PACIFIC JOURNAL OF MATHEMATICS
Vol 68, No. 2, 1977

THE DEFICIENCY INDEX OF A THIRD ORDER OPERATOR

RICHARD C. GILBERT

Let L be a formally selfadjoint third order linear ordinary differential operator defined on $[r,\infty)$. Using a method of Fedorjuk, asymptotic formulas are found for the solutions of $Ly = i\sigma y$, $\sigma \neq 0$. These formulas are used to determine the deficiency index of L when L has polynomial coefficients. As a consequence, the deficiency index is determined for values of the parameters involved for which it has not previously been determined.

1. Introduction. The general form of a third order formally selfadjoint linear ordinary differential operator L can be written

$$(1)\quad Ly = (ib_2y'')' + [(2^{-1}ib_2' + a_1)y']' + ib_1y' + (2^{-1}ib_1' + a_0)y,$$

where a_0, a_1, b_1, b_2 are real functions of x and $b_2(x) \neq 0$. (See [**4**, Ch. 1, §1.5]. We have assumed sufficient differentiability on the coefficients so that the Dunford and Schwartz form can be written in the form (1).) Unsworth [**12**] considered the case that $b_2(x) = 2$, $b_1(x) = 2ax^\alpha$, $a_1(x) = bx^\beta$, $a_0(x) = cx^\gamma$, $1 \leqq x < \infty$. Using the asymptotic methods of Devinatz [**3**], Unsworth deduced the deficiency index of L for various values of the parameters a, b, c, α, β, γ. Pfeiffer [**10**] considered the case $b_2(x) = 1$, $b_1(x) = ax^\alpha$, $a_1(x) = 0$, $a_0(x) = cx^\gamma$. The purpose of the present article is to obtain by the method of Fedorjuk [**6**] asymptotic formulas for the solutions of $Ly = i\sigma y$, $\sigma \neq 0$, and to apply these formulas to finding the deficiency index of L for the case $b_2(x) = 1$, $b_1(x) = ax^\alpha$, $a_1(x) = bx^\beta$, $a_0(x) = cx^\gamma$. Although Fedorjuk applied his method only to even order operators, it can be used for odd order operators as well. Shirikyan [**11**] applied the Fedorjuk method to a certain class of odd order operators. It turns out that the Fedorjuk method applied to the above case yields the deficiency index for values of the parameters different from Unsworth and Pfeiffer.

It is known that, except for a first order operator, a differential operator of order n cannot have deficiency index (n, p) or (p, n), where $p < n$. (See Atkinson [**1**] or Kogan and Rofe-Beketov [**7**], [**8**].) Further, for an operator of order $n = 2\nu - 1$ it is known that the deficiency numbers n_+ and n_- satisfy the inequalities $\nu \leqq n_+ \leqq 2\nu - 1$, $\nu - 1 \leqq n_- \leqq 2\nu - 1$, or the same inequalities with n_+ and n_- interchanged. (See

Everitt **[5]** or Kogan and Rofe-Beketov **[8]**.) It follows that the deficiency indices (2, 1), (1, 2), (2, 2) and (3, 3), obtained in this paper and by Unsworth and Pfeiffer, are the only possible deficiency indices for a third order operator.

2. Asymptotic formulas for the solutions of $Ly = i\sigma y$. We shall make the following assumptions on the coefficients a_0, a_1, b_1, b_2 of L. The need for the various assumptions will be seen as we go along.

In all that follows in this article, it will be necessary in various places to require that x is sufficiently large. We shall therefore assume once and for all that x_0 is chosen so large that if $x \geqq x_0$, then x is sufficiently large in all places where this is needed. We shall also often omit the stipulation $x \geqq x_0$ when it is clear from the context that this is needed.

ASSUMPTION I. $b_1(x), b_2(x) \in C^3[r, \infty)$. $a_0(x), a_1(x) \in C^2[r, \infty)$. $b_2(x) \neq 0$ for $x \geqq r$, $b_2(x) = 1 + o(1)$ as $x \to +\infty$. $a_0(x) \neq 0$ for $x \geqq r$. Either $a_0(x) \to +\infty$ and $a_0'(x) > 0$ for $x \geqq x_0$, or else $a_0(x) \to -\infty$ and $a_0'(x) < 0$ for $x \geqq x_0$.

ASSUMPTION II. $\lim_{x\to\infty} a_1/a_0^{1/3} = d \neq 3/2^{2/3}$, $b_1/a_0^{2/3} = o(1)$, $b_1'/a_0 = o(1)$, $b_2'/a_0^{1/3} = o(1)$.

ASSUMPTION III. $b_2''/a_0^{2/3} = o(1)$, $a_1'/a_0^{2/3} = o(1)$, $b_1''/a_0^{4/3} = o(1)$, $a_0'/a_0^{4/3} = o(1)$.

