_{n+1}$ for each n.

Let $X_n = K_n \cap X$. Obviously, $X = \bigcup_{n=1}^{\infty} X_n$; $X_n \subset X_{n+1}$ for all n;

and X_n is a closed and regular subspace of X. Moreover, X_n is strongly regular with respect to X; for suppose that A is closed and contained in X_n and $x \in (X_n \backslash A)$. Then $x \notin \bar{A}^{K_n}$. Thus, there exist open sets U, V in E such that $x \in U$; $\bar{A}^{K_n} \subset V$; and $U \cap V = \varnothing$. Let $0 = U \cap X$ and $P = V \cap X$. Then $x \in 0$ and $A \subset P$ and $0 \cap P = \varnothing$.

Sufficiency. By hypothesis, X is a Hausdorff Lindelof space; $X = \bigcup_{n=1}^{\infty} X_n$; and for all n, $X_n \subset X_{n+1}$, where X_n is closed and strongly regular with respect to X.

After Lemma 2.2, X_n is strongly normal with respect to X. Let βX_n be the Stone-Cech compactification of X_n (for $n = 1, 2 \cdots$). For all n, βX_n has a canonical embedding in βX_{n+1}. Since X_n is strongly normal with respect to X for all n, then $\beta X_n =$ closure of X_n in βX_{n+1}. (This is a consequence of the theorem of Tietze [2].)

For all n, let $K_n = \beta X_n$. Using the canonical embedding of βX_n in βX_{n+1}, we can consider $K_n \subset K_{n+1}$ for each n. Let $E = \bigcup_{n=1}^{\infty} K_n$ (or more precisely, the inductive limit of the K_n's).

To define the required topology on E, it is necessary to prove the following lemma.

LEMMA 3.2. *Let $X = \bigcup_{n=1}^{\infty} X_n$ be a Hausdorff Lindelof space, where X_n satisfies conditions* (1), (2), *and* (3) *of Theorem* 3.1 *for each n. Let $K_n = \beta X_n$ for each n (from which it follows that $K_n \subset K_{n+1}$). For any open subspace* 0 *of X and for each index n, there exists a subset 0_n^* of K_n such that*:

(i) 0_n^* *is an open subspace of K_n*,

(ii) $0_n^* \cap X_n = 0 \cap X_n$,

and (iii) $0_n^* \cap K_{n-1} = 0_{n-1}^*$.

Proof. Let $A = (X \backslash 0)$. Then A is closed in X. Let $A_n = A \cap X_n$ for all n. Then $A = \bigcup_{n=1}^{\infty} A_n$ and $A_{n+1} \cap X_n = A_n$ for all A_n.

Consider $\bar{A}_1^{K_1} =$ the closure of A_1 in K_1. Obviously, $\bar{A}_1^{K_1} \cap X_1 = A_1$ and $\bar{A}_1^{K_1} \cap (X_2 \backslash X_1) = \varnothing$ (since $K_1 \cap (X_2 \backslash X_1) = \varnothing$).

If we consider $\bar{A}_2^{K_2}$, then $\bar{A}_1^{K_1} \subset \bar{A}_2^{K_2}$ and $\bar{A}_2^{K_2} \cap X_2 = A_2$. It is necessary to show that $\bar{A}_2^{K_2} \cap K_1 = \bar{A}_1^{K_1}$. Obviously, $\bar{A}_1^{K_1} \subset (K_1 \cap \bar{A}_2^{K_2})$. Suppose that there exists a $y \in K_1 \cap (K_1 \backslash \bar{A}_1^{K_1}) = K_1 \cap (K_2 \backslash \bar{A}_1^{K_1})$. Then there exists an open neighbourhood U of y in K_2 such that:

$$\bar{U}^{K_2} \cap \bar{A}_1^{K_1} = \varnothing .$$

That is
$$\bar{U}^{K_2} \cap \overline{(A_2 \cap X_1)}^{K_2} = \varnothing .$$

This implies that $(\bar{U}^{K_2} \cap A_2) \cap (\bar{U}^{K_2} \cap X_1) = \varnothing$.

Since X_2 is strongly normal with respect to X and since $K_2 = \beta X_2$, then:

$$\overline{(\bar{U}^{K_2} \cap A_2)}^{K_2} \cap \overline{(\bar{U}^{K_2} \cap X_1)}^{K_2} = \varnothing \ .$$

(This is a consequence of the fact that two closed disjoint subspaces in a normal space Y have disjoint closures in βY; and this is a consequence of Urysohn's lemma [2].)

Since $y \in K_1$, then $y \in \overline{(\bar{U}^{K_2} \cap X_1)}^{K_2}$. Thus $y \notin \overline{(\bar{U}^{K_2} \cap A_2)}^{K_2}$. Thus, $y \notin \bar{A}_2^{K_2}$. This shows that $\bar{A}_2^{K_2} \cap K_1 \subset \bar{A}_1^{K_1}$; from which it follows that $\bar{A}_2^{K_2} \cap K_1 = \bar{A}_1^{K_1}$.

In the same way, it can be shown that for all n:

$$\bar{A}_n^{K_n} \cap K_{n-1} = \overline{A_{n-1}}^{K_{n-1}} \ .$$

If we let $0_n^* = K_n \backslash \bar{A}_n^{K_n}$; the sequence $\{0_n^*\}_{n=1}^{\infty}$ has the required properties.

