PACIFIC JOURNAL OF MATHEMATICS
Vol. 54, No. 1, 1974

CERTAIN REPRESENTATIONS OF INFINITE GROUP ALGEBRAS

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For any group G, let ρ be an irreducible representation of the group algebra $\mathfrak{F}G$ over a field $\mathfrak{F}$. Then by Schur's lemma, the center Δ of its commuting ring, is a field containing $\mathfrak{F}$. If ρ is finite-dimensional over Δ, then it is called finite and if it is finite-dimensional over $\mathfrak{F}$ itself, then it is called strongly finite. In this paper, certain conditions are given for finiteness of ρ. Also it is shown that for some types of groups, finiteness of ρ is related to the existence of abelian subgroups of finite index in certain quotient of the group. Conditions under which finiteness and strongly finiteness are equivalent, are given. Finally, consequences of ρ being faithful on G, or being faithful on $\mathfrak{F}G$, are studied.

Study of finiteness of irreducible representations was initiated by Kaplansky in [3], and later carried to a great extent by Passman, Issacs, and others: {see [5] and relevent references therein}. Finiteness and strong finiteness were studied in [6]. Using a slight modification of the technique of [4] to suit our nonsemisimple case, we get Theorem 1 which includes the results of [4] and gives us Theorem 2 whose corollaries contains the result of [3].

We further recall the well-known result that for a finite group G, if the kernel of an irreducible representation ρ contains the commutator subgroup G', then the representation is 1-dimensional over Δ. As corollary to our Theorem 3, we prove that in general, if G' is contained in the kernel of ρ, then ρ is finite whether G is finite or not.

2. Finiteness of representation. In this section we study conditions under which a given irreducible representation is finite, and also the conditions under which all irreducible representations are finite. We need the following:

DEFINITIONS. 1. Let ρ be a representation of $\mathfrak{F}G$. Then $G_\rho = \{g \in G \mid \rho(g) = 1\}$, and Kern ρ = kernel $\rho = \{x \in \mathfrak{F}G \mid \rho(x) = 0\}$.

Thus ρ is G-faithful if $G_\rho = 1$, while ρ is $\mathfrak{F}G$-faithful if Kern $\rho = 0$.

2. Let $\mathfrak{B} \leqq \mathrm{Aut}\, G$. For $S \leqq G$, we shall write $\mathfrak{A}_{\mathfrak{B}}(S)$ for the left-ideal $\{\sum x_i(\mathscr{S}_i^{\beta_i} - 1) \mid x_i \in \mathfrak{F}G, \mathscr{S}_i \in S, \beta_i \in \mathfrak{B}\}$. {For a general study of such ideals we may refer to [6] and [8].} We write $\mathfrak{A}(S)$, if $\mathfrak{B} = \{\text{identity}\}$.

3. We also define the $\mathfrak{B}$-kernel of ρ in $H \leqq G$ to be

$$\{h \in H \mid \rho(h^\beta) = 1,\ \forall \beta \in \mathfrak{B}\}\ ,$$

and set $K_n^{\mathfrak{B}}(H) = \bigcap \{\mathfrak{B}\text{-kernels of } \rho \text{ in } H\}$, where the intersection runs through all irreducible representations ρ of G for which $\dim_\Delta \rho > n^2$, where Δ is the center of the commuting ring of ρ. If no such ρ exists, then we put $K_n^{\mathfrak{B}}(H) = G$.

4. Let S_{2n} be the symmetric group of degree $2n$. Then for an algebra A, the sums

$$\sigma_n = \sum_{\sigma \in S_{2n}} (\operatorname{sgn} \sigma) x_{\sigma(1)} \cdot x_{\sigma(2)} \cdots x_{\sigma(2n)},\ x_\rho \in A\ ,$$

are called the Standard Monomial Sums (of parity n).

