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A NEW APPROACH TO REPRESENTATION THEORY FOR CONVOLUTION TRANSFORMS

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There are two different ways by which one obtains representation theorems for the Laplace transform. One way is to impose integral conditions on the inverse operator; and the other way is to impose summation conditions without referring to the inverse operator. Representation theorems for the convolution transform have hitherto been obtained by imposing integral conditions on the inverse operator, and no attempt has been made to impose summation conditions. We obtain here some representation theorems, which involve summation conditions, for convolution transforms with kernels in Class II. A representation theorem for convolution transforms of Class II with determining functions of bounded variation in $(-\infty, \infty)$, is given. Also, representation theorems involving determining functions which are integrals of functions in the Orlicz class $L_M(-\infty, \infty)$ are obtained.

In the sequel we follow the notation of Hirschman and Widder [2].

2. Notation and an auxiliary lemma. Let the sequence $\{a_n\}$ $(n \geqq 1)$ satisfy

$$0 < a_1 \leqq a_2 \leqq \cdots \leqq a_n \leqq \cdots ; \sum_{n=1}^{\infty} \frac{1}{a_n} = \infty ; \sum_{n=1}^{\infty} \frac{1}{a_n^2} < \infty . \tag{2.1}$$

Set

$$E(s) = \prod_{k=1}^{\infty}\left(1 - \frac{s}{a_k}\right)e^{s/a_k}$$

where s is complex and where the convergence of the infinite product is insured by (2.1) (see [2], p. 11). Our representation theorems will be concerned with convolution kernels of the form

$$G(t) = (2\pi i)^{-1}\int_{-i\infty}^{i\infty}[E(s)]^{-1}e^{st}ds , \quad -\infty < t < \infty .$$

Also set

$$E_m(s) = \prod_{k=m+1}^{\infty}\left(1 - \frac{s}{a_k}\right)e^{s/a_k} , \quad m \geqq 0 ;$$

$$G_m(t) = (2\pi i)^{-1}\int_{-i\infty}^{i\infty}[E_m(s)]^{-1}e^{st}ds , \quad m \geqq 0 ;$$

$$P_n(s) = \prod_{k=1}^{n}\left(1 - \frac{s}{a_k}\right)e^{s/a_k} , \quad n \geqq 1 ;$$

and

$$H_n(t) = (2\pi i)^{-1}\int_{-i\infty}^{i\infty}[P_n(s)]^{-1}e^{st}ds\ , \quad n \geqq 1\ .$$

Let D be differentiation with respect to x and let

$$P_n(D) = \prod_{k=1}^{n}\Big(1 - \frac{D}{a_k}\Big)e^{D/a_k}\ , \quad n \geqq 1\ .$$

With a function $f(x)$ differentiable infinitely often in (a, ∞), associate the sequence of functions defined on (a, ∞) by

$$\begin{aligned} f_0(x) &= f(x) \\ f_n(x) &= P_n(D)f(x)\ , \quad n \geqq 1\ . \end{aligned}$$

Then a proof similar to that of [2] p. 151 (4) gives

$$f_{n-1}(x) = -a_n e^{a_n(x-(1/a_n))}\int_{a-(1/a_n)}^{x-(1/a_n)} e^{-a_n t}f_n(t)dt + a_n C_n e^{a_n(x-(1/a_n))} \tag{2.2}$$

where C_n is a constant independent of x. Change of variable under the sign of the integral yields for all $x, u,\ a < x,\ u < \infty$,

$$e^{-a_n x}f_{n-1}(x) = e^{-a_n u}f_{n-1}(u) + a_n\int_x^u e^{-a_n t}f_n\Big(t - \frac{1}{a_n}\Big)dt\ . \tag{2.3}$$

Denote $\lambda_0 = 0$ and $\lambda_n = \sum_{k=1}^{n} 1/a_k$, $n \geqq 1$; and denote $a_0! = 1$ and $a_n! = a_1 \cdots a_n$, $n \geqq 1$. Then the following result plays a central role in the sequel.

