

ON LOEWY LENGTH OF RINGS

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Associated with each ring R over which every nonzero right module has a minimal submodule is an ordinal number called its right (lower) Loewy length. The concern here is with the various possible left and right Loewy lengths of such rings with zero radical and with the possible right-left symmetry of this minimal submodule condition. In particular, if R has finite right Loewy length n then R has left Loewy length $\leq 2^n - 1$.

All rings are associative rings with identity. Denoting the socle of a module M by $\text{Soc}(M)$, the (lower) Loewy series for M is defined transfinitely by: $S_0 = 0$, $S_{\alpha+1}/S_\alpha = \text{Soc}(M/S_\alpha)$ and, if α is a limit ordinal, $S_\alpha = \bigcup_{\beta < \alpha} S_\beta$. (See Bass [1, p. 470].) If $M = S_\alpha$ for some ordinal number α then M is called a (lower) Loewy module. The Loewy length of such a module is $L(M) = \gamma$, the least ordinal γ with $M = S_\gamma$. We call a ring R a right (resp., left) Loewy ring in case the regular representation R_r (resp., ${}_r R$) is a Loewy module. (Năstăsescu and Popescu [6] use the term "semi-artinian" to denote such a ring.) A ring R is easily seen to be a right Loewy ring if and only if each of its right modules has a nontrivial (hence, essential) socle. Over such a ring each right module M is a Loewy module of length $L(M) \leq L(R_r)$ and, since R_r is finitely generated, $L(R_r)$ cannot be a limit ordinal.

As part of his Theorem P, Bass [1] proved that a ring R is left perfect (i.e., its (Jacobson) radical $J = J(R)$ is left T -nilpotent and R/J is semisimple) if and only if R is right Loewy and contains no infinite orthogonal set of idempotents. No doubt inspired by this result, Năstăsescu and Popescu [6] proved that a ring R is right Loewy if and only if its radical J is left T -nilpotent and R/J is right Loewy. Thus we are led to study Loewy rings with zero radical. After first modifying an example of Osofsky [7] to show that there are primitive left and right Loewy rings of arbitrary infinite left and right lengths and that a primitive right Loewy ring need not be left Loewy, we prove that a right Loewy ring of finite length must also be left Loewy of finite length, but that these two lengths are neither independent nor necessarily equal. Then, recalling that the left and right Loewy series for a (von Neumann) regular ring are the same, we show that there exist both commutative and primitive regular Loewy rings of arbitrary length.

1. Right Loewy length vs. left. Osofsky [7], in answer to a

question of Bass, proved that, given any two infinite ordinals φ and γ , there exists a left and right perfect ring R with $L(R_e) = \gamma + 1$ and $L({}_R R) = \varphi + 1$; and that there exists a left perfect ring (necessarily right Loewy) with $L(R_e) = \gamma + 1$ and $\text{Soc}({}_R R) = 0$. In such rings R all the “action” must take place in the radical (i.e., if R is right perfect, $L(R_e) \leq L(J_e) + L(R/J) = L(J) + 1$). Indeed it was apparently heretofore not known whether a ring with zero radical could be Loewy with $L({}_R R) \neq L(R_e)$. The following lemma allows us to build such Loewy rings with zero radical by “putting Osofsky’s rings on top of” the socle of the ring of linear transformation of a vector space. Before proceeding to the lemma we recall (see [3, Chapter IV]) the ring of linear transformations of a vector space is a *PMI* ring. That is, it is a primitive ring containing a minimal right (equivalently, left) ideal; and so its left and right socles (the transformations of finite rank) coincide, are homogeneous, and, as left and right ideals are both faithful and essential.

LEMMA 1.1. *Let B be an algebra over a field K . Then there is a *PMI* algebra R over K with $\text{Soc}(R) = S$ and subalgebra $\hat{B} \cong B$ such that*

$$R = \hat{B} + S \quad \text{and} \quad \hat{B} \cap S = 0;$$

and hence $R/S \cong B$.

