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EXISTENCE, UNIQUENESS AND LIMITING BEHAVIOR OF SOLUTIONS OF A CLASS OF DIFFERENTIAL EQUATIONS IN BANACH SPACE

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Let X be a Banach space (real or complex) and A_n and B be linear operators in X with $D(B) \subseteqq D(A_n)$, $n = 1, 2, \cdots$. The following note is concerned with existence and uniqueness of solutions of the problem

$$\frac{d}{dt}[(I - A_n)u(t)] - Bu(t) = 0, \quad (t > 0), \quad u(0) = u_0\,, \tag{1.1}$$

and the limiting behavior of these solutions as the operators A_n tend to zero in a sense to be specified. We will show that for a large class of operators the problem (1.1) is well posed and that its solutions tend to the solution of the problem

$$\frac{du(t)}{dt} - Bu(t) = 0, \quad (t > 0), \quad u(0) = u_0\,. \tag{1.2}$$

In particular, we obtain an extension to Banach spaces of a result of R. E. Showalter [5] to the effect that (1.1) is well posed when X is a Hilbert space and A_n and B are maximal dissipative operators in X which satisfy the algebraic condition

$$\operatorname{Re}((I - A_n)x, Bx) \leqq 0\,, \qquad x \in D(B) \subseteqq D(A_n)\,. \tag{1.3}$$

In the next section we give sufficient conditions for (1.1) to be well posed. We note that these conditions do *not* guarantee that (1.2) is well posed. In § 3 we show that if, in addition, $\{A_n\}$ tends to zero in a certain sense, then (1.2) is well posed and the solutions u_n of (1.1) tend to the solution of (1.2). In particular, it will follow that if A and B are densely defined maximal dissipative operators in a Hilbert space and if (1.3) is satisfied with $A_n = n^{-1}A$, then

$$\frac{d}{dt}[(I - n^{-1}A)u_n(t)] - Bu_n(t) = 0\,, \quad (t > 0)\,, \quad u_n(0) = u_n \in D(B)\,,$$

is well posed and as $n \to \infty$, u_n converges strongly to the unique solution of (1.2). Two examples are discussed in § 4.

We emphasize that throughout this paper it is assumed that $D(B) \subseteqq D(A_n)$. The question of limiting behavior of solutions of (1.1) when X is a Hilbert space, $A_n = n^{-1}A$ and $D(A) \subseteqq D(B)$ has been considered previously [2], and it is interesting to compare the results of [2] with those of the present note in the case $D(A) = D(B)$. In [2] it was assumed that A and B were maximal dissipative operators arising from certain densely defined, strongly coercive sesequilinear forms and that A was self-adjoint. On the other hand the algebraic condition (1.3) which is the most restrictive assumption of the present note, was not assumed in [2] and the convergence results are somewhat stronger than those obtained here. Thus while the results of [2] do not apply to perturbations of hyperbolic problems, they are in some respects more satisfactory as far as perturbations of parabolic problems are concerned when $D(A) = D(B)$. We note that the methods used here are completely different from those of [2].

2. Existence and uniqueness of solutions. A *solution* of the problem (1.1) is a function $u: [0, \infty) \to D(B)$ such that $(I - A_n)u \in C([0, \infty); X) \cap C'((0, \infty), X)$ and (1.1) is satisfied. The initial condition in (1.1) is supposed to hold in the sense that $(I - A_n)u(t) \to (I - A_n)u_0$ strongly in X as $t \to 0_+$. While we will always assume that $I - A_n$ in invertible, the inverse need not be bounded and so we do not know in general that $u(t) \to u_0$ strongly in X.

THEOREM 2.1. *Let X be a Banach space and A_n and B linear operators in X which satisfy the following*

$$I - A_n \text{ is one-to-one} . \tag{2.1}$$

$$D(B) \subseteqq D(A_n) . \tag{2.2}$$

$$\| x - A_n x - \zeta B x \| \geqq \| x - A_n x \| \text{ for all } x \in D(B) \text{ and } \zeta > 0 . \tag{2.3}$$

$$\text{For some } \zeta_n > 0, \operatorname{Rg}(I - A_n - \zeta_n B) = X . \tag{2.4}$$