ASSUMPTION IV. b_2' and $b_1'/a_0^{2/3}$ are absolutely integrable on $[r, \infty)$.

Let

$$f(\lambda, x) = -\lambda^3 + im(x)b_2^{-1}(x)\lambda^2 - b_1(x)b_2^{-1}(x)\lambda + in(x)b_2^{-1}(x), \tag{2}$$

where

$$m(x) = 2^{-1}ib_2'(x) + a_1(x), \tag{3}$$

$$n(x) = 2^{-1}ib_1'(x) + a_0(x) - i\sigma. \tag{4}$$

Here σ is a real constant, $\sigma \neq 0$.

Let

$$\tau(x) = [a_0(x)b_2^{-1}(x)]^{1/3}[1 + (b_1'(x) - 2\sigma)(2a_0(x))^{-1}i]^{1/3}, \tag{5}$$

where if $z = \rho e^{i\theta}$, $-\pi < \theta \leqq \pi$, then we take $z^{1/3} = \rho^{1/3}e^{i\theta/3}$. Then,

$\tau^3 = nb_2^{-1}$, and $\tau(x) \neq 0$ for $x \geqq r$.

Putting

$$\lambda = i\eta\tau(x), \tag{6}$$

then

$$f(\lambda, x) = 0 \tag{7}$$

becomes

$$h(\eta, x) = 0, \tag{8}$$

where

$$h(\eta, x) = \eta^3 - m(x)[b_2(x)\tau(x)]^{-1}\eta^2 - b_1(x)[b_2(x)\tau^2(x)]^{-1}\eta + 1. \tag{9}$$

An essential part of the Fedorjuk method is that we should have

$$\lim_{x \to \infty} m(x)[b_2(x)\tau(x)]^{-1} = d + ie_1, \tag{10}$$

$$\lim_{x \to \infty} b_1(x)[b_2(x)\tau^2(x)]^{-1} = d_2 + ie_2, \tag{11}$$

where $d + ie_1$ and $d_2 + ie_2$ are complex constants. Then, as $x \to \infty$, $h(\eta, x)$ approaches a polynomial $h_0(\eta)$ with constant coefficients. We also want $h_0(\eta) = 0$ to have distinct roots. For reasons that will appear later we further want as $x \to \infty$ that $|a_0(x)| \to \infty$ and that

$$\tau(x) = a_0^{1/3}(x)[1 + o(1)]. \tag{12}$$

In I and II we have assumed $a_0(x) \to \pm\infty$, $b_2 = 1 + o(1)$, $b_1'/a_0 = o(1)$ in order that (12) and $|a_0(x)| \to \infty$ might be true. In order to explain the remaining assumptions in I and II, let us note that if (10) and (11) are to be true, we must have

$$\lim_{x \to \infty} (b_2'/a_0^{1/3}) = 2e_1, \tag{13}$$

$$\lim_{x \to \infty} (a_1/a_0^{1/3}) = d, \tag{14}$$

$$\lim_{x \to \infty} (b_1/a_0^{2/3}) = d_2, \tag{15}$$

and $e_2 = 0$. But then (13) and our assumptions that $|a_0| \to \infty$ and $b_2 = 1 + o(1)$ imply that $e_1 = 0$. Further, (15) and the assumptions on a_0 in I and the assumption that $b_1'/a_0^{2/3}$ is absolutely integrable on $[r, \infty)$ in IV imply that $d_2 = 0$. Thus, we have explained the reasons for all the limit assumptions in I and II.

From Assumptions I and II we have that

$$m(x)[b_2(x)\tau(x)]^{-1} = d + f_1(x), \tag{16}$$

$$b_1(x)[b_2(x)\tau^2(x)]^{-1} = f_2(x), \tag{17}$$

where $f_1(x) = o(1)$, $f_2(x) = o(1)$, and $f_1(x)$ and $f_2(x)$ are continuously differentiable on $[r, \infty)$. It follows that

$$h(\eta, x) = h_0(\eta) - \eta^2 f_1(x) - \eta f_2(x), \tag{18}$$

where

$$h_0(\eta) = \eta^3 - d\eta^2 + 1. \tag{19}$$

Since we have assumed in II that $d \neq 3/2^{2/3}$, $h_0(\eta) = 0$ has three distinct nonzero roots. If $d < 3/2^{2/3}$, then $h_0(\eta) = 0$ has one real negative root and two complex conjugate nonreal roots. If $d > 3/2^{2/3}$, then $h_0(\eta) = 0$ has three distinct real roots, one of which is negative and the other two positive. We denote the roots by η_{01}, η_{02}, η_{03}, where $\eta_{01} < \eta_{02} < \eta_{03}$ in the case of three real roots, and η_{01} is real and $\operatorname{Im} \eta_{02} > 0$, $\operatorname{Im} \eta_{03} < 0$ in the case of one real root. In the case of three real roots, $h'(\eta_{01}) > 0$, $h'(\eta_{02}) < 0$, $h'(\eta_{03}) > 0$. In the case of one real root, $h'(\eta_{01}) > 0$. In every case, $h'(\eta_{0k}) \neq 0$, $k = 1, 2, 3$.