Proof. Let $V_K = B^{(\mathbb{N})}$ be a countable direct sum of copies of B , considered as a vector space over K . Let $T = \text{End}(V_K)$ the ring of linear transformations of V_K , viewed as left operators. Then the mapping $b \mapsto \hat{b}$ where

$$\hat{b}: (b_1, \dots, b_n, 0, \dots) \longmapsto (bb_1, \dots, bb_n, 0, \dots)$$

is an injective algebra homomorphism $B \rightarrow T$ such that \hat{b} has infinite rank whenever $b \neq 0$. Now let $S = \text{Soc}(T)$, the transformations of V_K of finite rank, and let

$$R = \hat{B} + S.$$

Then, since S is an ideal in T , R is a subring of T , and clearly $\hat{B} \cap S = 0$. By the Structure Theorem for *PMI* rings [3, p. 75], R is a *PMI* ring with socle S .

Now, letting B be one of the rings of Osofsky discussed above (B is an algebra), we have at once:

PROPOSITION 1.2. *Let φ and γ be infinite ordinal numbers. Then there is a primitive left and right Loewy ring R with*

$$L(R_R) = \gamma + 1 \quad \text{and} \quad L({}_R R) = \varphi + 1.$$

Moreover, there exist primitive right Loewy rings, of arbitrary infinite Loewy length, that are not left Loewy.

Of course the model for Loewy rings is the semiprimary rings, which one may characterize as left perfect rings of finite right (or equivalently, left) Loewy length. The left and right Loewy lengths of a semiprimary ring are the same—one greater than the nilpotency index of its radical. We shall now show that finite Loewy length on one side of a ring implies it on the other, but that if the ring is not semisimple modulo its radical (i.e., perfect) then the two lengths can differ.

THEOREM 1.3. *If R is a right Loewy ring of finite Loewy length n , then R is a left Loewy ring of Loewy length at most $2^n - 1$.*

Proof. The proof is by induction on the right Loewy length. Let $L(R_R) = n$, $S = \text{Soc}(R_R)$, and $N = J \cap S$, where J is the Jacobson radical of R . Then, since $N \subset J$ and S_R is semisimple, we have $SN = 0$. Thus ${}_R N$ is an R/S module. Since R/S has right Loewy length $n - 1$, it has left Loewy length at most $2^{n-1} - 1$ by inductive hypothesis. Clearly $L({}_R R)$ is an upper bound for the Loewy length of any left R module, so the Loewy length of ${}_R N$ is less than or equal to $2^{n-1} - 1$.

We claim that S/N is semisimple as a left R module. To see this, note that $S_R = (\Sigma \oplus e_i R) \oplus N$, where $e_i^2 = e_i$ and the $e_i R$ are simple right R modules. This is true because N is a summand of S_R , since S_R is semisimple; and the simple components of any complementary summand are generated by idempotents, since they are not nilpotent. Now, the proof of [3, Proposition 1, p. 65] shows that if $e_i R$ is simple, and Re_i contains no nonzero nilpotent left ideals then Re_i is also simple. But $(R/N)(e_i + N)$ cannot contain any nonzero nilpotent left ideals, because, since $N^2 = 0$, any such left ideal would lift back to a nilpotent left ideal in $Re_i + N \subseteq S$, which would be contained in $J \cap S = N$. Thus each $(e_i + N)$ generates a simple left R module, so $(Re_i R + N)/N$ is semisimple as a left R/N module. But $S = \Sigma Re_i R + N$ so that S/N is left semisimple.

Now it is clear that $L(M) \leq L(N) + L(M/N)$ for any submodule $N \subset M$. Thus

$$\begin{aligned} L(R) &\leq L(R/S) + L(S/N) + L(N) \\ &\leq (2^{n-1} - 1) + 1 + (2^{n-1} - 1) = 2^n - 1. \end{aligned}$$

The case $n = 1$ being straightforward from the Wedderburn-Artin

Theorem, the proof is complete.