LEMMA. *Let $f(x)$ be differentiable infinitely often in (a, ∞) and let $a < x < u < \infty$. Then for every $n \geqq 0$,*

$$f(x) = \sum_{k=0}^{n}\frac{1}{a_{k+1}}f_k(u - \lambda_k)H_{k+1}(x - u + \lambda_{k+1}) + R_n(x, u)\ , \tag{2.4}$$

where

$$R_n(x, u) = \int_x^u f_{n+1}(t - \lambda_{n+1})H_{n+1}(x - t + \lambda_{n+1})dt\ . \tag{2.5}$$

REMARK. It should be noted that formulae (2.4) and (2.5) bear strong resemblance to Badaljan's formulae for the generalized Taylor expansion with remainder (see for example [1], (1.23) through (1.25)).

Proof of the lemma. Let $a < x < u < \infty$. Then it follows by (2.3) for $n = 1$, that

$$f(x) = e^{a_1(x-u)}f(u) + a_1\int_x^u e^{a_1(x-t)}f_1(t-\lambda_1)dt$$
$$= \frac{1}{a_1}f(u)H_1(x-u+\lambda_1) + R_0(x, u) .$$

Now proceed by induction assuming (2.4) and (2.5) for $n \geqq 0$ and proving it for $n+1$. To this end we have by (2.3) for $n+2$ and by (2.4) and (2.5) for n,

$$f(x) = \sum_{k=0}^{n}\frac{1}{a_{k+1}}f_k(u-\lambda_k)H_{k+1}(x-u+\lambda_{k+1})$$

$$+ \int_x^u e^{a_{n+2}(t-u)}f_{n+1}(u-\lambda_{n+1})H_{n+1}(x-t+\lambda_{n+1})dt \tag{2.6}$$

$$+ \int_x^u a_{n+2}\int_t^u e^{a_{n+2}(t-v)}f_{n+2}(v-\lambda_{n+2})dvH_{n+1}(x-t+\lambda_{n+1})dt .$$

Now it is well-known that

$$\int_x^u e^{a_{n+1}(t-u)}H_{n+1}(x-t-\lambda_{n+1})dt = \frac{1}{a_{n+2}}H_{n+2}(x-u+\lambda_{n+2}) . \tag{2.7}$$

Also changing of the order of integration in the third term on the right-hand side of (2.6), shows that it is equal to

$$\int_x^u f_{n+2}(v-\lambda_{n+2})H_{n+2}(x-v+\lambda_{n+2})dv = R_{n+1}(x, u) .$$

This completes our proof.

3. Determining functions of bounded variation.

In this section we prove the following

THEOREM 1. *Necessary and sufficient conditions in order that $f(x)$ possess the representation*

$$f(x) = \int_{-\infty}^{\infty} G(x-t)d\alpha(t) , \quad \gamma < x < \infty , \tag{3.1}$$

where $\alpha(t)$ is of bounded variation in $(-\infty, \infty)$, are that $f(x)$ is differentiable infinitely often in (γ, ∞) and that

$$\sup_{\gamma<x<\infty} \sum_{n=0}^{\infty} \frac{1}{a_{n+1}}|f_n(x-\lambda_n)| = H < \infty . \tag{3.2}$$

Furthermore

$$\int_{-\infty}^{\infty} |d\alpha(t)| = H .$$

REMARK. A kernel in Class II has usually one more parameter, a real number b. Also the definition of the kernels $G_m(t)$ and $H_n(t)$ involves a sequence of parameters $\{b_m\}$ tending to 0 as $m \to \infty$, (see [2], p. 125). We have restricted ourselves to the case $b = b_m = 0$, $m \geqq 1$, for convenience in both the writing of the article and the use of references [1, 5] where this case is the only one discussed. The results of this paper can be easily extended to the general case and this is left for the reader.

A representation theorem involving integral conditions can be found in [2], p. 156, Th. 5.2 a. However, in this result the representation (3.1) is assumed to hold throughout the real line while we allow representation in a half line.