According to Bellman [2, p. 26], for $x \geqq x_0$, (8) has three distinct roots $\eta_k(x)$, $k = 1, 2, 3$ which are given by the formula

$$\eta_k(x) = (2\pi i)^{-1} \int_{C_k} \eta h_\eta(\eta, x)[h(\eta, x)]^{-1} d\eta, \tag{20}$$

where C_k is a small circle around η_{0k}. $\eta_k(x)$ is continuously differentiable, and

$$\eta_k(x) = \eta_{0k}[1 + o(1)]. \tag{21}$$

We have that $h_\eta(\eta_k(x), x) \neq 0$, and that $\eta_k(x) \neq 0$, for $x \geqq x_0$. From (6) one sees that (7) has for $x \geqq x_0$, three distinct continuously differentiable nonzero roots $\lambda_k(x)$ given by

$$\lambda_k(x) = i\eta_k(x)\tau(x), \qquad k = 1, 2, 3, \tag{22}$$

and

$$\lambda_k(x) = ia_0^{1/3}(x)\eta_{0k}[1 + o(1)]. \tag{23}$$

We have that $f_\lambda(\lambda_k(x), x) \neq 0$.

ASSUMPTION V. $(b_2')^2/a_0^{1/3}$, $(b_2'')^2/a_0$, $(a_1')^2/a_0$, $(b_1')^2/a_0^{5/3}$, $(b_1'')^2/a_0^{7/3}$, $(a_0')^2/a_0^{7/3}$, $b_2''/a_0^{1/3}$, $b_2'''/a_0^{2/3}$, $a_1''/a_0^{2/3}$, b_1''/a_0, $b_1'''/a_0^{4/3}$, $a_0''/a_0^{4/3}$ are all absolutely integrable on $[r, \infty)$.

ASSUMPTION VI. For each pair j, k, one of the following is true:
(a) $\operatorname{Re}(\lambda_j(x) - \lambda_k(x)) \geqq 0$ for $x \geqq x_0$;
(b) $\operatorname{Re}(\lambda_j(x) - \lambda_k(x)) \leqq 0$ for $x \geqq x_0$, and

$$\int_{x_0}^{\infty} \operatorname{Re}(\lambda_j(x) - \lambda_k(x))\, dx = -\infty;$$

(c) $\displaystyle\int_{x_0}^{\infty} \operatorname{Re}(\lambda_j(x) - \lambda_k(x))\, dx$ is convergent.

Using Assumptions I–VI, it is now possible to obtain asymptotic formulas for the solutions of the equation

$$Ly = i\sigma y. \tag{24}$$

Let w be the column vector with components $w_1 = y$, $w_2 = y'$, $w_3 = ib_2 y'' + my'$. (24) is then equivalent to the system

$$w' = A(x)w, \tag{25}$$

where

$$A(x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & imb_2^{-1} & -ib_2^{-1} \\ -n & -ib_1 & 0 \end{pmatrix}. \tag{26}$$

The eigenvalues of $A(x)$ are the roots of (7), i.e., $\lambda_k(x)$, $k = 1, 2, 3$.

Let us now make the transformation

$$w = T_0(E + T_2)z, \tag{27}$$

where z is a column vector with components z_1, z_2, z_3, and T_0 and T_2 are matrices to be determined, and E is the identity matrix. Then, (25) becomes

$$z' = \Lambda_0 z + (\Lambda_0 T_2 - T_2\Lambda_0 - T_0^{-1}T_0')z + B(x)z, \tag{28}$$

where

$$\begin{aligned} B(x) = {} & (E + T_2)^{-1}[(T_2^2\Lambda_0 + T_2T_0^{-1}T_0')(E + T_2) - T_2'] \\ & - T_2\Lambda_0T_2 - T_0^{-1}T_0'T_2, \end{aligned} \tag{29}$$

and

$$\Lambda_0 = T_0^{-1}AT_0.$$

We shall show that we can choose T_0 and T_2 such that for $x \geqq x_0$, T_0^{-1} and $(E + T_2)^{-1}$ exist, and $T_0^{-1}AT_0$ and $\Lambda_0T_2 - T_2\Lambda_0 - T_0^{-1}T_0'$ are diagonal. To that end, we choose T_0 to be a matrix whose columns are eigenvectors for A, namely,

$$T_0 = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ [ib_2\lambda_1 + m]\lambda_1 & [ib_2\lambda_2 + m]\lambda_2 & [ib_2\lambda_3 + m]\lambda_3 \end{pmatrix}, \tag{30}$$