Next we show that, even though the preceding theorem severely limits the possible disparity between $L(R_R)$ and $L({}_R R)$ for a ring R of finite Loewy length, the two lengths can be different. To do so, we employ the following variation of the standard construction used in adjoining an identity to a ring.

LEMMA 1.4. *Let S be an ideal in a ring T . Let K be a subring of T . Let $\varphi: {}_K S_T \rightarrow {}_K S_T$ be a bimodule homomorphism with kernel N . Let $R = K \times S$, and define addition in R coordinatewise and multiplication by*

$$(k, s)(k', s') = (kk', sk' + ks' + s\varphi(s')).$$

Then R is a ring with identity $(1, 0)$ and $0 \times S$ and $0 \times N$ are ideals in R with $(0 \times S)(0 \times N) = 0$.

Proof. The fact that R is a ring is a simple calculation. Note that if φ is the identity map, one is simply adjoining a unit to S , and if not, the linearity of φ allows us to move the φ through a product to preserve associativity. The fact that $(0 \times S)(0 \times N) = 0$ is straightforward.

Note that, in order to be of minimal Loewy length, the counterexample that we are seeking must have nonzero radical, because in a ring R with zero radical $\text{Soc}({}_R R) = \text{Soc}(R_R)$. However, an application of (1.1) to our following example will yield a Loewy ring with zero radical and differing finite Loewy lengths.

EXAMPLE 1.5. *A Loewy ring R with $L(R_R) = 2$ and $L({}_R R) = 3$.*

DEMONSTRATION. Let T be the ring of linear transformations $\text{End}(V_K)$ of an infinite dimensional vector space over a field K . Let $S = \text{Soc}(T)$. Then S_T is an infinite direct sum of isomorphic minimal right ideals and so there is an epimorphism

$$\varphi: {}_K S_T \longrightarrow {}_K S_T \quad \text{with} \quad N = \text{Ker } \varphi \neq 0.$$

Let $R = K \times S$ be the ring constructed from this data via Lemma 1.4. Then since $(0, s)(k', s') = (0, s(k' + \varphi(s')))$, φ is epic, and $MS = M$ for every $M_T \leq S_T$, the right R -submodules of $0 \times S$ are just those of the form $0 \times M$ where $M \leq S_T$. Thus $0 \times S$ is a non-artinian semisimple right ideal in R and so, since $R/(0 \times S) \cong K$, we have $L(R_R) = 2$. Now to see that $L({}_R R) = 3$, first note that, since by (1.4) $(0 \times S)(0 \times N) = 0$, the left R -action on $0 \times N$ is just that of K .

So $0 \times N$ is left semisimple over R . Moreover, being a nonzero right ideal in the PMI ring T , N_T is faithful. So if $k \neq 0$ or $s \in N$ then there is an $n \in N$ with

$$(0, n)(k, s) = (0, n(k + \varphi(s))) \neq 0$$

because no transformation of finite rank can be scalar unless it is zero. Thus it follows that $0 \times N$ is essential in ${}_R R$. Hence $0 \times N = \text{Soc}({}_R R)$. But $0 \times N \neq 0$ and clearly $R/(0 \times N)$ is not artinian, so by (1.3) we do have $L({}_R R) = 3$.

Note that, according to the inequality of (1.3), $L({}_R R) = 3$ is the largest possible for a Loewy ring with $L(R_R) = 2$. We do not know whether the bound $2^n - 1$ on $L({}_R R)$ (given $L(R_R) = n$) can be achieved for any $n > 2$.

