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COMPARISON OF THE STATES OF CLOSED LINEAR TRANSFORMATIONS

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Let X and Y be Banach spaces and T, respectively S, be a bounded linear transformation mapping X into Y, respectively Y into X. It is well-known that a nonzero complex number λ belongs to the spectrum of ST precisely when λ belongs to the spectrum of TS. The main result of § 2 shows that for $\lambda \neq 0$ the states of the operators $ST - \lambda I_X$, $TS - \lambda I_Y$ agree.

Sufficient conditions are obtained for this same result to hold when T and S are unbounded closed linear transformations from X into Y and Y into X respectively. Section 4 compares spectral decompositions of ST and TS when these sufficient conditions are satisfied.

Throughout this paper $D(A)$ and $R(A)$ will denote the domain and range of A. The resolvent of A will be denoted $\rho(A)$, the spectrum $\sigma(A)$, the point spectrum $p(A)$ and the approximate point spectrum $a(A)$. $[X, Y]$ will denote the set of all bounded linear transformations, defined on the Banach space X into the Banach space Y. Any other notation used will agree with that of [3]. When no confusion will arise the identity operator will be denoted by I regardless of the space. The following preliminary result can be easily varified.

PROPOSITION 1.1. *If $T: D(T) \subset X \to Y$, $S: D(S) \subset Y \to X$ and $\lambda \neq 0$, then $\lambda \in p(TS)$ if and only if $\lambda \in p(ST)$.*

2. Continuous transformations.

PROPOSITION 2.1. *If $\lambda \neq 0$ then $\overline{R(ST - \lambda I)} = X$ precisely when $\overline{R(TS - \lambda I)} = Y$.*

Proof. $\overline{R(ST - \lambda I)} \neq X$ implies that there exists an $x' \in X'$, $x' \neq 0$ such that $x'((ST - \lambda I)(x)) = 0$ for all $x \in X$. Consequently for all $x \in X$, $0 = (ST - \lambda I)'(x'(x)) = (T'S' - \lambda I)(x'(x))$ and $\lambda \in p(T'S')$. By Proposition 1.1, $\lambda \in p(S'T')$ so $y' \in Y'$, $y' \neq 0$ exists with the property that for each $y \in Y$, $0 = (S'T' - \lambda I)(y'(y)) = y'((TS - \lambda I)(y))$. Thus $\overline{R(TS - \lambda I)} \neq Y$.

The following is a construction of a "generalized" Banach space in the manner of that of Berberian [2].

Denote by glim a fixed "generalized Banach limit" defined for all bounded sequences of complex numbers and having properties: ([1] — page 34)

(i) $\operatorname{glim}(\lambda_n + \mu_n) = \operatorname{glim} \lambda_n + \operatorname{glim} \mu_n$;

(ii) $\operatorname{glim}(\lambda \lambda_n) = \lambda \operatorname{glim} \lambda_n$;

(iii) $\operatorname{glim} \lambda_n = \lim \lambda_n$ if $\{\lambda_n\}$ converges;

(iv) $\operatorname{glim} \lambda_n \geqq 0$ whenever $\lambda_n \geqq 0$ for each n.

For a Banach space X, denote by $\mathscr{B}(X)$ the set of all sequences $\{x_n\}$ of elements of X for which $\sup \|x_n\| < \infty$. If for $s = \{x_n\}$ and $t = \{y_n\}$ in $\mathscr{B}(X)$ and complex λ we define $s + t = \{x_n + y_n\}$, $\lambda s = \{\lambda x_n\}$ and $\|s\|_1 = \operatorname{glim} \|x_n\|$ it is clear that $\mathscr{B}(X)$ is a prenormed space. If $\mathscr{N}(X) = \{s \in \mathscr{B}(X): \|s\|_1 = 0\}$ then $\mathscr{P}(X) = \mathscr{B}(X)/\mathscr{N}(X)$ is a normed vector space whose completion will be denoted by $\mathscr{K}(X)$. Since $x \mapsto \{x\} + \mathscr{N}(X)$ is an isomorphism of X into a closed linear subspace X' of the Banach space $\mathscr{K}(X)$, X can be identified with this subspace and X' is called the generalized extension of X.