$$T_0^{-1} = \begin{pmatrix} n/\lambda_1F_\lambda(\lambda_1, x) & -i\lambda_1b_2/F_\lambda(\lambda_1, x) & -1/F_\lambda(\lambda_1, x) \\ n/\lambda_2F_\lambda(\lambda_2, x) & -i\lambda_2b_2/F_\lambda(\lambda_2, x) & -1/F_\lambda(\lambda_2, x) \\ n/\lambda_3F_\lambda(\lambda_3, x) & -i\lambda_3b_2/F_\lambda(\lambda_3, x) & -1/F_\lambda(\lambda_3, x) \end{pmatrix}, \tag{31}$$

where

$$F(\lambda, x) = ib_2f(\lambda, x). \tag{32}$$

Then, for $x \geqq x_0$,

$$T_0^{-1}AT_0 = \Lambda_0 = \text{diagonal}[\lambda_j]. \tag{33}$$

We note that

$$\lim_{x\to\infty} a_0^{-2/3}(x)F_\lambda(\lambda_j(x), x) = ih_0'(\eta_{0j}) = \rho_{0j}\exp[i\theta_{0j}], \tag{34}$$

where $-\pi < \theta_{0j} \leqq \pi$, $\rho_{0j} > 0$. Let

$$a_0^{-2/3}(x)F_\lambda(\lambda_j(x), x) = \rho_j(x)\exp[i\theta_j(x)], \tag{35}$$

where $\rho_j(x)$ and $\theta_j(x)$ are chosen so that $\lim_{x\to\infty}\rho_j(x) = \rho_{0j}$, and $\lim_{x\to\infty}\theta_j(x) = \theta_{0j}$. We choose that branch of log such that for $x \geqq x_0$,

$$\log F_\lambda(\lambda_j(x), x) = (2/3)\operatorname{Log}|a_0(x)| + \operatorname{Log}\rho_j(x) + i\theta_j(x). \tag{36}$$

Then, for $t, x \geqq x_0$,

$$\begin{aligned}(d/dx)\log F_\lambda(\lambda_j(x),x) \\ = [F_{\lambda\lambda}(\lambda_j(x),x)\lambda_j'(x)+F_{\lambda x}(\lambda_j(x),x)][F_\lambda(\lambda_j(x),x)]^{-1},\end{aligned} \tag{37}$$

$$\int_t^x [(d/ds)\log F_\lambda(\lambda_j(s),s)]\,ds = \log F_\lambda(\lambda_j(x),x) - \log F_\lambda(\lambda_j(t),t), \tag{38}$$

$$\exp[(1/2)\log F_\lambda(\lambda_j(x),x)] = [1+o(1)]\,|\,a_0^{1/3}(x)|\,\rho_{0j}^{1/2}\exp(i\theta_{0j}/2). \tag{39}$$

Now we note that the elements $(T_0^{-1}T_0')_{jk}$ of the matrix $T_0^{-1}T_0'$ are given for $x \geqq x_0$, by

$$\begin{aligned}(T_0^{-1}T_0')_{jj} = (1/2)[F_{\lambda\lambda}(\lambda_j(x),x)\lambda_j'(x)+F_{\lambda x}(\lambda_j(x),x) \\ + ib_2'(x)\lambda_j^2(x)+ib_1'(x)][F_\lambda(\lambda_j(x),x)]^{-1},\end{aligned} \tag{40}$$

or,

$$\begin{aligned}(T_0^{-1}T_0')_{jj} = (1/2)(d/dx)\log F_\lambda(\lambda_j(x),x) \\ + 2^{-1}i[b_2'(x)\lambda_j^2(x)+b_1'(x)][F_\lambda(\lambda_j(x),x)]^{-1};\end{aligned} \tag{41}$$

and

$$\begin{aligned}(T_0^{-1}T_0')_{jk} = [\lambda_k(ib_2'\lambda_k\lambda_j+ib_1')+m'\lambda_k\lambda_j+n'] \\ \times[(\lambda_k-\lambda_j)F_\lambda(\lambda_j(x),x)]^{-1}, \qquad k\neq j.\end{aligned} \tag{42}$$

Let

$$\lambda_j^{(1)} = -(T_0^{-1}T_0')_{jj}, \tag{43}$$

$$\Lambda_1 = \text{diagonal}\,[\lambda_j^{(1)}]. \tag{44}$$

We note that the $\lambda_j^{(1)}(x)$ are continuous for $x \geqq x_0$. Let the matrix T_2 be defined by the equations

$$(T_2)_{jj} = 0, \tag{45}$$

$$(T_2)_{jk} = -(T_0^{-1}T_0')_{jk}(\lambda_k-\lambda_j)^{-1}, \qquad k\neq j. \tag{46}$$