Then for any $u_0 \in D(B)$ the problem (1.1) *has a unique solution $u(t)$ and*

$$\| (I - A_n)u(t) \| \leqq \| (I - A_n)u_0 \|, \ t \geqq 0 . \tag{2.5}$$

Proof. Set $\tilde{A}_n = A_{n|D(B)}$ and $B_n = B(I - \tilde{A}_n)^{-1}$ with $D(B_n) = \operatorname{Rg}(I - \tilde{A}_n)$. A function u is a solution of (1.1) if and only if $(I - A_n)u = v \in C([0, \infty); X) \cap C'((0, \infty); X)$ and

$$\frac{dv(t)}{dt} - B_n v(t) = 0, \ (t > 0), \ v(0) = v_0 \tag{2.6}$$

where $v_0 = (I - A_n)u_0 \in D(B_n)$. From (2.3) we obtain

$$\| y - \zeta B_n y \| \geqq \| y \|, \ y \in D(B_n), \ \zeta > 0 ,$$

which means that B_n is a dissipative operator in X, and from (2.4) we have $\operatorname{Rg}(I - \zeta B_n) = X$ from some $\zeta > 0$ (hence for all $\zeta > 0$). From these facts it follows that $D(B_n)$ is dense in X (Goldstein [1]; c.f. [4]). We may now apply the Lumer-Phillips theorem [3] to the effect that B_n is the infinitesimal generator of a (C_0)-semigroup $\{e^{tB_n}: t \geqq 0\}$ of contractions on X. Thus for any $v_0 \in D(B_n)$, (2.6) has a unique solution given by $v(t) = e^{tB_n}v_0$ and $|v(t)| \leqq |v_0|$. The conclusions of the theorem now follow by setting

$$u(t) = (I - A_n)^{-1}e^{tB_n}(I - A_n)u_0 . \tag{2.7}$$

COROLLARY 2.1. *Let X be a Hilbert space and A_n and B be densely defined, maximal dissipative linear operators in X such that $D(B) \subseteqq D(A_n)$ and which satisfy* (1.3). *Then the conclusions of Theorem* 2.1 *hold. Moreover, $B \in C([0, \infty), X) \cap C'((0, \infty); X)$ and $u(t) \to u_0$ strongly in X as $t \to 0_+$.*

Proof. Since A_n is densely defined and maximal dissipative, $(I - A_n)$ is a bijection of $D(A_n)$ onto X and $\|(I - A_n)^{-1}\| \leqq 1$. Also, R. E. Showalter proved [5] that under the stated hypotheses, $A_n + B$ is a densely defined, maximal dissipative operator in X. From this fact follows that $\mathrm{Rg}\,(I - A_n - B) = X$. For a Hilbert space, conditions (1.3) and (2.3) are equivalent. The conclusions of the corollary now follow from (2.7) and Theorem 2.1.

REMARK. Suppose (2.1)-(2.4) hold and that in addition there is a constant $C > 0$ such that

$$\|x - A_nx - \zeta Bx\| \geqq C\|x - A_nx\| \tag{2.8}$$

for each $x \in D(B)$ and all ζ with $\mathrm{Re}\,(\zeta) > 0$. Then the semigroup $\{e^{tB_n}: t \geqq 0\}$ has a strong holomorphic extension into some sector $|\arg t| < \alpha$, and therefore (2.6) (respectively, (1.1)) is uniquely solvable for any $v_0 \in X$ (respectively, $u_0 \in D(A_n)$). In fact, since B_n generates a (C_0)-semigroup of contractions, the open right half-plane lies in the resolvent set of B_n and from (2.8) we obtain $\|(\lambda - B_n)^{-1}\| \leqq (C|\lambda|)^{-1}$ whenever $\mathrm{Re}\,\lambda > 0$, which implies the desired conclusion. When X is a Hilbert space, a sufficient condition for (2.8) is that all of the values of $z = (x - A_nx, Bx)$ lie in some fixed sector

$$|\arg z - \pi| \leqq \frac{\pi}{2} - \varepsilon , \qquad \varepsilon > 0 .$$

To prove this, write $z = |z|e^{i\theta}$ and $\zeta = |\zeta|e^{i\phi}$. (2.8) is equivalent to

$$(1 - C^2)\|x - A_nx\|^2 - 2|\zeta||z|\cos(\phi - \theta) + |\zeta|^2\|Bx\|^2 \geqq 0 .$$

If $|\theta - \pi| \leqq \pi/2 - \varepsilon$, there is a $\delta > 0$ such that $\cos(\phi - \theta) \leqq 1 - \delta$ for all $\phi \in (-\pi/2, \pi/2)$ and therefore

$$\begin{aligned}(1 - \delta)^2\|x - A_nx\|^2 &- 2|\zeta||z|\cos(\phi - \theta) + |\zeta|^2\|Bx\|^2 \\ &\geqq (1 - \delta)^2\|x - A_nx\|^2 - 2|\zeta|(1 - \delta)\|x - A_nx\|\|Bx\| \\ &+ |\zeta|^2\|Bx\|^2 = [(1 - \delta)\|x - A_nx\| - |\zeta|\|Bx\|]^2 \geqq 0 .\end{aligned}$$

Thus (2.8) holds with $C^2 = 2\delta - \delta^2$.

