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SIMILARITIES INVOLVING NORMAL OPERATORS ON HILBERT SPACE

MARY R. EMBRY

The primary purpose of this note is to exhibit a proof and several corollaries of the following theorem concerning continuous linear operators on a complex Hilbert space X.

THEOREM 1. If H and K are commuting normal operators and $AH = KA$, where 0 is not in the numerical range of A, then $H = K$.

In the entire paper A, E, H and K represent continuous linear operators on X, A^* is the adjoint of $A, W(A)$ is the numerical range of A and $\sigma(A)$ is the spectrum of A. The terms self-adjoint, normal and unitary are used in the standard fashion. A is quasinormal if and only if A commutes with A^*A. A unitary operator is called cramped if and only if its spectrum is contained in an arc of the unit circle with central angle less than π.

In § 1 a proof of Theorem 1 will be given, as well as several corollaries. In § 2 corollaries of Theorem 1, which are valid if either $0 \notin W(A)$ or $\sigma(A) \cap \sigma(-A) = \emptyset$, are presented.

1. A proof of Theorem 1. Let h and k be the spectral resolutions of H and K respectively. Since $AH = KA$, $Ah(\alpha) = k(\alpha)A$ for each complex Borel set α by [10]. This last equation together with the fact that $h(\alpha)$ and $k(\alpha)$ are commuting projections implies that

(1) $p(\alpha)^*Ap(\alpha) = q(\alpha)^*Aq(\alpha) = 0$ for each Borel set α, where

$$(2)\quad \begin{aligned} p(\alpha) &= (I - h(\alpha))Ah(\alpha) \\ q(\alpha) &= h(\alpha)A(I - h(\alpha)). \end{aligned}$$

(I denotes the identity operator on X.) Since $0 \notin W(A)$, equation (1) implies that $p(\alpha) = q(\alpha) = 0$. Thus by (2) $Ah(\alpha) = h(\alpha)A$ for each Borel set α and consequently, $AH = HA$. Finally, $HA = KA$ and since $0 \notin W(A)$, $H = K$.

The following two examples show that if H and K are normal and $AH = KA$, then H and K may differ if $0 \in W(A)$ or if H and K do not commute, even if A is unitary.

EXAMPLE 1. If

$$K = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \; A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and } H = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix},$$

then H and K are normal, commute and $AH = KA$, but $H \neq K$.

EXAMPLE 2. If

$$K = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},\ A = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix},\ \text{and } H = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix},$$

then H and K are normal, $AH = KA$ and $0 \notin W(A)$, but $H \neq K$.

COROLLARY 1. [5]. *If AA^* and A^*A commute and $0 \notin W(A)$, then A is normal.*

Proof. Let $H = A^*A, K = AA^*$ and note that $AH = KA$, so that Theorem 1 is applicable.

The technique used in the proof of Theorem 1 is essentially the same as that used in [5] to prove a slightly stronger version of Corollary 1.

COROLLARY 2. *If $0 \notin W(A)$ and there exist real numbers r and s such that $r^2 + s^2 \neq 0$ and A commutes with $rAA^* + sA^*A$, then A is normal.*

Proof. In this case AA^* commutes with A^*A and Corollary 1 may be applied.

Several special cases of Corollary 2 are known. If A is quasi-normal and $0 \notin W(A)$, then A is normal [4]. If A commutes with $AA^* - A^*A$, then A is normal [11]. This last follows from Corollary 2 by applying the corollary to $A - zI$ (which commutes with

$$(A - zI)(A - zI)^* - (A - zI)^*(A - zI))$$

for $z \notin W(A)$.

In [12] C. R. Putnam proved a stronger version of the next corollary.

COROLLARY 3. [12]. *If A^2 is normal and $0 \notin W(A)$, then A is normal.*

Proof. By [7], [8], or [10] $A^*A^2 = A^2A^*$ if A^2 is normal. Thus AA^* and A^*A must commute and Corollary 1 is applicable.

We note that the condition $0 \notin \sigma(A)$ is not sufficiently strong to guarantee that A is normal when A^2 is normal. (For example take any nonnormal square root of the identity operator I.) However, we recall that if A^2 is normal and $\sigma(A) \cap \sigma(-A) = \emptyset$, then A is normal [6]. This suggests that perhaps Theorem 1 and Corollary 1 remain valid if the hypothesis $\sigma(A) \cap \sigma(-A) = \emptyset$ is substituted for the hypothesis $0 \notin W(A)$. Example 3 provides a counterexample to this proposition.

EXAMPLE 3. Let $A = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & 0 \end{pmatrix}$. Direct computation shows that AA^* and A^*A commute and differ from one another. Moreover, $\sigma(A) \cap \sigma(-A) = \emptyset$ since $z \in \sigma(A)$ if and only if $z^3 = 6$. If we take $H = A^*A$ and $K = AA^*$, then $AH = KA$, H and K are normal and commute, but $H \neq K$.