For $T \in [X, Y]$ define $\mathscr{B}(T): \mathscr{B}(X) \to \mathscr{B}(Y)$ as $\mathscr{B}(T): s = \{x_n\} \mapsto \{Tx_n\}$. T is bounded so $\mathscr{B}(T)$ is bounded and $\|\mathscr{B}(T)\|_1 = \|T\|$. Moreover, $\mathscr{B}(T): \mathscr{N}(X) \to \mathscr{N}(Y)$ so $\mathscr{B}(T)$ may be extended to $\mathscr{P}(X)$ and consequently to X' to obtain a unique extension $T' \in [X', Y']$ of T with $\|T'\| = \|T\|$.

For $T_1, T_2 \in [X, Y]$ and $S \in [Y, X]$ the following properties can be verified directly:

(i) $(T_1 + T_2)' = T_1' + T_2'$;

(ii) $(\lambda T_1)' = \lambda T_1'$;

(iii) $(ST_1)' = S'T_1'$.

The next proposition gives the results which necessitated the preceding construction. For the Hilbert space analogue of this proposition, see Berberian [2], Theorem 1.

PROPOSITION 2.2. *Let $A \in [X, X]$ then $a(A) = a(A') = p(A)$.*

Proof. $\lambda \in a(A')$ implies that for each $\varepsilon > 0$ an $s \in X'$ exists with $\|(A' - \lambda I)s\| < \varepsilon \|s\|$. Since $\mathscr{P}(x)$ is dense in X', it may be assumed that $s = \{x_n\} \in \mathscr{P}(X)$. Thus $\|(A' - \lambda I)s\| = \operatorname{glim} \|(A - \lambda I)x_n\| < \varepsilon \operatorname{glim} \|x_n\|$ so $0 > \operatorname{glim} [\varepsilon \|x_n\| - \|(A - \lambda I)x_n\|]$. By property (iv) of glim it must be true that for at least one n, $0 < \varepsilon \|x_n\| - \|(A - \lambda I)x_n\|$ and hence for some $x_n \in X$, $\|(A - \lambda I)x_n\| < \varepsilon \|x_n\|$, which implies $\lambda \in a(A)$.

To complete the proof of this proposition, if suffices to show that $a(A) \subset p(A')$. For $\lambda \in a(A)$, a sequence $\{x_n\}$ in X exists with $\|x_n\| = 1$ for all n and $\|(A - \lambda I)x_n\| \to 0$. $\{x_n\}$ is bounded in norm so $s = \{x_n\} \in X'$, $\|s\| = 1$ and $\|(A' - \lambda I)s\| = \operatorname{glim} \|(A - \lambda I)x_n\| = 0$. Hence $\lambda \in p(A')$.

Considering $T \in [X, Y]$ and $S \in [Y, X]$ we obtain:

COROLLARY 2.1. *If* $\lambda \neq 0$ *then* $\lambda \in a(TS)$ *if and only if* $\lambda \in a(ST)$.

Proof. By Proposition 2.2, $\lambda \in a(TS)$ implies $\lambda \in p((TS)') = p(T'S')$. Hence by Proposition 1.1, $\lambda \in p(S'T') = a(ST)$.

The preceding corollary together with the result of Propositions 1.1 and 2.1 prove the following theorem. The classification of states of a linear operator may be found in [5].

THEOREM 2.1. *If* $T \in [X, Y]$, $S \in [Y, X]$ *and* $\lambda \neq 0$, *then the states of* $TS - \lambda I$ *and* $ST - \lambda I$ *agree.*

Proof. To show that one of the operators cannot be in state I_3 while the other is in state II_3, a theorem of Goldberg [4], Theorem II 4.4, is used which in our case states:

(i) T has a bounded inverse if and only if $R(T^*) = X^*$.

(ii) T^* has a bounded inverse if and only if $R(T) = Y$.