3. Limiting behavior of solutions. We first prove that if B is closed and A_n tends to zero in a certain way then (1.2) is well posed.

THEOREM 3.1. *Let X be a Banach space and A_n and B be linear operators in X which satisfy* (2.1)–(2.4). *Suppose in addition*

$$B \text{ is closed .} \tag{3.1}$$

$$\lim_{n\to\infty} \sup_{\substack{x \in D(B)\\ x\neq 0}} \|A_n x\|/(\|Bx\| + \|x\|) = 0 . \tag{3.2}$$

Then B is the infinitesimal generator of a (C_0)-semigroup of contractions on X.

Proof. We have to show that B is a dissipative operator such that $\operatorname{Rg}(I - B) = X$. From (2.3) and (3.2) we obtain, upon letting $n \to \infty$,

$$\|x - \zeta Bx\| \geqq \|x\|,\ x \in D(B),\ \zeta > 0 , \tag{3.3}$$

and so B is dissipative. For each n and $\zeta > 0$, B_n is dissipative and $\operatorname{Rg}(I - \zeta B_n) = X$. Let $y \in X$ and $x_n \in D(B)$ such that

$$x_n - A_n x_n - Bx_n = y , \qquad n = 1, 2, \cdots .$$

By (2.3), $\|x_n - A_n x_n\| \leqq \|y\|$ and therefore $\{Bx_n\}$ is bounded. Let

$$C_n = \sup_{\substack{x \in D(B)\\ x\neq 0}} \|A_n x\|/(\|Bx\| + \|x\|) .$$

$C_n \to 0$ as $n \to \infty$ according to (3.2). From (3.3)

$$\begin{aligned} \|x_n\| &\leqq \|x_n - Bx_n\| = \|y + A_n x_n\| \\ &\leqq \|y\| + C_n(\|Bx_n\| + \|x_n\|) \end{aligned}$$

so that

$$(1 - C_n)\|x_n\| \leqq \|y\| + C_n\|Bx_n\| .$$

Hence $\{x_n\}$ is also bounded. It follows from (3.2) that $A_n x_n \to 0$ strongly in X as $n \to \infty$. Therefore

$$\begin{aligned} \|x_n - x_m\| &\leqq \|(x_n - x_m) - B(x_n - x_m)\| \\ &\leqq \|A_n x_n - A_m x_m\| \longrightarrow 0 \end{aligned}$$

as $n, m \to \infty$. Let $x = \lim x_n$. We have that $x_n \to x$, $Bx_n \to x - y$. Since B is closed, $x \in D(B)$ and $x - Bx = y$, that is, $\operatorname{Rg}(I - B) = X$. This fact, together with the dissipativity of B, implies $D(B)$ is dense in X. The conclusion of the theorem now follows from the Lumer-Phillips theorem.

THEOREM 3.2. *Let X be a Banach space and A_n and B be linear operators in X which satisfy* (2.1)-(2.4), (3.1) *and* (3.2). *Then as $n \to \infty$, $e^{tB_n} \to e^{tB}$ strongly and uniformly on bounded subsets of* $[0, \infty)$.

Proof. We apply the Trotter convergence theorem [6]. To do this we show that for each $\zeta > 0$,

$$\lim_{n\to\infty} (I - \zeta B_n)^{-1} = (I - \zeta B)^{-1} \tag{3.4}$$

in the uniform operator topology of $\mathscr{L}(X)$ (= the linear space of bounded linear operators on X).

We may write

$$\begin{aligned}(I - \zeta B_n)^{-1} &= (I - \zeta B(I - \tilde{A}_n)^{-1})^{-1} \\ &= (I - A_n)(I - \zeta B)^{-1}(I - A_n(I - \zeta B)^{-1})^{-1} .\end{aligned} \tag{3.5}$$

For each $x \in X$,

$$\begin{aligned}\| A_n(I - \zeta B)^{-1}x \| &\leqq C_n(\| B(I - \zeta B)^{-1}x \| + \| (I - \zeta B)^{-1}x \|) \\ &\leqq C_n\Big(1 + \frac{2}{\zeta}\Big)\| x \| .\end{aligned} \tag{3.6}$$