2. Regular Loewy rings. In [6] Năstăsescu and Popescu proved that a commutative Loewy ring with zero radical is (von Neumann) regular. Subsequently, Năstăsescu [5] proved that a right Loewy ring is regular iff $J(R/S_\alpha) = 0$ for each $\alpha < L(R_R)$. Since the left and right socles of a ring with zero radical are equal [3, Theorem 1, p. 65], the left and right Loewy series for such a ring must be identical. Our concluding results show how to construct both commutative and primitive regular Loewy rings of any length. Unlike our first example there is, of course, no T -nilpotence involved. The basic construction is based upon the following lemma about partitions of an uncountable set.

LEMMA 2.1. *Let β be an ordinal, c be an infinite cardinal and let A be a set such that $\text{card } A > c \geq \text{card } \beta$. Then there is a sequence of partitions $\{\mathcal{P}^\alpha \mid \alpha \leq \beta\}$ of the set A such that:*

- (1) *Whenever $\alpha_1 < \alpha_2$, \mathcal{P}^{α_1} is a refinement of \mathcal{P}^{α_2} but no member of \mathcal{P}^{α_2} is a finite union of members of \mathcal{P}^{α_1} .*
- (2) *Each element of \mathcal{P}^α has cardinality $\leq c$ and $\text{card } \mathcal{P}^\alpha = \text{card } A$ for all α .*

Proof. Let \mathcal{P}^0 be the discrete partition of A . If $a \in A$ let P_a^α be the member of \mathcal{P}^α containing a . The proof is by induction.

First, suppose α is a successor. Then, since $\text{card } \mathcal{P}^{\alpha-1} = \text{card } A$ and $c \leq \text{card } A$, we can write $\mathcal{P}^{\alpha-1}$ as a disjoint union of $\text{card } A$ sets each of cardinality c . If X is an element of the partition of $\mathcal{P}^{\alpha-1}$ described in the previous sentence, we take $Q(X) = \bigcup_x P_x^{\alpha-1}$. The set of all such $Q(X)$ is a partition of A , clearly having the desired properties.

If α is a limit ordinal, we define $P_a^\alpha = \bigcup_{\alpha' < \alpha} P_a^{\alpha'}$ and let $\mathcal{P}^\alpha = \{P_a^\alpha\}$. First, \mathcal{P}^α is a partition. For suppose $P_a^\alpha \cap P_b^\alpha \neq \emptyset$. Then there are ordinals $\alpha_1 \leq \alpha_2 < \alpha$ with $P_a^{\alpha_1} \cap P_b^{\alpha_2} \neq \emptyset$. But since \mathcal{P}^{α_1} is a refinement of \mathcal{P}^{α_2} , we have $P_a^{\alpha_1} \subset P_a^{\alpha_2}$, so $P_a^{\alpha_2} \cap P_b^{\alpha_2} \neq \emptyset$. Thus, $P_a^{\alpha_2} = P_b^{\alpha_2}$. Now, if $y \in P_a^\alpha$, then $y \in P_a^{\delta'}$, where $\delta' < \alpha$. Thus, $y \in P_a^{\delta'}$ for all $\delta' \geq \delta$. Thus, we may assume $\delta \geq \alpha_2$. But since $P_a^{\alpha_2} = P_b^{\alpha_2}$, $P_a^\delta = P_b^\delta$ so that $y \in P_b^\delta \subset P_b^\alpha$, and, since y was arbitrary $P_a^\alpha \subset P_b^\alpha$, and since the argument is symmetric, the P_a^α are pairwise disjoint. Thus, since clearly $A = \bigcup_{a \in A} P_a^\alpha$, we have partitions $\{\mathcal{P}^\alpha \mid \alpha \leq \beta\}$. For these part (1) is clear by construction, and (2) also follows for successor ordinals by construction. If, however, α is a limit ordinal then $P_a^\alpha = \bigcup_{\delta < \alpha} P_a^\delta$. So $\text{card } P_a^\alpha \leq \sup_{\delta < \alpha} (\text{card } P_a^\delta)(\text{card } \alpha)$. But $\text{card } P_a^\delta \leq c$ by induction, and $\text{card } \alpha \leq c$ so $\text{card } P_a^\alpha \leq c^2 = c$. Now, $\text{card } A \leq (\text{card } \mathcal{P}^\alpha)(\sup_{a \in A} \text{card } P_a^\alpha)$. But $\text{card } P_a^\alpha \leq c$, so $\text{card } A \leq (\text{card } \mathcal{P}^\alpha) \cdot c$, while $c < \text{card } A$. So, $\text{card } A \leq \text{card } \mathcal{P}^\alpha$ but \mathcal{P}^α is a partition of A so $\text{card } \mathcal{P}^\alpha \leq \text{card } A$, and we have equality.