Proof of Theorem 1. First we prove sufficiency. We prove first that condition (3.2) implies that for $\gamma < x < u < \infty$,

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{a_{k+1}} f_k(u - \lambda_k) H_{k+1}(x - u + \lambda_{k+1}) . \tag{3.3}$$

Since $H_n(t) \to G(t)$, uniformly in $(-\infty, \infty)$, as $n \to \infty$ (see [2], p. 152, Lemma 4.1) and since $G(t)$ is bounded in $(-\infty, \infty)$ it follows that for n sufficiently large the functions $H_n(t)$ are uniformly bounded in $(-\infty, \infty)$. Hence the series on the righthand side of (3.3) converges. Therefore by virtue of (2.4) it suffices to prove that for some subsequence $\{n_j\}$, $R_{n_j}(x, u) \to 0$ as $j \to \infty$. To this end observe that by (3.2)

$$\sum_{n=0}^{\infty} \frac{1}{a_{n+1}} \int_x^u |f_n(t - \lambda_n)| dt < \infty ;$$

on the other hand by (2.1)

$$\sum_{n=0}^{\infty} \frac{1}{a_{n+1}} = \infty .$$

Consequently there is a subsequence $\{n_j + 1\}$ such that

$$\int_x^u |f_{n_j+1}(t - \lambda_{n_j+1})| \, dt \to 0 \quad \text{as} \quad j \to \infty . \tag{3.4}$$

Now by the uniform boundedness of the functions $H_n(t)$ for n sufficiently large it follows by (2.5) and (3.4) that $R_{n_j}(x, u) \to 0$ as $j \to \infty$. This completes the proof of (3.3). We have already noted that $H_n(t) \to G(t)$, uniformly in $(-\infty, \infty)$, as $n \to \infty$. Also for each fixed n, $H_n(t) \to 0$ as $t \to -\infty$ and $G(t) \to 0$ as $t \to -\infty$. Hence it follows by (3.2) and (3.3) that for $\gamma < x < \infty$

$$f(x) = \lim_{u\to\infty} \sum_{k=0}^{\infty} \frac{1}{a_{k+1}} f_k(u - \lambda_k) G(x - u + \lambda_{k+1}) . \tag{3.5}$$

Define now the functions $\alpha_u(t)$ $(u > \gamma)$ by

$$\alpha_u(t) = \sum_{u-\lambda_{n+1} \leqq t} \frac{1}{a_{n+1}} f_n(u - \lambda_n) , \quad -\infty < t < \infty . \tag{3.6}$$

Then the functions $\alpha_u(t)$ are of uniformly bounded variations and (3.5) can be rewritten as

$$f(x) = \lim_{u\to\infty} \int_{-\infty}^{\infty} G(x - t) d\alpha_u(t) , \quad \gamma < x < \infty . \tag{3.7}$$

By [2], p. 156, Th. 5.1 there exists a sequence $\{u_j\}$ and a function of bounded variation $\alpha(t)$ such that $\alpha_{u_j}(t) \to \alpha(t)$, $-\infty < t < \infty$, as $j \to \infty$, and for all $\gamma < x < \infty$,

$$f(x) = \lim_{j\to\infty} \int_{-\infty}^{\infty} G(x - t) d\alpha_{u_j}(t) = \int_{-\infty}^{\infty} G(x - t) d\alpha(t) .$$

Since $G(t) \to 0$ as $t \to \pm\infty$. This completes the proof of (3.1). Also it is obvious that

$$\int_{-\infty}^{\infty} | d\alpha(t) | \leqq H . \tag{3.8}$$

The converse follows by [1], (1.30) and [5] Th. 1, however, we will give a direct proof as we need some of the results for § 4.

First it follows by (3.1) and [2], p. 129, Th. 5. 2a that $f(x)$ is differentiable infinitely often in (γ, ∞) and that

$$f_n(x) = \int_{-\infty}^{\infty} G_n(x - t) d\alpha(t) , \quad \gamma < x < \infty .$$

Consequently

$$\sum_{n=0}^{\infty} \frac{1}{a_{n+1}} | f_n(x - \lambda_n) | \leqq \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{1}{a_{n+1}} G_n(x - \lambda_n - t) \, | d\alpha(t) | \tag{3.9}$$

and our proof will be complete when we show that for all x, t, $-\infty < x, t < \infty$,

$$\sum_{n=0}^{\infty} \frac{1}{a_{n+1}} G_n(x - \lambda_n - t) \leqq 1 . \tag{3.10}$$