5. Define $\sum_n (G)$ to be the $\mathfrak{F}$-space spanned by all the σ_n in $\mathfrak{F}G$. {We shall frequently write $\sum_n$ wherever the group in question is clear from context.}

This $\sum_n$ plays a significant role in determining the degrees of irreducible representations.

Specifically we have:

PROPOSITION 1. *Let ρ be an irreducible representation of $\mathfrak{F}G$. Then* $\dim_\Delta \rho \leqq n^2$ *if and only if* $\sum_n \subseteqq \operatorname{Kern} \rho$.

Proof. If $\dim_\Delta \rho \leqq n^2$ then $\mathfrak{F}G/\operatorname{Kern} \rho$ is a primitive algebra of matrices of $\dim n$ over Δ. By Theorem 1 of [1], for any $\sigma_n \in \mathfrak{F}G$, $\rho(\sigma_n) = 0$, whence $\sigma_n \in \operatorname{Kern} \rho$ so that $\sum_n \subseteqq \operatorname{Kern} \rho$.

Conversely, suppose $\sum_n \subseteqq \operatorname{Kern} \rho$. Then $\mathfrak{F}G/\operatorname{Kern} \rho$ is a primitive algebra satisfying $\sigma_n = 0$ for every σ_n in $\mathfrak{F}G/\operatorname{Kern} \rho$. Then by Theorem 1 of [2], p. 226, $\mathfrak{F}G/\operatorname{Kern} \rho$ is a central simple algebra of $\dim \leqq n^2$. Hence $\dim_\Delta \rho \leqq n^2$.

Using this result we obtain:

THEOREM 1. *Let $S \leqq H \leqq G$. Then $S \subseteqq K_n^{\mathfrak{B}}(H)$ if and only if* $\mathfrak{A}_{\mathfrak{B}}(S) \cdot \sum_n \subseteqq \operatorname{Rad} \mathfrak{F}G$.

Proof. We observe that $\mathfrak{A}_{\mathfrak{B}}(S) = \{\sum x_i(s_i^{\beta_i} - 1) \mid x_i \in \mathfrak{F}G, s_i \in S, \beta_i \in \mathfrak{B}\}$. Thus, to show that $\mathfrak{A}_{\mathfrak{B}}(S) \cdot \sum_n \subseteqq \operatorname{Rad} \mathfrak{F}G$, it suffices to show that $(s^\beta - 1) \cdot \sum_n \subseteqq \operatorname{Rad} \mathfrak{F}G$, for $\forall \beta \in \mathfrak{B}, \forall s \in S$. Now let $h \in K_n^{\mathfrak{B}}(H)$ and ρ be an irreducible representation of G. If $\dim_\Delta \rho > n^2$, then $\rho(h^\beta) = 1$ or $\rho(h^\beta - 1) = 0$, $\forall \beta \in \mathfrak{B}$, by the very difinition of $K_n^{\mathfrak{B}}(H)$. On the other hand, if $\dim_\Delta \rho \leqq n^2$, then by Proposition 1, $\rho(\sum_n) = 0$.

Thus, in both cases, $\rho[(h^\beta - 1)\cdot\sum_n] = 0$. Since ρ is arbitrary, so $(h^\beta - 1)\cdot\sum_n \subseteqq \mathrm{Rad}\,\mathfrak{F}G$. Hence $S \subseteqq K_n^{\mathfrak{B}}(H)$ implies $\mathfrak{A}_{\mathfrak{B}}(S)\cdot\sum_n \subseteqq \mathrm{Rad}\,\mathfrak{F}G$.