3. **Closed transformations.** Let T be a closed linear transformation with $D(T)$ and $R(T)$ both contained in the Banach space X. Suppose further that $\rho(T) \neq \phi$, that $\alpha \in \rho(T)$ is fixed and $A \in [X, X]$ is defined by $A = (T - \alpha I)^{-1}$. The following theorems are due to Taylor [6].

THEOREM 3.1. *Suppose* μ *and* λ *are complex numbers satisfying* $(\lambda - \alpha)\mu = 1$:

(i) *If* $x \in X$ *and* $(\mu I - A)x = y$ *then* $(T - \lambda I)(\mu x - y) = \mu^{-1}y$;

(ii) *If* $x \in D(T)$ *and* $(T - \lambda I)x = y$ *then* $(\mu I - A)x = \mu Ay$.

Furthermore, $\mu I - A$ *is* 1-1 *precisely when* $T - \lambda I$ *is* 1-1 *and on the common domain of their inverses* $(\mu I - A)^{-1} = \mu^{-2}[\mu I + (T - \lambda I)^{-1}]$ *and* $(T - \lambda I)^{-1} = \mu(\mu I - A)^{-1}A = \mu A(\mu I - A)^{-1}$.

THEOREM 3.2. *Let* λ *and* μ *satisfy* $(\lambda - \alpha)\mu = 1$. *Then* λ *belongs to* $\rho(T)$ *if and only if* μ *belongs to* $\rho(A)$.

The following lemma follows from the closed graph theorem and will be needed often in our development:

LEMMA 3.1. *If* $P: D(P) \subset Y \to Z$ *is a closed linear transformation and* $Q \in [X, Y]$ *where* X, Y, Z *are Banach spaces, then* PQ *is closed and if* $R(Q) \subset D(P)$ *then* $PQ \in [X, Z]$.

For T closed with $D(T)$ and $R(T)$ both in X and $0 \neq \alpha \in \rho(T)$ we define $A = (T - \alpha I)^{-1}$ and $B = T(T - \alpha I)^{-1}(T - \alpha I^{-1})$. (By Lemma 3.1, $B \in [X, X]$.)

The next three propositions give the substance for a method of referring a pair of closed operators to a pair which are continuous and everywhere defined.

PROPOSITION 3.1. *Consider* T, A, *and* B *as defined above and* $0 \neq \alpha \in \rho(T)$. *For* $0 \neq \lambda \neq \alpha$, *let* $\nu = (\lambda/(\lambda - \alpha)^2)$, $\mu = (1/(\lambda - \alpha))$. *Then* $R(B - \nu I) \subset R(T - \lambda I)$.

Proof. Suppose $y = (B - \nu I)x$. Then $y + \nu x = (T - \alpha I)^{-1}[x + \alpha(T - \alpha I)^{-1}x] \in D(T - \alpha I) = D(T - \lambda I)$ and $(T - \lambda I)(y + \nu x) + (\lambda - \alpha)(y + \nu x) = x + \alpha(T - \alpha I)^{-1}x$; so $-1/\alpha[(T - \lambda I)(y + \nu x) + (\lambda - \alpha)y] = \mu x - Ax$. If Theorem 3.1, part (i), is applied, we obtain that $1/\alpha[(T - \lambda I)(y + \nu x) + (\lambda - \alpha)y] \in R(T - \lambda I)$ so that

$$
\begin{aligned}
(T - \lambda I)\Big\{\mu x + \frac{1}{\alpha}[(T - \lambda I)(y + \nu x) + (\lambda - \alpha)y] \\
+ \frac{1}{\mu\alpha}(y + \nu x)\Big\} = \frac{-1}{\alpha\nu\mu} y \in R(T - \lambda I) .
\end{aligned} \tag{1}
$$

PROPOSITION 3.2. *If* $\lambda \neq 0$ *is such that for some* $0 \neq \alpha \in \rho(T)$, $\alpha^2/\lambda \in \rho(T)$ *also, then* $R(T - \lambda I) \subset R(B - \nu I)$.

Proof. We may assume, without loss of generality, that $\lambda \neq \alpha$, for if $0 \neq \alpha \in \rho(T)$ there exists some $a > 0$ with $0 \notin \{\mu \mid |\mu - \lambda| < a\} \subset \rho(T)$ and $\lambda = \alpha + (a/2)e^{i\theta}$, where θ is the argument of α, will satisfy our hypothesis.