THEOREM 2.2. *Let β be any ordinal, then there is a commutative von Neumann regular ring R_β such that every R_β module has non-trivial socle, and R_β has Loewy length $\beta + 1$.*

Proof. Let A be a set with $\text{card } A > \text{card } \beta$ and let $\{\mathcal{P}^\alpha \mid \alpha \leq \beta\}$ be the partitions of Lemma 2.1. Let K be a field and let $T = K^A$, the ring direct product of $\text{card } A$ copies of K with projections $\pi_a: T \rightarrow K$ ($a \in A$). For each ordinal $\alpha < \beta$ and each $P_a^\alpha \in \mathcal{P}^\alpha$ define $\mathfrak{m}(P_a^\alpha) \in T$

$$\pi_b(\mathfrak{m}(P_a^\alpha)) = 1 \quad \text{if } b \in P_a^\alpha$$

and

$$\pi_b(\mathfrak{m}(P_a^\alpha)) = 0 \quad \text{if } b \in A \setminus P_a^\alpha$$

and let 1_T be the identity of T . Then from (2.1.1) it is easy to see that

$$\{1_T\} \cup \{\mathfrak{m}(P_a^\alpha) \mid \alpha < \beta, P_a^\alpha \in \mathcal{P}^\alpha\}$$

is a linearly independent set of vectors in T_K . Let R_β be the subspace that it spans. We shall prove that R_β is a subalgebra of T with the desired properties. The proof is facilitated by the following observation: If $\alpha_1 \leq \alpha_2$, then

$$(*) \quad \mathfrak{m}(P_a^{\alpha_1})(\mathfrak{m}P_a^{\alpha_2}) = \begin{cases} \mathfrak{m}(P_a^{\alpha_1}), & \text{if } P_a^{\alpha_1} \subseteq P_a^{\alpha_2} \\ 0, & \text{otherwise.} \end{cases}$$

This is true because \mathcal{P}^{α_1} is a refinement of \mathcal{P}^{α_2} so that if $P_a^{\alpha_1} \not\subseteq P_a^{\alpha_2}$ then $P_a^{\alpha_1} \cap P_a^{\alpha_2} = \emptyset$. It shows immediately that R_β is a subalgebra of T . Now for each $\alpha < \beta$, let

$$L_\alpha = \sum_{\mathcal{P}^\alpha} \mathfrak{m}(P_a^\alpha)K$$

and for each $\alpha \leq \beta$ set

$$S_\alpha = \sum_{\delta < \alpha} L_\delta$$

(so $S_0 = 0$). Then from (*) it also follows that

$$S_\alpha \ (\alpha \leq \beta) \quad \text{and} \quad S_\alpha + \mathfrak{m}(P_a^\alpha)K \ (\alpha < \beta)$$

are ideals of R_β . But as K -spaces

$$S_{\alpha+1} = S_\alpha \oplus L_\alpha = S_\alpha \oplus (\bigoplus_{\mathcal{P}^\alpha} \mathfrak{m}(P_a^\alpha)K),$$

for each $\alpha < \beta$. Thus $S_{\alpha+1}/S_\alpha$ is a direct sum

$$S_{\alpha+1}/S_\alpha = \bigoplus_{\mathcal{P}^\alpha} (\mathfrak{m}(P_a^\alpha)K + S_\alpha)/S_\alpha$$