Thus for all sufficiently large n,

$$(I - A_n(I - \zeta B)^{-1})^{-1} = \sum_{k=0}^{\infty} (A_n(I - \zeta B)^{-1})^k$$

and

$$\| (I - A_n(I - \zeta B)^{-1})^{-1} - I \| \leqq \sum_{k=0}^{\infty} C_n^{k+1}\Big(1 + \frac{2}{\zeta}\Big)^{k+1}$$

which tends to zero as $n \to \infty$. Therefore

$$\lim_{n\to\infty} (I - \zeta B)^{-1}(I - A_n(I - \zeta B)^{-1})^{-1} = (I - \zeta B)^{-1} \tag{3.7}$$

in the uniform operator topology of $\mathscr{L}(X)$. From (3.6) we have

$$\begin{aligned}&\| A_n(I - \zeta B)^{-1}(I - A_n(I - \zeta B)^{-1})^{-1} \| \\ &\leqq C_n\Big(1 + \frac{2}{\zeta}\Big)\| (I - A_n(I - \zeta B)^{-1})^{-1} \| \\ &\leqq \sum_{k=0}^{\infty} C_n^{k+1}\Big(1 + \frac{2}{\zeta}\Big)^{k+1} \longrightarrow 0\end{aligned} \tag{3.8}$$

as $n \to \infty$. (3.4) now follows from (3.5), (3.7), and (3.8).

THEOREM 3.3. *Let X be a Banach space and A_n and B be linear operators in X which satisfy* (2.1)-(2.4), (3.1) *and* (3.2). *Suppose in* *addition that* $\operatorname{Rg}(I - A_n) = X$, $n = 1, 2, \cdots$, *and* $\sup_n \| (I - A_n)^{-1} \| < \infty$. *Let* $u_0 \in D(B)$ *and* $u_n(t)$ *be the unique solution of* (1.1). *Then as* $n \to \infty$, $u_n(t) \to e^{tB}u_0$ *strongly in* X, *uniformly on bounded subsets of* $[0, \infty)$.

Proof. From (2.7) we obtain

$$\begin{aligned} \| u_n(t) - e^{tB}u_0 \| &\leqq \| (I - A_n)^{-1}(e^{tB_n} - e^{tB})u_0 \| \\ &\quad + \| (I - A_n)^{-1}A_n e^{tB}u_0 \| + \| (I - A_n)^{-1}e^{tB_n}A_n u_0 \| \\ &\leqq (\text{const.}) [\| e^{tB_n}u_0 - e^{tB}u_0 \| + C_n(\| Bu_0 \| + \| u_0 \|)] \end{aligned}$$

and the right side tends to zero as $n \to \infty$, uniformly on bounded subsets of $[0, \infty)$.

COROLLARY 3.1. *Let* X *be a Hilbert space and* A_n *and* B *be densely defined, maximal dissipative operators in* X *such that* $D(B) \subseteqq D(A_n)$ *and which satisfy* (1.3) *and* (3.2). *Then for each* $u_0 \in D(B)$ *the problem* (1.1) *has a unique solution* u_n *and* $u_n(t) \to e^{tB}u_0$ *as* $n \to \infty$, *uniformly on bounded subsets of* $[0, \infty)$.

Proof. As noted in the proof of Corollary 2.1, A_n and B satisfy (2.1)-(2.4) and moreover, $\operatorname{Rg}(I - A_n) = X$ with $\| (I - A_n)^{-1} \| \leqq 1$. In addition B, being a densely defined, maximal dissipative operator in a Hilbert space, is closed. The corollary now follows from Theorem 3.3.

REMARK. When $A_n = n^{-1}A$, (3.2) is automatically satisfied provided A and B are closed operators with $D(B) \subseteqq D(A)$. Thus in this case hypothesis (3.2) may be omitted in Corollary 3.1. In fact, as a rather well-known consequence of the closed graph theorem we have

$$\| Ax \| \leqq C(\| Bx \| + \| x \|)\,, \qquad x \in D(B)$$

where the constant C does not depend on x. Therefore

$$\sup_{\substack{u \in D(B) \\ u \neq 0}} \| A_n x \| / (\| Bx \| + \| x \|) \leqq Cn^{-1}\,.$$

4. Examples. As a first example we consider the problem

$$\frac{\partial}{\partial t}\left(u - a_n^1(x)u - a_n(x)\frac{\partial u}{\partial x}\right) - \left(b^1(x)u + b(x)\frac{\partial u}{\partial x}\right) = 0\,, \qquad 0 < x < 1\,, \quad t > 0\,, \tag{4.1}$$

$$u(x, 0) = u_0(x)\,, \quad 0 < x < 1\,; \quad u(0, t) = cu(1, t)\,, \quad t > 0\,, \tag{4.2}$$

where c is a complex constant satisfying certain conditions and the coefficients in (4.1) are real-valued and of class $C'([0, 1])$. Let X be the complex Hilbert space $L_2(0, 1)$ and $H_2^1(0, 1)$ be the subclass of