Now set $a_0 = 0$ and define the functions $\Lambda_{Nn}(t)$, $0 \leqq n \leqq N$, by

$$\int_{-\infty}^{\infty} \Lambda_{Nn}(t) e^{-st} dt = \left[(a_n - s) \prod_{k=n+1}^{N} \left(1 - \frac{s}{a_k}\right) \right]^{-1}, \ \mathrm{Re}\, s < a_n ,$$

where for $n = N$ the right hand side is interpreted as $(a_n - s)^{-1}$. It then follows similar to [2], p. 241–242 (although some of our a_k may not be different from one another) that

$$\Lambda_{Nn}(t) = 0 \;, \quad 0 < t < \infty$$

and

$$\sum_{n=0}^{N} \Lambda_{Nn}(t) = 1 \;, \quad -\infty < t \leqq 0 \;. \tag{3.11}$$

Convoluting both sides of (3.11) by $G_N(t)$ we obtain

$$\int_{-\infty}^{0} G(t + \lambda_N - u)du + \sum_{k=0}^{N-1} \frac{1}{a_{k+1}} G_k(t + \lambda_N - \lambda_k) = \int_{-\infty}^{0} G_N(t - u)du \;.$$

Hence

$$\int_{-\infty}^{0} G(t - u)du + \sum_{k=0}^{N-1} \frac{1}{a_{k+1}} G_k(t - \lambda_k) \leqq 1 \;,$$

and since $\int_{-\infty}^{0} G(t - u)du \geqq 0$, (3.10) is evident. This completes the proof of (3.2); and to conclude the proof of the theorem we observe that by (3.9) and (3.10),

$$H \leqq \int_{-\infty}^{\infty} | d\alpha(t) | \;.$$

Our proof is now complete.

4. **Determining functions in Orlicz classes.** Let the function $M(u)$ be an even, continuous, convex function satisfying (1) $M(u)/u \to 0$ as $u \to 0$, (2) $M(u)/u \to \infty$ as $u \to \infty$. Let $L_M(-\infty, \infty)$ denote the class of measurable functions $\varphi(x)$ in $(-\infty, \infty)$ which satisfy

$$\int_{-\infty}^{\infty} M[\varphi(x)]dx < \infty \;.$$

$L_M(-\infty, \infty)$ is known as the Orlicz class associated with $M(u)$ and is not necessarily a linear space (see [3]). The space $L^p(-\infty, \infty)$, $1 < p < \infty$, is obtained as the orlicz class associated with $M(u) = | u |^p$.

First we give a necessary condition.

THEOREM 2. *If $f(x)$ possesses the representation*

$$f(x) = \int_{-\infty}^{\infty} G(x - t)\varphi(t)dt \;, \quad \gamma < x < \infty \;, \tag{4.1}$$

where $\varphi(t) \in L_M(-\infty, \infty)$, then $f(x)$ is differentiable infinitely often in (γ, ∞) and

$$\sup_{\gamma<x<\infty} \sum_{n=0}^{\infty} \frac{1}{a_{n+1}} M[f_n(x-\lambda_n)] \equiv H < \infty . \tag{4.2}$$

Furthermore

$$H \leqq \int_{-\infty}^{\infty} M[\varphi(x)]dx .$$

Proof. By (4.1) and [2], p. 129, Th. 5. 2a, $f(x)$ is differentiable infinitely often in (γ, ∞) and

$$f_n(x) = \int_{-\infty}^{\infty} G_n(x-t)\varphi(t)dt , \quad \gamma < x < \infty .$$

Thus by Jensen's inequality (see [8], p. 23 (10.8))

$$M[f_n(x-\lambda_n)] \leqq \int_{-\infty}^{\infty} G_n(x-\lambda_n-t)M[\varphi(t)]dt ,$$

since

$$\int_{-\infty}^{\infty} G_n(t)dt = 1 , \quad n \geqq 0 .$$

Consequently by (3.10),

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{a_{n+1}} M[f_n(x-\lambda_n)] &\leqq \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{1}{a_{n+1}} G_n(x-\lambda_n-t)M[\varphi(t)]dt \\ &\leqq \int_{-\infty}^{\infty} M[\varphi(t)]dt < \infty .\end{aligned}$$

This proves (4.2) and also that

$$H \leqq \int_{-\infty}^{\infty} M[\varphi(t)]dt .$$

A partial converse of Theorem 2 is the following

THEOREM 3. *If $f(x)$ possesses the representation* (3.1) *and if* (4.2) *holds, then $f(x)$ possesses the representation* (4.1) *where $\varphi(t) \in L_M(-\infty, \infty)$. Furthermore $\int_{-\infty}^{\infty} M[\varphi(t)]dt \leqq H$.*

REMARK. Theorem 3 is not a satisfactory converse of Theorem 2 since not every function in $L_M(-\infty, \infty)$ is also integrable. One would conjecture that (4.2) with the fact that $f(x)$ is differentiable in (γ, ∞) would suffice for (4.1) as is the case with the Laplace transform (see [4]).