The end of the proof of Theorem 3.1. For each open set 0 in X, let $0^* = \bigcup_{n=1}^{\infty} 0_n^*$, where the 0_n^* were defined above. To show that the 0^*'s define a base for a topology on $\bigcup_{n=1}^{\infty} K_n$, it is necessary to show that $(0 \cap P)^* = 0^* \cap P^*$ for any two open subspaces 0, P of X. Since $0^* \cap P^* = \bigcup_{n=1}^{\infty} (0_n^* \cap P_n^*)$, it suffices to show for each n that:

$(0_n \cap P_n)^* = 0_n^* \cap P_n^*$, where $0_n = 0 \cap X_n$ and $P_n = P \cap X_n$ for each n.

Let $A_n = X_n \backslash 0_n$ and $B_n = X_n \backslash P_n$. Then $0_n^* = K_n \backslash \bar{A}_n^{K_n}$ and $P_n^* = K_n \backslash \bar{B}_n^{K_n}$.

Then $0_n^* \cap P_n^* = K_n \backslash (\bar{A}_n^{K_n} \cup \bar{B}_n^{K_n}) = K_n \backslash \overline{(A_n \cup B_n)}^{K_n}$.

Since $A_n \cap B_n = X_n \backslash (0_n \cup P_n)$; then, by definition $(0_n \cap P_n)^* = K_n \backslash \overline{(A_n \cup B_n)}^{K_n}$.

Thus, the 0^*'s define a base for a topology τ on $\bigcup_{n=1}^{\infty} K_n$. From now on, let E designate the topological space $(\bigcup_{n=1}^{\infty} K_n, \tau)$. By the definition of τ, X is embedded in E.

To show that E is Hausdorff, let x, y be in E. Then there exists an n such that $x, y \in K_n$. Without loss of generality, let us suppose that $n = 1$. Let us choose U, V open in K_1 such that $x \in U$; $y \in V$ and $\bar{U}^{K_1} \cap \bar{V}^{K_1} = \varnothing$. Let $0_1 = U \cap X_1$, $P_1 = V \cap X_1$; then $\bar{0}_1^{X_1} \cap \bar{P}_1^{X} = \varnothing$.

From Lemma 2.2, it follows that X_1 is strongly normal with respect to X. Thus, there exist 0, P open in X such that $\bar{0}_1^{X_1} \subset 0'$; $\bar{P}_1^{X_1} \subset P'$ and $0' \cap P' = \varnothing$. Thus, there exist 0, P open in X such that $0 \cap P = \varnothing$; $0 \cap X_1 = 0_1$ and $P \cap X_1 = P_1$.

Let $A = (X \backslash 0)$; $B = (X \backslash P)$; and for each n:

$$A_n = X_n \backslash (0 \cap X_n) \ ,$$
$$B_n = X_n \backslash (P \cap X_n) \ .$$

For each n, define $0_n^* = K_n \backslash \bar{A}_n^{K_n}$; $P_n^* = K_n \backslash \bar{B}_n^{K_n}$. Let $0^* = \bigcup_{n=1}^{\infty} 0_n^*$

and $P^* = \bigcup_{n=1}^{\infty} P_n^*$. Since $(K_1 \backslash U) \supset \overline{A}_1^{K_1}$; then $U \subset (K_1 \backslash \overline{A}_1^{K_1})$; then 0^* is an open neighborhood of x in E. In the same way, it can be shown that P^* is a neighborhood of y in E. We have already shown that $0^* \cap P^* = (0 \cap P)^*$. Since $0 \cap P = \varnothing$ and X is dense in E, then $0^* \cap P^* = \varnothing$. Thus, E is a Hausdorff space.

Since E is Hausdorff and $0^* \cap K_n = 0_n^*$ for each n; then K_n is a compact subspace of E. Thus X is embeddable in a Hausdorff $\mathscr{K}_\sigma$.

4. An example of a nonregular space satisfying the conditions of the theorem. To show that the theorem is not trivial, we shall give an example of a space satisfying the hypotheses of Theorem 3.1 without being regular.

Let $[0, 1]$ have the topology $\mathscr{T}$ already used in the example that follows Definition 2.1. From now on, let Y denote the topological space $([0, 1], \mathscr{T})$; I the set of irrationals in $[0, 1]$; and C the Cantor set in $[0, 1]$.

Consider the following subspace X of Y, defined by $X = Z \cup A \cup B$ where:

$Z = C \cap I$,

$A = Q \cap ([0, 1] \backslash C)$,

$B =$ a subset of I which is countable and dense in $[0, 1]$ with respect to the usual topology.

For convenience, let $B = \bigcup_{n=1}^{\infty} \{b_n\}$ and $A = \bigcup_{n=1}^{\infty} \{a_n\}$. (Note that A is not finite.)

Now, let $X = \bigcup_{n=1}^{\infty} X_n$, where

$$X_n = Z \cup (\bigcup_{j=1}^{n} \{b_j\}) \cup (\bigcup_{j=1}^{n} \{a_j\}) .$$

It can be shown that X is Lindelof and that X_n is strongly normal with respect to X. But X is not regular, since $(Z \cup B)$ is closed in X and all open sets that contain $(Z \cup B)$ are everywhere dense in X.

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