For $x \in D(T)$, $BTx = x + \alpha Ax + \alpha Bx$. Consequently, if $(T - \lambda I)x = y$, then $(y - \alpha)Bx = x + \alpha Ax - By$. Theorem 3.1, part (ii), implies $(\mu I - A)x = \mu Ay$ so $(\lambda - \alpha)Bx = \lambda\mu x - \alpha\mu Ay - By$. Thus

$$
\begin{aligned}
(B - \nu I)(x + \mu y) &= -\frac{\alpha\nu}{\lambda} Ay - \nu\mu y \\
&= -\frac{\alpha\nu}{\lambda}\Big[A - \frac{1}{(\alpha^2/\lambda - \alpha)} I\Big]y .
\end{aligned}
$$

By hypothesis, $\alpha^2/\lambda \in \rho(T)$, so Theorem 3.2 may be used to obtain

$$
\frac{1}{(\alpha^2/\lambda - \alpha)} \in \rho(A)
$$

and

$$
-\frac{\alpha\nu}{\lambda} y = \Big[A - \frac{\lambda}{(\alpha^2 - \alpha\lambda)} I\Big]^{-1} (B - \nu I)(x + \mu y) .
$$

Since $\Big[A - \dfrac{\lambda}{(\alpha^2 - \alpha\lambda)} I\Big]^{-1}$ and $B - \nu I$ commute,

$$(2)\quad -\frac{\alpha\nu}{\lambda} y = (B - \nu I)\left[A - \frac{1}{(\alpha^2/\lambda - \alpha)} I\right]^{-1} (x + \mu y) \in R(B - \nu I)\ .$$

The following proposition follows easily by considering equations (1) and (2), together with the result of Theorem 3.2.

PROPOSITION 3.3. *Suppose* $\lambda \neq 0$ *and for some* $\alpha \neq 0$ *in* $\rho(T)$, $\alpha^2/\lambda \in \rho(T)$ *also. Then* $T - \lambda I$ *is* 1-1 *precisely when* $B - \nu I$ *is* 1-1.

The following theorem is an immediate consequence of the Propositions 3.1, 3.2, 3.3, and the closed graph theorem.

THEOREM 3.3. *Let* T *be a closed linear operator with* $D(T)$ *and* $R(T)$ *both contained in the Banach space* X. *Suppose* $\lambda \neq 0$ *is a complex number with the property that for some* $\alpha \in \rho(T)$, $\alpha^2/\lambda \in \rho(T)$ *also, then the state of* $T - \lambda I$ *is the same as the state of* $B - \nu I$.

For the remainder of this section, we consider a pair of closed linear transformations, $T: D(T) \subset X \to Y$ and $S: D(S) \subset Y \to X$, with the property that ST and TS are both closed on their respective domains. We assume moreover that $\rho(TS) \cap \rho(ST) \neq \varphi$ and for $\alpha \in \rho(TS) \cap \rho(ST)$ fixed we define:

$$A(ST) = (ST - \alpha I)^{-1}\ , \qquad A(TS) = (TS - \alpha I)^{-1}$$

and

$$\begin{aligned} B(ST) &= ST(ST - \alpha I)^{-1}(ST - \alpha I)^{-1} \\ B(TS) &= TS(TS - \alpha I)^{-1}(TS - \alpha I)^{-1}\ . \end{aligned}$$

When $x \in D(T)$, $TA(ST)x = A(TS)Tx$; thus $B(ST)$ and $B(TS)$ may be rewritten:

$$\begin{aligned} B(ST) &= S(TS - \alpha I)^{-1}T(ST - \alpha I)^{-1} = SA(TS)TA(ST) \\ B(TS) &= T(ST - \alpha I)^{-1}S(TS - \alpha I)^{-1} = TA(ST)SA(TS)\ . \end{aligned}$$

Since $R(A(ST)) \subset D(T)$ and $R(A(TS)) \subset D(S)$, Lemma 3.1 shows that $TA(ST) \in [X, Y]$ and $SA(TS) \in [Y, X]$. By Theorem 2.1, whenever $\nu \neq 0$, the state of $B(TS) - \nu I$ is the same as the state of $B(ST) - \nu I$, which gives the main result in this section:

THEOREM 3.4. *If* $\lambda \neq 0$ *is such that for some* $\alpha \in \rho(ST) \cap \rho(TS)$, $\alpha^2/\lambda \in \rho(ST) \cap \rho(TS)$ *also, the state of* $ST - \lambda I$ *is the same as the state of* $TS - \lambda I$.