Conversely, suppose $\mathfrak{A}_{\mathfrak{B}}(S)\cdot\sum_n \subseteqq \mathrm{Rad}\,\mathfrak{F}G$. Then, in particular,

$$(s^\beta - 1)\cdot\textstyle\sum_n \subseteqq \mathrm{Rad}\,\mathfrak{F}G,\ \forall\beta \in \mathfrak{B},\ s \in S\ .$$

We define the left-idealisor [7], of $\sum_n$ into $\mathrm{Rad}\,\mathfrak{F}G$, by $L_{\mathrm{Rad}}(\sum_n)_{\mathfrak{F}G} = L(\sum_n) = \{x \in \mathfrak{F}G \mid x\cdot\sum_n \subseteqq \mathrm{Rad}\,\mathfrak{F}G\}$. This is clearly a left-ideal. Also $[L(\sum_n)\cdot g]\cdot\sum_n = L(\sum_n)[g\cdot\sum_n\cdot g^{-1}]\cdot g = [L(\sum_n)\cdot\sum_n]\cdot g \subset \mathrm{Rad}\,\mathfrak{F}G\cdot g = \mathrm{Rad}\,\mathfrak{F}G$, for $\forall g \in G$. Hence $L(\sum_n)$ is a two-sided ideal of $\mathfrak{F}G$.

Now let ρ be an irreducible representation of G, afforded by the $\mathfrak{F}G$-module $\mathfrak{M}$. Since $L(\sum_n)$ is a two-sided ideal, so $\mathrm{Ann}\,L(\sum_n) = \{m \in \mathfrak{M} \mid L(\sum_n)m = 0\}$ is an $\mathfrak{F}G$-submodule of $\mathfrak{M}$. Thus either $\mathrm{Ann}\,L(\sum_n) = 0$ or $\mathfrak{M}$. Now assume that $\dim_\Delta \rho > n^2$. Again, by Proposition 1, $\rho(\sum_n) \neq 0$ so that $\sum_n\cdot\mathfrak{M} \neq 0$, whence $\sum_n \not\subseteqq \mathrm{Rad}\,\mathfrak{F}G$. But $L(\sum_n)\cdot[\sum_n\cdot\mathfrak{M}] = [L(\sum_n)\cdot\sum_n]\cdot\mathfrak{M} = 0$, since $L(\sum_n)\cdot\sum_n \subseteqq \mathrm{Rad}\,\mathfrak{F}G$. Thus $\mathrm{Ann}\,L(\sum_n) = \mathfrak{M}$. Then, as by hypothesis $s^\beta - 1 \in L(\sum_n)$, so $(s^\beta - 1)\cdot\mathfrak{M} = 0$; or $\rho(s^\beta - 1) = 0$. As ρ was arbitrary with $\dim_\Delta \rho > n^2$; so $s \in K_n^{\mathfrak{B}}(H)$.

Letting $\mathfrak{F}$ = complex-field, we have $\mathrm{Rad}\,\mathfrak{F}G = 0$. Taking $\mathfrak{B} = \{1\}$ in this case, we obtain the result of Passman [4]:

COROLLARY. *$g \in K_n(G)$ if and only if $(g - 1)\cdot\sum_n = 0$.*

We also deduce:

THEOREM 2. *Let $S \leqq G$ and $\mathfrak{B} \leqq \mathrm{Aug}\,G$ such that $S^{\mathfrak{B}} = G$. Then $\mathfrak{A}_{\mathfrak{B}}(S)\cdot\sum_n \subseteqq \mathrm{Rad}\,\mathfrak{F}G$ if and only if $\dim_\Delta \rho \leqq n^2$ for every irreducible representation ρ of G. {Of course, Δ depends on ρ.}*

Proof. By Theorem 1, $\mathfrak{A}_{\mathfrak{B}}(S)\cdot\sum_n \subseteqq \mathrm{Rad}\,\mathfrak{F}G$ if and only if $S \subseteqq K_n^{\mathfrak{B}}(G)$: $\{G = H\}$.

The latter condition is equivalent to the statement that for every irreducible representation ρ with $\dim_\Delta \rho > n^2$, we have $\rho(s^\beta) = 1$, $\forall\beta \in \mathfrak{B}$, $s \in S$. Since $S^{\mathfrak{B}} = G$, so we deduce that $\rho = 1$.