of minimal ideals of R/S_α . Therefore, $R_\beta = S_\beta + 1_T K$ is a Loewy ring; and to see that it has the desired length we need only show that $S_{\alpha+1}/S_\alpha$ is essential in R/S_α for each $\alpha < \beta$. So let $\alpha < \beta$ and

$$r = \sum_{i=1}^n \mathfrak{m}(P_{a_i}^{\alpha_i})k_i \in R \setminus S_\alpha$$

with the $k_i \neq 0$, the $P_{a_i}^{\alpha_i}$ distinct members of $\bigcup_{\delta < \beta} \mathcal{P}^\delta$ and

$$\alpha_1 \leq \dots \leq \alpha_{k-1} < \alpha \leq \alpha_{k+1} \leq \dots \leq \alpha_n.$$

Then by (2.1.1) $P_{a_n}^{\alpha_n} \neq \bigcup_{i=1}^{n-1} P_{a_i}^{\alpha_i}$, but $P_{a_n}^{\alpha_n}$ is a union of members of \mathcal{P}^α . So there exists a $P_a^\alpha \in \mathcal{P}^\alpha$ such that $P_a^\alpha \subseteq P_{a_n}^{\alpha_n}$ but $P_a^\alpha \not\subseteq P_{a_i}^{\alpha_i}$ ($i = 1, \dots, n - 1$). Now by (*) we have

$$r\mathfrak{m}(P_a^\alpha) = \sum_{i=1}^{k-1} \mathfrak{m}(P_{a_i}^{\alpha_i})\mathfrak{m}(P_a^\alpha)k_i + \mathfrak{m}(P_a^\alpha)k_n \in S_{\alpha+1} \setminus S_\alpha,$$

which shows that $S_{\alpha+1}/S_\alpha$ is indeed essential in R_β/S_α . Thus $L(R_\beta) = \beta + 1$. Finally, since R_β is commutative, to see that it is regular we need only, by [6, Theorem 3.1], observe that $J(R_\beta) = 0$. But this is the case because each $\mathfrak{m}(P_a^\alpha)$ is an idempotent in R_β .

COROLLARY 2.3. *There are primitive regular Loewy rings of Loewy length $\beta + 1$ for each ordinal β .*

Proof. Use the ring R_β and Lemma 1.1 to construct a PMI ring $R = \hat{R}_\beta + S$ with

$$\text{Soc}(R) = S, \quad \hat{R}_\beta \cap S = 0 \quad \text{and} \quad \tilde{R}_\beta \cong R_\beta;$$

and observe that R has zero radical modulo each term in its Loewy series. Now apply the result of Năstăsescu [5] cited above.

3. **Remarks.** (1) Since minimal one-sided ideals are either nilpotent or idempotent generated, semiprime (left or right) Loewy rings have zero Jacobson radical.

(2) Over a direct product of rings $\prod_A R_\alpha$, the factor $\prod_A R_\alpha / \bigoplus_A R_\alpha$ contains no minimal one-sided ideals. Thus no infinite direct product of rings is a Loewy ring.

(3) Let R be a left or right self-injective right Loewy ring with $J(R) = 0$. Then R is a direct product of endomorphism rings of vector spaces (see [2]). By (2) this product must be finite. But also the vector spaces must be finite dimensional (for otherwise R would have a non-Loewy factor ring). Thus R is a semisimple ring.

(4) Left perfect (=right Loewy with $R/J(R)$ semisimple) rings have the property that each of their nonzero left modules contains a maximal submodule. So do commutative Loewy rings (see [6] and [4]) and right Loewy rings of finite length (see [6], Proposition 5.3, and our Theorem 1.3). We suspect that every right Loewy ring has this property.

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Added in proof. T. Shores has kindly informed us that L. Fuchs had obtained Theorem 2.2 earlier in Theorem 6 of [J. Reine Angew. Math. 239/240 (1970), 169-179].

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