Proof of Theorem 3. The functions $\alpha_u(t)$ defined in (3.6) are, by virtue of Theorem 1, of uniformly bounded variations. Hence, as in the proof of Theorem 1, there exists a function $\alpha(t)$ such that $\alpha_{u_j}(t) \to \alpha(t)$ $-\infty < t < \infty$, as $j \to \infty$ and

$$f(x) = \int_{-\infty}^{\infty} G(x-t)d\alpha(t)\ , \quad \gamma < x < \infty\ . \tag{4.3}$$

In order to complete our proof we show that $\alpha(t)$ is the indefinite integral of a function in $L_M(-\infty, \infty)$. To this end, let

$$-\infty < t_0 < t_1 < \cdots < t_n < \infty$$

be a fixed subdivision of $(-\infty, \infty)$. Then by Jensen's inequality (see [8], p. 24 (10.10))

$$\begin{aligned} M\left[\frac{\alpha_u(t_i) - \alpha_u(t_{i-1})}{\sum_{t_{i-1}<u-\lambda_{n+1}\leqq t_i} \frac{1}{a_{n+1}}}\right] &= M\left[\frac{\sum_{t_{i-1}<u-\lambda_{n+1}\leqq t_i} \frac{1}{a_{n+1}} f_n(u-\lambda_n)}{\sum_{t_{i-1}<u-\lambda_{n+1}\leqq t_i} \frac{1}{a_{n+1}}}\right] \\ &\leqq \frac{\sum_{t_{i-1}<u-\lambda_{n+1}\leqq t_i} \frac{1}{a_{n+1}} M[f_n(u-\lambda_n)]}{\sum_{t_{i-1}<u-\lambda_{n+1}\leqq t_i} \frac{1}{a_{n+1}}}\ . \end{aligned}$$

Hence by (4.2)

$$\begin{aligned} &\sum_{i=1}^{n}\left[\sum_{t_{i-1}<u-\lambda_{n+1}\leqq t_i} \frac{1}{a_{n+1}}\right] M\left[\frac{\alpha_u(t_i) - \alpha_u(t_{i-1})}{\sum_{t_{i-1}<u-\lambda_{n+1}\leqq t_i} \frac{1}{a_{n+1}}}\right] \\ &\leqq \sum_{n=0}^{\infty} \frac{1}{a_{n+1}} M[f_n(u-\lambda_n)] \leqq H < \infty\ . \end{aligned} \tag{4.4}$$

Now $\lim_{j\to\infty} \{\alpha_{u_j}(t_i) - \alpha_{u_j}(t_{i-1})\} = \alpha(t_i) - \alpha(t_{i-1})$ and, as is readily seen,

$$\lim_{u\to\infty} \sum_{t_{i-1}<u-\lambda_{n+1}\leqq t_i} \frac{1}{a_{n+1}} = t_i - t_{i-1}\ .$$

So (4.4) implies

$$\sum_{i=1}^{n} (t_i - t_{i-1}) M\left[\frac{\alpha(t_i) - \alpha(t_{i-1})}{t_i - t_{i-1}}\right] \leqq H\ . \tag{4.5}$$

Since (4.5) holds for any finite subdivision of $(-\infty, \infty)$ it follows by a slight modification of Medvedev's theorem [6] that

$$\alpha(t) = c + \int_0^t \varphi(v)dv\ , \quad -\infty < t < \infty\ ,$$

where $\varphi(v) \in L_M(-\infty, \infty)$. This in turn implies, together with (4.3), that $f(x)$ has the representation (4.1). Also

$$\int_{-\infty}^{\infty} M[\varphi(t)]dt \leqq H .$$

This completes our proof.

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