It is conjectured that the hypothesis of Theorem 3.4 can be weakened to simply requiring that $\rho(ST) \cap \rho(TS) \neq \{0\}$. A different method of proof would likely be needed, however.

In the remainder of this section we consider conditions on the transformations S and T which will ensure that the hypotheses of Theorem 3.4 are fulfilled. We first need the following propositions:

PROPOSITION 3.4. *If T, S, TS, and ST are closed and $\lambda \neq 0$ is such that $\lambda \in \sigma(TS) \cap \rho(ST)$, then whenever $\alpha \in \rho(TS) \cap \rho(ST)$,*

$$\frac{\alpha^2}{\lambda} \in \sigma(ST) .$$

Proof. Since $\lambda \in \rho(ST)$, $ST - \lambda I$ is 1–1; so by Theorem 3.4, $TS - \lambda I$ is also 1–1, and $\lambda \in \sigma(TS)$ implies $\overline{R(TS - \lambda I)} \neq Y$. For $0 \neq \alpha \in \rho(TS)$ and $\nu = \lambda/(\lambda - \alpha)^2$ we have by Proposition 3.1

$$R(B(TS) - \nu I) \subset R(TS - \lambda I)$$

and consequently

$$\overline{R(B(TS) - \nu I)} \neq Y .$$

By Theorem 3.3 $R(B(ST) - \nu I) \neq X$. If $\alpha^2/\lambda \in \rho(ST)$, then

$$R(ST - \lambda I) \subset R(B(ST) - \nu I) ,$$

so

$$\overline{R(ST) - \lambda I)} \neq X .$$

This clearly contradicts our assumption of $\lambda \in \rho(ST)$.

PROPOSITION 3.5. *If T, S, TS, and ST are closed and ρ_1, respectively ρ_2, are connected components of $\rho(ST)$, respectively $\rho(TS)$, then $(\rho_1 - \rho_2) \cup (\rho_2 - \rho_1) \subset \{0\}$.*

Proof. It suffices to show that both $\rho_1 \cap \partial\rho_2 \subset \{0\}$, where $\partial\rho_2$ denotes the boundary of ρ_2, and $\rho_2 \cap \partial\rho_1 \subset \{0\}$.

To prove the former, suppose $0 \neq \lambda \in \rho_1 \cap \partial\rho_2$. Then $\lambda \in \sigma(TS)$ and there is an open set N with $\lambda \in N \subset \rho_1$. We may therefore construct a sequence $\lambda_n \in \rho_1 \cap \rho_2$ for all n with the property that λ_n converges to λ. By Proposition 3.4 $\mu^2/\lambda \in \sigma(ST)$ whenever $\mu \in \rho(ST) \cap \rho(TS)$. In particular $(\lambda_n)^2/\lambda \in \sigma(ST)$ for all n. This is clearly impossible since $(\lambda_n)^2/\lambda$ converges to λ and eventually $(\lambda_n)^2/\lambda \in N$.

The next two propositions give sufficient conditions for the hypothesis of Theorem 3.4 to be fulfilled.

PROPOSITION 3.6. *If T, S, TS, and ST are closed and such that there exists a neighborhood of zero intersected with an open half plane about the origin which is a subset of $\rho(ST) \cap \rho(TS)$ then the state of $TS - \lambda I$ is the same as the state of $ST - \lambda I$ whenever $\mu \neq 0$.*

Proof. Suppose $D = \{\mu \mid |\mu| < r\} \subset U$ is contained in $\rho(ST) \cap \rho(TS)$, where U denotes the open upper half plane.