$L_2(0, 1)$ consisting of those functions whose first derivative in the sense of distributions is again in $L_2(0, 1)$. The norms in X and in $H_2^1(0, 1)$ will be denoted by $\|\cdot\|_0$ and $\|\cdot\|_1$ respectively and the inner product in X by $(\cdot, \cdot)$; we have

$$\|u\|_1 = \left(\|u\|_0^2 + \left\|\frac{du}{dx}\right\|_0^2\right)^{1/2}, \qquad u \in H_2^1(0, 1) .$$

Each function in $H_2^1(0, 1)$ is continuous, i.e., coincides with a function in $C([0, 1])$ up to a set of Lebesgue measure zero, and the injection of $H_2^1(0, 1)$ into $C([0, 1])$ is continuous.

We define operators A_n and B in X as follows:

$$D(A_n) = D(B) = \{u \in H_2^1(0, 1) : u(0) = cu(1)\}$$

and for $u \in D(A_n) = D(B)$,

$$A_n u = a_n^1 u + a_n \frac{du}{dx}, \qquad Bu = b^1 u + b\frac{du}{dx} .$$

From our preceding remarks it is easy to see that $D(B)$ is a closed subspace of $H_2^1(0, 1)$ and $D(B)$ is dense in $L_2(0, 1)$.

By a solution of (4.1), (4.2) we mean a solution of (1.1) in which A_n and B are the operators defined above. In order to apply the theory developed in §§2 and 3 to the problem (4.1) and (4.2) we shall have to verify in particular condition (1.3). Concerning this we have

LEMMA 4.1. *Suppose $a_n b \geqq 0$ and that*

$$b^1 - \frac{1}{2}\frac{db}{dx} - a_n^1 b^1 + \frac{1}{2}\frac{d}{dx}(a_n b^1 + a_n^1 b) \leqq 0 ; \tag{4.3}$$

$$\alpha_n(1) - |c|^2 \alpha_n(0) \leqq 0 \tag{4.4}$$

where $\alpha_n = b - a_n b^1 - a_n^1 b$. Then (1.3) is satisfied.

Proof. For $u \in D(B)$ we have

$$\operatorname{Re}(u - A_n u, Bu) = \int_0^1 (b^1 - a_n^1 b^1)|u|^2 dx - \int_0^1 a_n b\left|\frac{du}{dx}\right|^2 dx$$
$$+ \operatorname{Re}\int_0^1 (b - a_n^1 b - a_n b^1)\bar{u}\frac{du}{dx}dx .$$

The following identity is easily obtained by an integration by parts: For $u \in D(B)$ and $f \in C'([0, 1])$,

$$\operatorname{Re}\int_0^1 f\bar{u}\frac{du}{du}dx = \frac{1}{2}(f(1) - |c|^2 f(0))|u(1)|^2 - \frac{1}{2}\int_0^1 \frac{df}{dx}|u|^2 dx .$$

From this identity we obtain for $u \in D(B)$

$$\operatorname{Re}(u - A_n u, Bu) = \frac{1}{2}(\alpha_n(1) - |c|^2\alpha_n(0))|u(1)|^2 - \int_0^1 a_n b \left|\frac{du}{dx}\right|^2 dx$$
$$+ \int_0^1 \left[b^1 - \frac{1}{2}\frac{db}{dx} - a_n^1 b^1 + \frac{1}{2}\frac{d}{dx}(a_n^1 b + a_n b^1)\right]|u|^2 dx \leqq 0 .$$

REMARK. If $\{a_n\}$ and $\{a_n^1\}$ tend to zero in the topology of $C'([0, 1])$ and if for some $\varepsilon > 0$ we have

$$b^1 - \frac{1}{2}\frac{db}{dx} \leqq -\varepsilon , \qquad b(1) - |c|^2 b(0) \leqq -\varepsilon ,$$

then (4.3) and (4.4) are easily seen to be satisfied for all sufficiently large n.