2. The condition $0 \notin W(A)$ or $\sigma(A) \cap \sigma(-A) = \emptyset$. Although the two conditions $0 \notin W(A)$ and $\sigma(A) \cap \sigma(-A) = \emptyset$ do not yield the same results, as seen by Example 3, several corollaries of Theorem 1 remain valid if the hypothesis $0 \notin W(A)$ is replaced by

$$\sigma(A) \cap \sigma(-A) = \emptyset .$$

In the remainder of the paper we let D be the set of all operators A for which either $0 \notin W(A)$ or $\sigma(A) \cap \sigma(-A) = \emptyset$.

Because of the importance of Theorem 2 in the following corollaries, we restate it here.

THEOREM 2. [6]. *If $\sigma(A) \cap \sigma(-A) = \emptyset$, then A and A^2 commute with exactly the same operators.*

COROLLARY 4. *If $A \in D$ and $AE = -EA$, where either A or E is normal, then $E = 0$.*

Proof. If $\sigma(A) \cap \sigma(-A) = \emptyset$, then by Theorem 2 $AE = EA$ since $A^2E = EA^2$. Therefore $E = 0$. Assume now that $0 \notin W(A)$. If E is normal, we apply Theorem 1 and have $E = -E$ or $E = 0$. If A is normal, then $A^*E = -EA^*$ by [10] and thus $A(E - E^*) = -(E - E^*)A$. Since $E - E^*$ is normal, $E = E^*$ by Theorem 1. Consequently, E is normal and a second application of Theorem 1 yields $E = -E = 0$.

COROLLARY 5. *If A is a normal element of D, then A and A^2 commute with exactly the same operators.*

Proof. Assume that $A^2E = EA^2$ and let $H = AE - EA$. Then $AH = -HA$ and by Corollary 4, $H = 0$.

COROLLARY 6. *If $AE = E^*A$ and $AE^* = EA$, where $A \in D$, then*

E is self-adjoint.

Proof. Under these hypotheses $A(E-E^*)=-(E-E^*)A$ and Corollary 4 can be applied to the normal operator $E-E^*$, resulting in $E=E^*$.

COROLLARY 7. *If $AE=E^*A$, where $A \in D$ and either A is unitary or E is normal, then E is self-adjoint.*

Proof. If E is normal, then $AE^*=EA$ by [10]; if A is unitary, then $EA^*=A^*E^*$ and consequently, $AE^*=EA$. Thus in either case Corollary 6 may be applied.

Corollary 7 includes a slight improvement of a result of J. P. Williams. In [13] Williams proved that $\sigma(E)$ is real if $AE=E^*A$, where 0 is not in the closure of $W(A)$. Thus if E is normal, E is self-adjoint. In particular, Williams noted that if E is normal and $AE=E^*A$, where A is a cramped unitary operator, then E is self-adjoint. More generally, in [1] W. A. Beck and C. R. Putnam and in [2] S. K. Berberian proved this same result without the hypothesis that A is normal. Finally, in [9] C. A. McCarthy obtained a generalization from which it follows that if $AE=E^*A$, A unitary and $\sigma(A)\cap\sigma(-A)=\varnothing$, then E is self-adjoint. All of these results are included in Corollary 7.

For completeness we include the following special case of Theorem 1.

COROLLARY 8. *If H and K are commuting normal operators and $H=A^*KA$, where A is a cramped unitary operator, then $H=K$.*

Proof. $AH=KA$ since A is unitary and $0 \notin W(A)$ since A is cramped [3]. Thus Theorem 1 is applicable.

In Corollary 9, we have a result similar to that of Theorem 1. The hypothesis that H and K commute is replaced by $A^*H=KA^*$.

COROLLARY 9. *Let $AH=KA$ and $A^*H=KA^*$, where $A \in D$. If A is unitary or H and K are normal, then $H=K$.*

Proof. If H and K are normal, we also have $AH^*=K^*A$ and $A^*H^*=K^*A^*$ by [10]; if A is unitary, these equations also hold since $HA^*=A^*K$ and $HA=AK$. If we now define

$$\mathscr{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \text{ and } \mathscr{E} = \begin{pmatrix} 0 & H \\ K^* & 0 \end{pmatrix},$$

direct computation shows that $\mathscr{A}\mathscr{E} = \mathscr{E}^*\mathscr{A}$ and $\mathscr{A}\mathscr{E}^* = \mathscr{E}\mathscr{A}$. Since $W(\mathscr{A}) = W(A)$ and $\sigma(\mathscr{A}) = \sigma(A)$, Corollary 6 may be applied to show $\mathscr{E} = \mathscr{E}^*$. Thus $H = K$.

A rather curious result can be obtained by using the technique of proof in Corollary 9. Note that $\mathscr{E}$ (as defined in the proof of Corollary 9) is normal if and only if $HH^* = KK^*$ and $H^*H = K^*K$. But by Corollary 7 if $\mathscr{E}$ is normal, $\mathscr{A} \in D$ and $\mathscr{A}\mathscr{E} = \mathscr{E}^*\mathscr{A}$, then $\mathscr{E}$ is self-adjoint and $H = K$. Thus we have:

COROLLARY 10. *Let H and K be operators such that $HH^* = KK^*$ and $H^*H = K^*K$. If there exists an element A of D such that $AH = KA$ and $A^*H = KA^*$, then $H = K$.*

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THE UNIVERSITY OF NORTH CAROLINA AT CHARLOTTE