T_2 has been defined so that $\Lambda_0T_2 - T_2\Lambda_0 - T_0^{-1}T_0'$ is a diagonal matrix; indeed,

$$\Lambda_0T_2 - T_2\Lambda_0 - T_0^{-1}T_0' = \Lambda_1. \tag{47}$$

Thus, T_0 and T_2 in the transformation (27) have been chosen so that for $x \geqq x_0$, equation (28) is

$$z' = (\Lambda_0 + \Lambda_1)z + Bz. \tag{48}$$

We shall now show that for $x \geqq x_0$, $B(x)$ exists and is continuous, and $\|B(x)\|$ is integrable on $[x_0, \infty)$. To do this will require a series of lemmas whose proofs are mostly straightforward or else contained in Fedorjuk **[6]** and are therefore omitted. For $x \geqq x_0$, let

$$\lambda(x) = \max_j |\lambda_j(x)|. \tag{49}$$

Then,

$$\lambda(x) = |\tau(x)| \max_j |\eta_j(x)| = |a_0^{1/3}(x)|[1 + o(1)] \max_j |\eta_j| > 0.$$

In the following, the capital letters C and D denote suitably chosen positive constants.

LEMMA 1. $D_1|a_0^{1/3}(x)| \leqq \lambda(x) \leqq D_2|a_0^{1/3}(x)|$.

LEMMA 2. $C_1\lambda(x) \leqq |\lambda_j(x) - \lambda_k(x)| \leqq C_2\lambda(x)$, $j \neq k$.

Let

$$\alpha(x) = \max\{|b_2'|, |m'|/\lambda, |b_1'|/\lambda^2, |n'|/\lambda^3\}, \tag{50}$$

$$\beta(x) = \max\{|b_2''|, |m''|/\lambda(x), |b_1''|/\lambda^2(x), |n''|/\lambda^3(x)\}, \tag{51}$$

$$\delta(x) = \max\{|b_2'|, |m'|/|a_0^{1/3}|, |b_1'|/|a_0^{2/3}|, |n'|/|a_0|\}, \tag{52}$$

$$\gamma(x) = \max\{|b_2''|, |m''|/|a_0^{1/3}|, |b_1''|/|a_0^{2/3}|, |n''|/|a_0|\}. \tag{53}$$

LEMMA 3. $\alpha(x) \leqq C\delta(x)$.

LEMMA 4. $\beta(x) \leqq C\gamma(x)$.

LEMMA 5. $C_1\lambda^2(x) \leqq |F_\lambda(\lambda_j(x), x)| \leqq C_2\lambda^2(x)$.

LEMMA 6. $|F_x(\lambda_j(x), x)| \leqq C\lambda^3(x)\alpha(x)$.

LEMMA 7. $|[\lambda_k(x) - \lambda_j(x)]F_\lambda(\lambda_j(x), x)| \geqq C\lambda^3(x)$.

LEMMA 8. $|[\lambda_k(x) - \lambda_j(x)]^2 F_\lambda(\lambda_j(x), x)| \geqq C\lambda^4(x)$.

LEMMA 9. $|F_{\lambda\lambda}(\lambda_j(x), x)| \leqq C\lambda(x)$.

LEMMA 10. $|F_{\lambda x}(\lambda_j(x), x)| \leqq C\lambda^2(x)\alpha(x)$.

LEMMA 11. $|\lambda_j'(x)| \leqq C\alpha(x)\lambda(x)$.

If $A = (A_{jk})_{j,k=1}^n$ is an $n \times n$ matrix, we define the norm $\|A\|$ by $\|A\| = n \max_{j,k} |A_{jk}|$.

LEMMA 12. $\|\Lambda_1(x)\| \leqq C\alpha(x)$.

LEMMA 13. $\|T_0^{-1}(x)T_0'(x)\| \leqq C\alpha(x)$.

LEMMA 14. $\|T_2(x)\| \leqq C\alpha(x)/\lambda(x)$.

LEMMA 15. $\|T_2'(x)\| \leqq C\{\alpha^2(x) + \beta(x)]/\lambda(x)$.

LEMMA 16. *$[E + T_2(x)]^{-1}$ exists and is continuous for $x \geqq x_0$, and $\|[E + T_2(x)]^{-1}\| \leqq C$.*

LEMMA 17. *$B(x)$ exists and is continuous for $x \geqq x_0$, and $\|B(x)\| \leqq C[\alpha^2(x) + \beta(x)]/\lambda(x)$.*

We note that Lemmas 16 and 17 depend on the fact that $\lim_{x\to\infty} \alpha(x)/\lambda(x) = 0$, which follows from Assumptions II and III.

LEMMA 18. *$\|B(x)\|$ is integrable on $[x_0, \infty)$.*

We note that Lemma 18 follows from Lemma 17, and Assumption V.