THEOREM 4.1. *Assume* (4.3) *and* (4.4), *that* $a_n b \geqq 0$ *and* $a_n^2 + b^2 > 0$. *In addition suppose*

(4.5) $$a_n(1) - |c|^2 a_n(0) \leqq 0 ;$$

(4.6) $$a_n^1 - \frac{1}{2}\frac{da_n}{dx} < 1 , \qquad 0 \leqq x \leqq 1 ;$$

(4.7) *There exists* $\zeta_n > 0$ *such that*

$$c \exp\left[\int_0^1 \frac{1 - a_n^1(\xi) - \zeta_n b^1(\xi)}{a_n(\xi) + \zeta_n b(\xi)} d\xi \neq 1 .\right.$$

Then the hypotheses of Theorem 2.1 *are satisfied.*

Thus, in particular, for each $u_0 \in D(B)$ the problem (4.1), (4.2) has a unique solution.

THEOREM 4.2. *Assume* (4.3)–(4.5), *that* $a_n b \geqq 0$ *and* $b \neq 0$. *In addition suppose*

(4.8) $\{a_n\}$ *and* $\{a_n^1\}$ *tend to zero in the topology of* $C'([0, 1])$ *as* $n \to \infty$.

Then the hypotheses of Theorem 3.2 *are satisfied for all sufficiently large* n.

THEOREM 4.3. *Assume* (4.3)–(4.5), (4.8) *and that* $a_n b > 0$. *Then the hypotheses of Theorem* 3.3 *are satisfied for all sufficiently large* n.

Thus if u_n is the unique solution of (4.1), (4.2), as $n \to \infty$ $u_n(t)$ converges in $L_2(0, 1)$ to the unique solution of

$$\frac{\partial u}{\partial t} - b(x)\frac{\partial u}{\partial x} - b^1(x)u = 0 , \quad 0 < x < 1 , \quad t > 0 ,$$

$$u(x, 0) = u_0(x)\ , \quad 0 < x < 1\ ; \quad u(0, t) = cu(1, t)\ , \quad t > 0\ ,$$

uniformly on bounded subsets of $[0, \infty)$.

Proof of Theorem 4.1. We have already verified (2.3). To check (2.1) consider the equation

$$u - A_n u = f \in X\ .$$

Multiplying by $\bar{u}$ and integrating gives

$$\int_0^1 (1 - a_n^1)|\,u\,|^2 dx - \mathrm{Re}\int_0^1 a_n \bar{u}\frac{du}{dx}dx = \mathrm{Re}\int_0^1 \bar{u} f\, dx$$

and this may be written

$$\begin{aligned}(4.9)\qquad \int_0^1\Big(1 - a_n^1 + \frac{1}{2}\frac{da_n}{dx}\Big)|\,u\,|^2 &= \frac{1}{2}(a_n(1) - |\,c\,|^2 a_n(0))|\,u(1)\,|^2 \\ &\quad + \mathrm{Re}\int_0^1 \bar{u} f\, dx\ .\end{aligned}$$

From (4.5), (4.6) and (4.9) follows that $u = 0$ if $f = 0$.

We next verify (2.4). Let $f \in L_2(0, 1)$. We have to solve

$$(4.10)\quad u - A_n u - \zeta_n B u = (1 - a_n^1 - \zeta_n b^1)u - (a_n + \zeta_n b)\frac{du}{dx} = f$$

where $\zeta_n > 0$ is to be determined.

Since $a_n b \geqq 0$ and $a_n^2 + b^2 > 0$, we have $a_n + \zeta b \neq 0$ for every $\zeta > 0$ and therefore (4.10) is equivalent to

$$u(x) = k_n \exp\int_0^x K_n(\xi)d\xi + \int_0^x \tilde{K}_n(x, \xi)f(\xi)d\xi$$

where

$$K_n(\xi) = (1 - a_n^1(\xi) - \zeta_n b^1(\xi))/(a_n(\xi) + \zeta_n b(\xi))\ ,$$

$$\tilde{K}_n(x, \xi) = -\Big[\exp\int_\xi^x K_n(\eta)d\eta\Big]\Big/(a_n(\xi) + \zeta_n b(\xi))\ .$$

The constant k_n must be such that $u(0) = cu(1)$. This condition leads to

$$k_n = k_n c \exp\int_0^1 K_n(\xi)d\xi + c\int_0^1 \tilde{K}_n(1, \xi)f(\xi)d\xi$$

and this equation is solvable for k_n for arbitrary $f \in L_2(0, 1)$ if and only if

$$c \exp\int_0^1 K_n(\xi)d\xi \neq 1\ .$$

This last condition is satisfied provided ζ_n is chosen according to condition (4.7). Thus (2.4) is satisfied.