Given $\lambda \neq 0$, choose α satisfying

(i) $0 < |\alpha| < \min\{r, |\lambda|\}$;

(ii) argument of α, $\arg \alpha$, is as follows:

(a) $\pi/4$ if $\arg \lambda = 0$;

(b) $\arg \lambda$ if $0 < \arg \lambda < \pi$;

(c) $3\pi/4$ if $\arg \lambda = \pi$;

(d) $\pi/2 + \arg \lambda/4$ if $\pi < \arg \lambda < 2\pi$.

By direct calculation, it can be shown that both α and α^2/λ belong to $D \subset \rho(ST) \cap \rho(TS)$ and consequently by Theorem 3.4, the states of $TS - \lambda I$ and $ST - \lambda I$ agree. It is clear, by the method in which α was chosen, that our assumption of U being the open upper half plane involves no loss of generality. Any other open half plane about the origin would simply introduce a change in $\arg \alpha$.

Note that if S, T, ST, and TS are closed operators in a Hilbert space with both ST and TS self-adjoint, the hypotheses of Proposition 3.6 hold.

PROPOSITION 3.7. *Let T, S, TS, and ST be closed and such that there exists a half plane entirely contained in $\rho(ST) \cap \rho(TS)$. Then the state of $ST - \lambda I$ is the same as the state of $TS - \lambda I$ whenever $\lambda \neq 0$.*

Proof. Suppose that U is a half plane contained in $\rho(ST) \cap \rho(TS)$. We may assume, without loss of generality, that

$$U = \{\mu \mid IM(\mu) > R\} \text{ where } R > 1 .$$

For $\lambda \neq 0$ we choose α as follows:

(i) If $\arg \lambda = 0$, then $\arg \alpha = \pi/4$ and $|\alpha| = \max\{\alpha R, |\lambda|\}$;

(ii) If $\arg \lambda = \pi$, then $\arg \alpha = 3\pi/4$ and $|\alpha| = \max\{\alpha R, |\lambda|\}$;

(iii) If $0 < \arg \lambda < \pi$, then $\arg \alpha = \arg \lambda$ and

$$|\alpha| = \max\left\{|\lambda|, \frac{\alpha R}{\sin(\arg \lambda)}\right\} ;$$

(iv) If $\pi < \arg \lambda < 2\pi$, then $\arg \alpha = \arg \lambda - \pi$ and

$$|\alpha| = \max\left\{|\lambda|, \frac{R}{\sin(\arg \lambda - \pi)}\right\} .$$

It can be demonstrated in a straight forward manner that both α and α^2/λ are in $\rho(ST) \cap \rho(TS)$ in each case.

4. **Spectral decompositions.** The notation in the following discussion is full explained in [5].

THEOREM 4.1. *If D is a bounded Cauchy domain satisfying*

$$\partial D \subset \rho(ST) \cap \rho(TS)$$

then there exists a pair of closed subspaces (X_1, X_2) of X and (Y_1, Y_2) of Y such that

(i) *(X_1, X_2) completely reduces ST;*

(ii) *(Y_1, Y_2) completely reduces TS;*

(iii) *$(ST)_1 = ST \mid X_1$ and $(TS)_1 = TS \mid Y_1$ are continuous with domains X_1, Y_1 respectively;*

(iv) *$T: X_i \to Y_i$, $S: Y_i \to X_i$, $i = 1, 2$.*

Proof. Let

$$\sigma_1 = D \cap \sigma_e(ST) ,$$
$$\sigma_2 = D \cap \sigma_e(TS) ,$$

where σ_e denotes the extended spectrum of the transformation. σ_1 and σ_2 are bounded spectral sets for ST and TS respectively. Let $\tau_1 = \sigma_e(ST) - \sigma_1$ and $\tau_2 = \sigma_e(TS) - \sigma_2$ be their complementary spectral sets.

If $E(\sigma_1)$, $E(\sigma_2)$, $E(\tau_1)$, and $E(\tau_2)$ are the projections associated with these spectral sets with ranges X_1, Y_1, X_2, and Y_2 respectively, it is well-known, see [5], that statements (i), (ii), and (iii) are satisfied.