It is now possible to show that (48) has three linearly independent solutions which satisfy certain specified boundary conditions at infinity. To that end, we observe that a fundamental matrix $Z_0(x_0, x)$ for the homogeneous equation

$$z' = (\Lambda_0 + \Lambda_1)z, \qquad x \geqq x_0, \tag{54}$$

is given by

$$Z_0(x_0, x) = \text{diagonal}\left[\exp \int_{x_0}^{x} (\lambda_j(t) + \lambda_j^{(1)}(t))\, dt\right]. \tag{55}$$

Putting

$$Z(x) = U(x)Z_0(x_0, x), \tag{56}$$

we find that $Z(x)$ is a matrix solution of (48) for $x \geqq x_0$ if $U(x)$ satisfies

$$U(x) = C + (KU)(x), \qquad x \geqq x_0, \tag{57}$$

where C is an arbitrary constant matrix, and K is a linear operator on matrices $U(x)$ such that

$$(KU(x))_{jk} = \int_{x_{jk}}^{x} (Z_0(t,x)B(t)U(t)Z_0(x,t))_{jk}\, dt, \tag{58}$$

x_{jk} being an arbitrary number in the interval $[x_0, \infty]$.

Let M be the Banach space of continuous matrices $V(x)$ on $[x_0, \infty)$, with $\|V\|_M = \sup_{x \geqq x_0} \|V(x)\| < \infty$. For reasons that will appear in Lemmas 19 and 20 below, if Assumption VI (a) or (c) holds, we take $x_{jk} = \infty$; if Assumption VI (b) holds, we take $x_{jk} = x_0$. Also, we take $C = E$.

LEMMA 19. *If x_0 is sufficiently large, then $K\colon M \to M$, and $\|K\|_M \leqq 1/2$.*

Proof. From (58) it follows that if $V \in M$ and if $x \geqq x_0$, then

$$\begin{aligned} |((KV)(x))_{jk}| \leqq \Bigg| \int_{x_{jk}}^{x} &\left[\exp \int_t^x \mathrm{Re}(\lambda_j(s) - \lambda_k(s))\, ds\right] \\ &\times \left[\exp \int_t^x \mathrm{Re}(\lambda_j^{(1)}(s) - \lambda_k^{(1)}(s))\, ds\right] \|B(t)\|\, dt \Bigg|\, \|V\|_M. \end{aligned} \tag{59}$$

By (41), (43) and (38),

$$\begin{aligned} &\int_t^x [\lambda_j^{(1)}(s) - \lambda_k^{(1)}(s)]\, ds \\ &\quad = (1/2)[\log F_\lambda(\lambda_j(x), x) - \log F_\lambda(\lambda_j(t), t)] \\ &\quad - (1/2)[\log F_\lambda(\lambda_k(x), x) - \log F_\lambda(\lambda_k(t), t)] \\ &\quad + (i/2) \int_t^x b_2'(s)\{\lambda_j^2(s)[F_\lambda(\lambda_j(s), s)]^{-1} - \lambda_k^2(s)[F_\lambda(\lambda_k(s), s)]^{-1}\}\, ds \\ &\quad + (i/2) \int_t^x b_1'(s)\{[F_\lambda(\lambda_j(s), s)]^{-1} - [F_\lambda(\lambda_k(s), s)]^{-1}\}\, ds. \end{aligned} \tag{60}$$

It now follows from (36), (49), Lemma 1, Lemma 5, and Assumption IV that $\left| \int_t^x (\lambda_j^{(1)}(s) - \lambda_k^{(1)}(s))\, ds \right|$ is bounded for t, $x \geqq x_0$. Hence, if $V \in M$, then

$$|((KV)(x))_{jk}|$$
$$\leqq C\left|\int_{x_{jk}}^{x}\left[\exp\int_{t}^{x}\operatorname{Re}(\lambda_j(s)-\lambda_k(s))\,ds\right]\|B(t)\|\,dt\right|\|V\|_M,\quad x\geqq x_0. \tag{61}$$

By our choice of x_{jk}, if Assumption VI (a) or (b) holds, $\exp\int_t^x \operatorname{Re}(\lambda_j(s)-\lambda_k(s))\,ds\leqq 1$. If Assumption VI (c) holds, then $\left|\int_t^x \operatorname{Re}(\lambda_j(s)-\lambda_k(s))\,ds\right|\leqq C_1$ for t, $x\geqq x_0$, and therefore $\exp\int_t^x \operatorname{Re}(\lambda_j(s)-\lambda_k(s))\,ds\leqq \exp C_1$. It follows from (61) that

$$|((KV)(x))_{jk}|\leqq C\int_{x_0}^{\infty}\|B(t)\|\,dt\,\|V\|_M,\qquad x\geqq x_0. \tag{62}$$

Hence,

$$\|(KV)(x)\|=3\max_{j,k}|((KV)(x))_{jk}|\leqq 3C\int_{x_0}^{\infty}\|B(t)\|\,dt\,\|V\|_M. \tag{63}$$

If we now choose x_0 so large that $\int_{x_0}^{\infty}\|B(t)\|\,dt\leqq 1/6C$, then $\|K\|_M\leqq 1/2$. This proves Lemma 19.