Proof of Theorem 4.2. We first note that (4.8) implies (4.6) for all sufficiently large n. Moreover, (4.7) is also satisfied for all large n if $\{\zeta_n\}$ is any sequence of positive numbers which tends to zero. Thus conditions (2.1)-(2.4) are satisfied. That (3.1) and (3.2) also hold is a consequence of the inequality

$$\| u \|_1 \leqq K(\| Bu \|_0 + \| u \|_0) , \qquad u \in D(B) \tag{4.11}$$

where the constant K is independent of u. In fact, suppose (4.11) holds and $\{u_n\} \subset D(B)$, $u_n \to u$, $Bu_n \to v$ in $L_2(0, 1)$. By (4.11), $\{u_n\}$ converges in $H_2^1(0, 1)$. Since $D(B)$ is a closed subspace of $H_2^1(0, 1)$ and $\| Bu_n \|_0 \leqq (\text{const.}) \| u_n \|_1$ it follows that $u \in D(B)$ and $Bu = v$, i.e., B is closed. Moreover, we have

$$\| A_n u \|_0 \leqq \sup_{0 \leqq x \leqq 1} (| a_n(x) | + | a_n^1(x) |) \| u \|_1$$

and therefore

$$\sup_{\substack{u \in D(B) \\ u \neq 0}} \| A_n u \|_0 / (\| Bu \|_0 + \| u \|_0) \leqq K \sup_{0 \leqq x \leqq 1} (| a_n(x) | + | a_n^1(x) |)$$

which tends to zero as $n \to \infty$. Thus (3.2) is satisfied.

It only remains to prove (4.11). We have

$$\| Bu \|_0^2 = \int_0^1 \left(b \frac{du}{dx} + b^1 u \right) \left(b \frac{d\bar{u}}{dx} + b^1 \bar{u} \right) dx .$$

Using the inequality

$$2 | yz | \leqq \delta | y |^2 + \frac{1}{\delta} | z |^2 , \qquad \delta > 0$$

we obtain

$$\| Bu \|_0^2 \geqq \inf_{0 \leqq x \leqq 1} | b(x) |^2 \left\| \frac{du}{dx} \right\|_0^2 - \varepsilon \left\| \frac{du}{dx} \right\|_0^2 - K_\varepsilon \| u \|_0^2 .$$

Choosing $\varepsilon = 1/2 \inf_{0 \leqq x \leqq 1} | b(x) |^2$ leads to (4.11).

Proof of Theorem 4.3. We have only to verify that $\operatorname{Rg}(I - A_n) = L_2(0, 1)$, $n \geqq N$, and

$$\sup_{n \geqq N} \| (I - A_n)^{-1} \| < \infty .$$

From (4.9) and the present hypotheses it follows that for all sufficiently large n,

$$\frac{1}{2} \| u \|_0^2 \leqq \| u - A_n u \| , \qquad u \in D(A_n) .$$

Let $f \in L_2(0, 1)$. Since $a_n \neq 0$, the equation

$$u - A_n u = (1 - a_n^1)u - a_n \frac{du}{dx} = f$$

is equivalent to

$$u(x) = k_n \exp \int_0^x \frac{1 - a_n^1}{a_n}(\xi) d\xi + F_n(x)$$

where $F_n(x)$ is a known function and the constant k_n must be such that $u(0) = cu(1)$. This is possible for arbitrary $f \in L_2(0, 1)$ if and only if

$$c \exp \int_0^1 \frac{1 - a_n^1(\xi)}{a_n(\xi)} d\xi \neq 1$$

and this last condition is obviously satisfied for all sufficiently large n in view of (4.8). Thus $\mathrm{Rg}(I - A_n) = X$, $n \geqq N$, and the proof is complete.

EXAMPLE 2. We consider, for $n = 1, 2, \cdots$, the problem

$$\frac{\partial}{\partial t}\left(u - \frac{1}{n}\frac{\partial u}{\partial x}\right) - \left(b(x)u + \frac{\partial^2 u}{\partial x^2}\right) = 0\,, \quad 0 < x < 1\,, \quad t > 0\,, \tag{4.12}$$

$$u(x, 0) = u_0(x)\,, \qquad 0 < x < 1\,, \tag{4.13}$$

$$u(0, t) = cu(1, t)\,, \quad \bar{c}\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(1, t)\,, \quad t > 0\,. \tag{4.14}$$

The function b is real-valued and of class $C'([0, 1])$ and c is a complex constant. Let $X = L_2(0, 1)$, $D(A_n)$ be as in the first example and $A_n = 1/n\, d/dx$. Let $H_2^2(0, 1)$ be the set of functions in $L_2(0, 1)$ whose first and second distributional derivatives are in $L_2(0, 1)$ and set

$$D(B) = \left\{u \in H_2^2(0, 1) \colon u(0) = cu(1),\ \bar{c}\frac{du}{dx}(0) = \frac{du}{dx}(1)\right\}\,,$$

$$Bu = bu + \frac{d^2u}{dx^2}\,, \quad u \in D(B)\,.$$

The norm in $H_2^2(0, 1)$ is denoted by $\|\cdot\|_2$ and defined by

$$\|u\|_2 = \left(\|u\|_0^2 + \left\|\frac{du}{dx}\right\|_0^2 + \left\|\frac{d^2u}{dx^2}\right\|_0^2\right)^{1/2}\,.$$