COROLLARY 1. *If $\sum_n \subseteqq \mathrm{Rad}\,\mathfrak{F}G$ for some n, then every irreducible representation of $\mathfrak{F}G$ is finite.*

COROLLARY 2. [3]. *If in $\mathfrak{F}G$, $\sum_n = 0$ for some n, then every irreducible representation of G is finite.*

Next recall that if $|G| < \infty$ then $G' \subseteqq G_\rho$ for any irreducible representation ρ, if and only if ρ is of dim. 1. A generalization of sort, is obtained in the corollary to:

THEOREM 3. *Let ρ be an irreducible representation of $\mathfrak{F}G$.*

(a) *If either* (i) $\sum_n(G/G_\rho) = 0$ *for some n, or* (ii) $\exists A \leqq G \ni \cdot G_\rho \leqq A$, $|G\colon A| < \infty$ *and A/G_ρ is abelian, then ρ is finite.*

(b) (Conversely) *If ρ is finite and $\mathfrak{F}G$ satisfies either of the following conditions*:

(i) *G/G_ρ is periodic and $\mathfrak{F}(G/G_\rho)$ is nonmodular;*

(ii) *G/G_ρ is periodic with a finite p-Sylow subgroup for* Char. $\mathfrak{F} = p \neq 0$;

(iii) *G/G_ρ satisfies minimum-condition on subgroups; then* $\exists A \leqq G \ni \cdot G_\rho \leqq A$, $|G\colon A| < \infty$ *and A/G_ρ is abelian.*

Proof. (a) Suppose (i) holds. In the notation of [6], $G_\rho = \mathfrak{A}^{-1}(\text{Kern}\,\rho)$ where for any ideal I of $\mathfrak{F}G$, $\mathfrak{A}^{-1}(I) = \{g \in G \mid g - 1 \in I\}$, and hence $\mathfrak{A}(G_\rho)$ is a sub-ideal in Kern ρ. Since $\mathfrak{A}(G_\rho)$ is the kernel of the linear extension of the cannonical map $G \rightarrow G/G_\rho$, so $\mathfrak{F}G/\mathfrak{A}(G_\rho) \cong \mathfrak{F}(G/G_\rho)$. Therefore, $\sum_n (G/G_\rho) = 0$ implies that the standard monomialsum, in $\mathfrak{F}G/\mathfrak{A}(G_\rho)$, all vanish. Now $\mathfrak{F}G/\text{Kern}\ \rho \cong \mathfrak{F}G/\mathfrak{A}(G_\rho)/\text{Kern}\,\rho/\mathfrak{A}(G_\rho)$; therefore, the same holds for $\mathfrak{F}G/\text{Kern}\,\rho$. In particular, $\sum_n(G) \subseteqq \text{Kern}\,\rho$. Then, by Proposition 1, ρ is finite. Next let (ii) hold. Then $|G/G_\rho\colon A/G_\rho| = n < \infty$, and A/G_ρ is abelian. Therefore, by the result of Kaplansky mentioned before, or by Theorems. 5.1, 8.1 of [5], all the irreducible representations of G/G_ρ are finite.

Now if ρ is afforded by the $\mathfrak{F}G$-module $\mathfrak{M}$, then putting $\bar{\rho}(\bar{g})\cdot m = \rho(g)\cdot m$, for $\bar{g} \in G/G_\rho$,and observing that $G_\rho = \{g \in G \mid \rho(g) = 1\}$, we get a representation $\bar{\rho}$ of G/G_ρ, such that $\bar{\rho}$ is irreducible and the commuting ring of $\bar{\rho}$ in $\text{Hon}_{\mathfrak{F}}(\mathfrak{M}, \mathfrak{M})$, is the same as that of ρ.

Thus the finiteness of $\bar{\rho}$ implies the finiteness of ρ.