For $x \in X$, $E(\sigma_1)x \in X_1$ and

$$\begin{aligned} TE(\sigma_1)x &= T\left[-\frac{1}{2\pi i}\int_{+\partial D}(ST - \lambda I)^{-1}d\lambda\right]x \\ &= \left[-\frac{1}{2\pi i}\int_{+\partial D}T(ST - \lambda I)^{-1}d\lambda\right]x \\ &= \left[-\frac{1}{2\pi i}\int_{+\partial D}(TS - \lambda I)^{-1}d\lambda\right]Tx \\ &= E(\sigma_2)Tx . \end{aligned}$$

so $T: X_1 \to Y_1$.

Similarly, if $x \in X$, $E(\tau_1)x \in X_2$ and

$$\begin{aligned} TE(\tau_1) &= T(I - E(\sigma_1))x = Tx - TE(\sigma_1)x \\ &= (I - E(\sigma_2))Tx = E(\tau_2)Tx . \end{aligned}$$

So $T\colon X_2 \to Y_2$.

In a similar manner $S\colon Y_1 \to X_1$, $S\colon Y_2 \to X_2$ which completes the proof of the theorem.

THEOREM 4.2. *If D is a bounded Cauchy domain with*

$$\partial D \subset \rho(ST) \cap \rho(TS)$$

and

$$\sigma_1 = D \cap \sigma_e(ST) , \qquad \sigma_2 = D \cap \sigma_e(TS) ,$$

then

$$(\sigma_1 - \sigma_2) \cup (\sigma_2 - \sigma_1) \subset \{0\} .$$

If in addition $0 \in D$, then

(i) *the complementary spectral sets τ_1 and τ_2 are equal;*

(ii) *the state of $ST - \lambda I$ is the same as the state of $TS - \lambda I$, whenever $\lambda \neq 0$.*

Proof. Using the notation of Theorem 4.1, let $T_i = T \mid X_i$, $S_i = S \mid Y_i$, $i = 1, 2$. Since T_i, S_i, T_iS_i, and S_iT_i, $i = 1, 2$, are restrictions of closed operators to closed subspaces, they are closed. Furthermore, $S_iT_i = (ST)_i$, $T_iS_i = (TS)_i$ for $i = 1, 2$.

By Theorem 4.1, $S_1T_1 \in [X_1, X_1]$ and $T_1S_1 \in [Y_1, Y_1]$ and therefore satisfy the hypotheses of Theorem 1.1. Thus for $\lambda \neq 0$, the state of $S_1T_1 - \lambda E(\sigma_1)$ agrees with the state of $T_1S_1 - \lambda E(\sigma_2)$.

When $0 \in D$ the sets $\sigma(S_2T_2) = \tau_1$ and $\sigma(T_2S_2) = \tau_2$ are bounded away from zero. Consequently by Proposition 3.6, the state of $S_2T_2 - \lambda E(\tau_1)$ agrees with the state of $T_2S_2 - \lambda E(\tau_2)$ whenever $\lambda \neq 0$.

It can be seen that the above is both necessary and sufficient for the state of $ST - \lambda I$ to be the same as the state of $TS - \lambda I$.

From the preceding theorems we obtain the final results:

THEOREM 4.3. *Suppose $0 \notin \rho(TS) \cap \sigma(ST)$ and a bounded Cauchy domain D exists satisfying:*

(i) $\partial D \subset \rho(ST) \cap \rho(TS)$;

(ii) $0 \in D$.

If $\sigma_1, \cdots, \sigma_n$ is a spectral decomposition of $\sigma_e(ST)$ then $\sigma_0, \sigma_1, \cdots, \sigma_n$ is a spectral decomposition of $\sigma_e(TS)$ where

$$\sigma_0 = \{0\}$$

whenever $0 \in \sigma(TS) \cap \rho(ST)$, and is empty otherwise.