Each function in $H_2^2(0, 1)$ is of class $C'([0, 1])$ and the injection of $H_2^2(0, 1)$ into $C'[(0, 1)]$ is continuous. It follows that $D(B)$ is a closed subspace of $H_2^2(0, 1)$, is dense in $L_2(0, 1)$ and as in the first example it is not difficult to verify that

$$\|u\|_2 \leqq K(\|Bu\|_0 + \|u\|_0)\,, \qquad u \in D(B)\,. \tag{4.15}$$

Let B^* be the adjoint of B. As is well-known, $D(B^*) \subset H_2^2(0, 1)$

and, since $b + d^2/dx^2$ is a formally self-adjoint differential operator,

$$B^*v = bv + \frac{d^2v}{dx^2}, \quad v \in D(B^*).$$

We show that $B^* = B$. If $v \in D(B^*)$ then for all $u \in D(B)$ we have

$$(Bu, v) = \int_0^1 \left(bu + \frac{d^2u}{dx^2}\right)\bar{v}dx = \frac{du}{dx}(0)(\bar{c}\bar{v}(1) - \bar{v}(0))$$
$$- u(1)\left(\frac{d\bar{v}}{dx}(1) - c\frac{d\bar{v}}{dx}(0)\right) + (u, B^*v).$$

Since the first two terms on the right must vanish for all $u \in D(B)$ we have $v(0) = cv(1)$, $\bar{c}(dv/dx)(0) = (dv/dx)(1)$, that is, $v \in D(B)$. Thus $B^* \subseteqq B$. On the other hand, $(Bu, v) = (u, Bv)$ for all u and v in $D(B)$ so that B is symmetric. Hence B is self-adjoint.

THEOREM 4.4. *Suppose* $b \leqq 0$, $db/dx \leqq 0$ *and*

$$b(1) - |c|^2 b(0) \geqq 0.$$

Then the hypotheses of Corollary 3.1 *are satisfied.*

Thus for each n and $u_0 \in D(B)$ the problem (4.12)-(4.14) has a unique solution u_n and, as $n \to \infty$, $u_n(t)$ converges in $L_2(0, 1)$ to the solution of

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - b(x)u = 0, \quad 0 < x < 1, \quad t > 0,$$

$$u(x, 0) = u_0(x), \qquad 0 < x < 1,$$

$$u(0, t) = cu(1, t), \quad \bar{c}\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(1, t), \quad t > 0,$$

uniformly on bounded subsets of $[0, \infty)$.

Proof of Theorem 4.4. We have for $u \in D(B)$

$$(Bu, u) = \int_0^1 b|u|^2 dx - \int_0^1 \left|\frac{du}{dx}\right|^2 dx \leqq 0.$$

B is therefore a self-adjoint, dissipative operator and, consequently, maximal dissipative. We also have

$$n \cdot \mathrm{Re}\,(A_n u, u) = \frac{1}{2}(1 - |c|^2)|u(1)|^2, \qquad u \in D(A_n).$$

Since $b \leqq 0$ and $db/dx \leqq 0$ we have $b(1) \leqq b(0) \leqq 0$. Since also $b(1) \geqq$

$|c|^2b(0)$ it follows that $|c|^2 \geqq 1$. Thus A_n is dissipative and one easily proves as in the first example that $Rg(I - A_n) = X$.

Next we verify (1.3). We have for $u \in D(B)$

$$\begin{aligned} n\cdot\operatorname{Re}(A_n u, Bu) &= \operatorname{Re}\int_0^1 \frac{du}{dx}\Big(b\bar{u} + \frac{d^2\bar{u}}{dx^2}\Big)dx \\ &= \frac{1}{2}(b(1) - |c|^2 b(0))|u(1)|^2 - \frac{1}{2}\int_0^1 \frac{db}{dx}|u|^2 dx \\ &\quad + \frac{1}{2}\Big|\frac{du}{dx}(0)\Big|^2(|\bar{c}|^2 - 1) \geqq 0\,. \end{aligned}$$

(1.3) follows from this inequality and the fact that B is dissipative. Finally, (3.2) is an immediate consequence of (4.15).

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