LEMMA 20. *If x_0 is sufficiently large, equation* (57) *has a unique solution $U(x)\in M$. It is true that $\|(KU)(x)\|=o(1)$ as $x\to\infty$. $U(x)$ can be written in the form*

$$U(x)=E+o(1),\qquad x\geqq x_0. \tag{64}$$

Proof. The existence and uniqueness of $U(x)$ follows from Lemma 18 and Banach's contraction mapping theorem or successive approximations. To prove that $\|(KU)(x)\|=o(1)$, we observe that if Assumption VI (a) or (c) holds (so that we take $x_{jk}=\infty$), then from (61), $|((KU)(x))_{jk}|\leqq C\int_x^{\infty}\|B(t)\|\,dt\,\|U\|_M=o(1)$. If Assumption VI (b) holds (so that we take $x_{jk}=x_0$), then from (61),

$$|((KU)(x))_{jk}|\leqq C\left\{\int_{x_0}^{x_1}\left[\exp\int_{x_1}^{x}\operatorname{Re}(\lambda_j(s)-\lambda_k(s))\,ds\right]\|B(t)\|\,dt\right.$$
$$\left.+\int_{x_1}^{x}\|B(t)\|\,dt\right\}\|V\|_M,$$

where $x\geqq x_1\geqq x_0$. From this inequality it is seen that $|((KU)(x))_{jk}|=$

$o(1)$ also when Assumption VI (b) holds. (64) follows from (57) and the fact that $\|(KU)(x)\| = o(1)$. This completes the proof of Lemma 20.

THEOREM 1. *Under Assumptions* I–VI, *the equation* $Ly = i\sigma y$, $x \geqq r$, $\sigma \neq 0$, *has three linearly independent solutions* y_k, $k = 1,2,3$, *of the form*

$$y_k = [1 + o(1)]a_0^{-1/3}(x)\exp\int_{x_0}^{x} \lambda_k(t)\,dt, \qquad x \geqq x_0, \tag{65}$$

where the $\lambda_k(t)$ *are given by equation* (22).

Proof. By (56) and (64), there is a solution matrix $Z(x)$ for (48) of the form

$$Z(x) = [E + o(1)]Z_0(x_0, x), \qquad x \geqq x_0. \tag{66}$$

If x_0 is sufficiently large, $\det[E + o(1)] \neq 0$ for $x \geqq x_0$ and therefore $Z(x)$ is a fundamental matrix for (48). By (66) and (27) a solution matrix for (25) is given by

$$W(x) = T_0(x)[E + T_2(x)][E + o(1)]Z_0(x_0, x), \qquad x \geqq x_0. \tag{67}$$

Since $[E + T_2(x)]^{-1}$ exists by Lemma 16 and $T_0^{-1}(x)$ exists by (31), $W(x)$ is a fundamental matrix. By Lemma 14 and the fact that $\lim_{x\to\infty} \alpha(x)/\lambda(x) = 0$, we see that

$$W(x) = T_0(x)[E + o(1)]Z_0(x_0, x), \qquad x \geqq x_0. \tag{68}$$

Let $y_k(x) = w_{1k}(x)$, $k = 1,2,3$, where $w_{1k}(x)$ is the element in the first row and kth column of $W(x)$. Then, by the equivalence of (24) and (25), y_k is a solution of (24), and by (68) and (30),

$$y_k = [1 + o(1)]\exp\int_{x_0}^{x} [\lambda_k(t) + \lambda_k^{(1)}(t)]\,dt, \qquad x \geqq a. \tag{69}$$

From the equations $y_k = w_{k1}$, $y' = w_{k2}$, $y_k'' = -(ib_2)^{-1}mw_{k2} + (ib_2)^{-1}w_3$, we see that $W(y_1, y_2, y_3)(x) = \det W(x) \neq 0$, $x \geqq x_0$, where $W(y_1, y_2, y_3)$ is the Wronskian of y_1, y_2, y_3. Hence, y_1, y_2, y_3 are linearly independent for $x \geqq x_0$. By (43), (41), (38), (39), (49), Lemma 5, Lemma 1 and Assumption IV we see that

$$\exp\int_{x_0}^{x} \lambda_k^{(1)}(t)\,dt = C_k\,|a_0^{-1/3}(x)|\,[1 + o(1)], \quad x \geqq x_0, \quad C_k \neq 0. \tag{70}$$

(65) now follows from (69), (70) and the fact that $|a_0| \to \infty$, so that $a_0(x) > 0$ or $a_0(x) < 0$ for $x \geqq x_0$. This finishes the proof of Theorem 1.