Moreover, if $E_i(ST)$ and $E_i(TS)$ are the projections associated with these spectral sets with ranges X_i and Y_i respectively, then

$$T: X_i \longrightarrow Y_i \ ;$$
$$S: Y_i \longrightarrow X_i$$

where $i = 1, \cdots, n$, *and when* $0 \in \sigma(TS) \cap \rho(ST)$,

$$S: Y_0 \longrightarrow \{0\} \ .$$

Proof. First note that by Theorem 4.2,

$$(\sigma_e(ST) - \sigma_e(TS)) \cup (\sigma_e(TS) - \sigma_e(ST)) \subset \{0\}$$

and since $\sigma_e(ST)$ and $\sigma_e(TS)$ are both closed subsets of the complex plane, if $0 \in \sigma(TS) \cap \rho(ST)$ it must be an isolated point in $\sigma(TS)$. This demonstrates that the spectral decomposition $\sigma_1, \cdots, \sigma_n$ of $\sigma_e(ST)$ gives rise to the spectral decomposition $\sigma_0, \sigma_1, \cdots, \sigma_n$ of $\sigma_e(TS)$.

If $\infty \in \sigma_e(ST)$, i.e., if $ST \notin [X, X]$, assume that $\infty \in \sigma_n$. Then $\sigma_1, \cdots, \sigma_{n-1}$ are bounded spectral sets for both ST and TS.

Let D_i be an admissible domain for σ_i, $i = 1, \cdots, n-1$. Then

$$E_i(ST) = -\frac{1}{2\pi i}\int_{+\partial D_i}(ST - \lambda I)^{-1}d\lambda$$

and

$$E_i(TS) = -\frac{1}{2\pi i}\int_{+\partial D_i}(TS - \lambda I)^{-1}d\lambda \ .$$

By Theorem 4.1,

$$T: X_i \longrightarrow Y_i \ ,$$
$$S: Y_i \longrightarrow X_i \ ,$$

$i = 1, \cdots, n-1$, moreover T, S are continuous and everywhere defined on these subspaces.

Further, if $0 \in \sigma(TS) \cap \rho(ST)$ and D_0 is an admissible domain for σ_0, let $y \in Y_0$. By Theorem 4.1, $y \in D(S)$ and

$$\begin{aligned} Sy &= SE_0(TS)y \\ &= S\Big(-\frac{1}{2\pi i}\int_{+\alpha D_0}(TS - \lambda I)^{-1}d\lambda\Big)y \\ &= \Big(-\frac{1}{2\pi i}\int_{+\alpha D_0}(ST - \lambda I)^{-1}d\lambda\Big)Sy \\ &= 0 \ . \end{aligned}$$

To show that $T: X_n \rightarrow Y_n$, $S: Y_n \rightarrow X_n$ observe that

$$E_n(ST) = I - \sum_{i=1}^{n-1} E_i(ST)$$

and

$$E_n(TS) = I - \sum_{i=0}^{n-1} E_i(TS)$$

where $E_0(TS) = 0$ if $0 \in \rho(TS)$.

When $T \in [X, Y]$, $S \in [Y, X]$ we clearly have a bounded Cauchy domain

$$D = \{\mu \mid |\mu| < \max(\|ST\|, \|TS\|) + 1\}$$

which satisfies the conditions of Theorem 4.3. Hence:

COROLLARY 4.1. *If $T \in [X, Y]$, $S \in [Y, X]$ and $0 \notin \rho(TS) \cap \sigma(ST)$ then a spectral decomposition $\sigma_1, \cdots, \sigma_n$ of $\sigma(ST)$ gives a spectral decomposition $\sigma_0, \sigma_1, \cdots, \sigma_n$ of $\sigma(TS)$ where*

$$\sigma_0 = \begin{cases} \{0\} & \text{whenever} \quad 0 \in \sigma(TS) \cap \rho(ST) \\ \phi & \text{otherwise}. \end{cases}$$

Moreover, if $E_i(ST)$ and $E_i(TS)$ are the projections associated with the spectral sets with ranges X_i and Y_i respectively, then $T: X_i \to Y_i$, $S: Y_i \to X_i$, $i = 1, \cdots, n$ and when $0 \in \sigma(TS) \cap \rho(ST)$, $S: Y_0 \to \{0\}$.

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