(b) $G/G_\rho \cong S \leqq GL(n, \Delta)$ and any such S satisfying either of the conditions (i), (ii) or (iii), is abelian by finite: {see [9], Corollaries 9.4, 9.7, 9.8, and 9.23}. We then get our A, by taking the complete inverse-image of the abelian part of G/G_ρ.

Since the group-algebra of an abelian group always satisfies $\sum_n = 0$, so we obtain:

COROLLARY. *If $G' \subseteqq G_\rho$, then ρ is finite.*

3. Strong finiteness of representations. In this section we give a result which shows the equivalence of finiteness and strong-finiteness in certain conditions.

THEOREM 4. *Under either of the following conditions, an irreducible representation ρ of G is finite if and only if it is strongly finite:*

(i) *G is finitely generated;*

(ii) *ρ is absolutely irreducible;*

(iii) *$\exists H \trianglelefteq G \ni |G: H| < \infty$ and ρ_H has a strongly finite constituent.*

Proof. (i) This is the content of Lemma 7 of [6].

(ii) Let the absolutely irreducible finite representation ρ, be afforded by the $\mathfrak{F}G$-module $\mathfrak{M}$. Since $\Delta \subseteqq \mathrm{Hom}_{\mathfrak{F}}(\mathfrak{M}, \mathfrak{M})$, $\mathfrak{F} \subseteqq \Delta$, so we can make $\Delta \otimes_{\mathfrak{F}} \mathfrak{M}$ into a ΔG-module by letting $g\cdot(d \otimes m) = d \otimes \rho(g)m$.

Define $\psi: \Delta \otimes_{\mathfrak{F}} \mathfrak{M} \to \mathfrak{M}$ by $\psi(d \otimes m) = dm$. Since ρ and d commute, so

$$\psi(g\cdot(d \otimes m)) = \psi(d \otimes \rho(g)m) = d(\rho(g)m) = \rho(g)(dm) .$$

Thus ψ is a ΔG-homomorphism. Since $\mathfrak{M}$ is absolutely irreducible, so $\Delta \otimes_{\mathfrak{F}} \mathfrak{M}$ is irreducible. So ψ is an isomorphism. Then $\dim_{\Delta}(\Delta \otimes_{\mathfrak{F}} \mathfrak{M}) = \dim_{\Delta} \mathfrak{M} < \infty$. Thus ρ is also finite-dimensional over $\mathfrak{F}$.

(iii) By Clifford's theorem, $\rho_H = \bigoplus \sum_{i=1}^{[G:H]} \rho_i$, where ρ_i are all conjugate irreducible-representations of H. Hence, if one of them is finite-dimensional over $\mathfrak{F}$, then so are all; and hence ρ.

4. Faithful representation. Finally, let ρ be a representation (not necessarily irreducible) of the group algebra $\mathfrak{F}G$. For any left-ideal I we shall write ρ^I for the representation afforded by the module $I\cdot\mathfrak{M}$, where ρ is afforded by $\mathfrak{M}$. We shall let $\mathfrak{A} = \mathfrak{A}(G)$ denote the augmentation-ideal of $\mathfrak{F}G$ and $J = [\mathfrak{F}G, \mathfrak{F}G]$. Let $\mathrm{Char}\, \mathfrak{F} = p \neq 0$.

We then investigate the consequences of ρ being faithful as a representation of G and as a representation of $\mathfrak{F}G$ respectively. Recalling that if $H \leqq G$, then $\mathfrak{A}(H)$ is the left-ideal in $\mathfrak{F}G$ generated by $\{h - 1 \mid h \in H\}$: [6], we have the following:

THEOREM 5. (a) *If ρ is faithful on G, then $A = \mathfrak{A}^{-1}(\mathrm{Kern}\, \rho^{\mathfrak{A}})$ is an elementary abelian normal p-subgroup which is central if $\mathfrak{A} \subseteqq \mathrm{Kern}\, \rho^{\mathfrak{A}}$.*

(b) *If ρ is faithful on $\mathfrak{F}G$, then*

(i) *$A = 1$ unless $|G| = 2$, $p = 2$ in which case $A = G$;*

(ii) *$B = \mathfrak{A}^{-1}(\mathrm{Kern}\, \rho^J) = 1$ unless $p = 2$ and $B = G$ is abelian, or $p = 2$, G is nonabelian and B is central.*

Proof. (a) Since $\mathrm{Kern}\, \rho^{\mathfrak{A}} \trianglelefteq \mathfrak{F}G$ so $A \trianglelefteq G$.

Let $\mathfrak{M}$ afford ρ so that $\mathfrak{A}\cdot\mathfrak{M}$ affords $\rho^{\mathfrak{A}}$. Hence $g \in A$ if and only if $(g-1)\cdot\sum_{\substack{x\in G\\ \lambda_x\in\mathfrak{F}}} \lambda_x(x-1)m = 0$ for each $m \in \mathfrak{M}$. Since $p \neq 0$, so $(g^p - 1)m = (g-1)^p m = (g-1)[(g-1)^{p-1}m] = 0$ as $(g-1)^{p-1}m \in \mathfrak{A}\cdot\mathfrak{M}$. Thus $g^p m = m$ and faithfulness implies that $g^p = 1$. Further, if $h \in A$ then $(h-1)m$ and $(g-1)m$ are both in $\mathfrak{A}\cdot\mathfrak{M}$. Then,

$$(g-1)(h-1)m = 0 = (h-1)(g-1)m$$

or

$$g^{-1}h^{-1}gh\, m = m\ .$$

Again faithfulness gives that $gh = hg$; i.e., A is abelian.

If $\mathfrak{A} \subseteqq \operatorname{Kern} \rho^{\mathfrak{A}}$, then $g \in G$, $h \in A$ implies

$$\begin{aligned}(gh-1)m &= [(g-1)(h-1) + (g-1) + (h-1)]m\\ &= (g-1)m + (h-1)m = (hg-1)m\ .\end{aligned}$$

Thus $gh\, m = hg\, m$ and faithfulness gives that $A \subseteqq Z(G)$.

(b) (i) Now let ρ be $\mathfrak{F}G$-faithful. Then the Kern $\rho^{\mathfrak{A}} = \operatorname{Ann} \mathfrak{A}$ in $\mathfrak{F}G$. It is well-known that this annihilator is 0 unless $|G| < \infty$ and $\operatorname{Ann}\mathfrak{A} = \mathfrak{F}\cdot(\sum_{g\in G} g)$. Now if $g_i \in A$, then $g_i - 1 = a \in \operatorname{Kern} \rho^{\mathfrak{A}}$ so that $g_i - 1 = k\cdot\sum g$, $k \in \mathfrak{F}$. Linear independence of the group elements, gives us that $g_i = 1$, and $k = 0$, or $|G| = 2$, $g_1 = 1$, $i = 2$, $k = 1$, and $+1 = -1$ in $\mathfrak{F}$.

(ii) Again by faithfulness $\operatorname{Kern} \rho^J = \operatorname{Ann} J$. So $g \in B$ implies that $(g-1)(hk - kh) = 0$, $\forall\, h, k \in G$, i.e., $ghk - gkh - hk + kh = 0$. If Char, $\mathfrak{F} \neq 2$, then we must either have $ghk = gkh$ in which case $hk = kh$, or $ghk = hk$ in which case $g = 1$.

In case $\operatorname{Char} \mathfrak{F} = 2$ and g is noncentral then choose $k \in G$ such that $gk \neq kg$. Put $h = g^{-1}$. Then the above identity gives,

$$k - gkg^{-1} - g^{-1}k + kg^{-1} = 0\ .$$

Since $gkg^{-1} \neq k$, so either $k = g^{-1}k$ or $k = kg^{-1}$, both leading to $g = 1$, a contradiction. Thus in this case $g \in Z(G)$.

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Received April 18, 1973 and in revised form October 16, 1973.

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