3. Asymptotic formulas for the $\lambda_k(x)$. In this section we take the coefficients of the operator L of equation (1) to be the following on the interval $[1, \infty)$:

$$b_2(x) \equiv 1, \tag{71}$$

$$b_1(x) = ax^{\alpha}, \qquad \alpha < 2\gamma/3, \tag{72}$$

$$a_1(x) = bx^{\gamma/3}, \tag{73}$$

$$a_0(x) = cx^{\gamma}, \qquad \gamma > 0, \quad c \neq 0. \tag{74}$$

LEMMA 21. *If $b/c^{1/3} \neq 3/2^{2/3}$, then the coefficients of L given by* (71)–(74) *satisfy Assumptions* I–V *with*

$$d = b/c^{1/3}. \tag{75}$$

The proof is straightforward. We note that it is required in (74) that $\gamma > 0$ and $c \neq 0$ in order that $a_0(x) \to +\infty$ or $a_0(x) \to -\infty$ (Assumption I). The exponent $\gamma/3$ occurs in (73) in order that $\lim_{x\to\infty} a_1/a_0^{1/3} = d$ (Assumption II) with the possibility that $d \neq 0$. The inequality $\alpha < 2\gamma/3$ is required in (72) in order that $b_1/a_0^{2/3} = o(1)$ (Assumption II).

LEMMA 22. *If $b/c^{1/3} < 3/2^{2/3}$, the coefficients of L given by* (71)–(74) *satisfy Assumptions* I–VI.

Proof. Since $d = b/c^{1/3} < 3/2^{2/3}$, $h_0(\eta) = 0$ has one real negative root and two complex conjugate nonreal roots. Suppose $\eta_{02} = p + iq$, $\eta_{03} = p - iq$, $q > 0$. Then from (23) one sees that Assumption VI is satisfied; in fact, (a) or (b) is true for each pair j, k. This proves the lemma.

If $d > 3/2^{2/3}$, then $h_0(\eta) = 0$ has three real roots. In this case in order to check Assumption VI it is necessary to have asymptotic formulas for the $\lambda_k(x)$ which are more precise than (23). We obtain these by use of (20).

LEMMA 23. *Suppose the coefficients of L are given by* (71)–(74) *and that $b/c^{1/3} \neq 3/2^{2/3}$. Then the roots $\lambda_k(x)$ of* (7) *are given by*

$$
\begin{aligned}
\lambda_k(x) = i a_0^{1/3}\Big\{ & \eta_{0k} + [\eta_{0k} - v_{11} d](6c)^{-1}(iD) \\
& + a c^{-2/3} v_{10} x^{-\nu} + [-(\eta_{0k} - v_{11} d) + v_{22} d^2](6c)^{-2}(iD)^2 \\
& - a c^{-2/3}[v_{10} + v_{21} d](6c)^{-1}(iD) x^{-\nu} \\
& + (a c^{-2/3})^2 v_{20} x^{-2\nu} \\
& + [(5/3)(\eta_{0k} - v_{11} d) - 3 v_{22} d^2 - v_{33} d^3](6c)^{-3}(iD)^3 \\
& + O(D^2 x^{-\nu}) + O(D x^{-2\nu}) + (a c^{-2/3})^3 v_{30} x^{-3\nu} \\
& + \sum_{j=4}^{n+2} \sum_{s=1}^{j} O(D^s x^{-(j-s)\nu}) + O(D x^{-(n+2)\nu}) \\
& + \sum_{j=4}^{n} v_{j0} (a c^{-2/3})^j x^{-j\nu} \\
& + w_{n+1,0}(x)(a c^{-2/3})^{n+1} x^{-(n+1)\nu} \\
& + w_{n+2,0}(x)(a c^{-2/3})^{n+2} x^{-(n+2)\nu} \Big\},
\end{aligned} \tag{76}
$$

where n is an integer, $n \geqq 4$, the v_{js} are constants which depend on η_{0k} and are real when η_{0k} is real, $w_{n+1,0}(x)$ and $w_{n+2,0}(x)$ are complex functions which are bounded as $x \to \infty$,

$$
\nu = 2\gamma/3 - \alpha > 0, \tag{77}
$$

$$
\begin{aligned}
D &= a\alpha x^{-(\nu+1+\gamma/3)} - 2\sigma x^{-\gamma} \\
&= o(1) \quad \textit{as} \quad x \to \infty.
\end{aligned} \tag{78}
$$

If η_{0k} is real,

$$
\begin{aligned}
\operatorname{Re}\lambda_k(x) = a_0^{1/3}\Big\{ & [v_{11} d - \eta_{0k}](6c)^{-1} D \\
& + a c^{-2/3}[v_{10} + d v_{21}](6c)^{-1} D x^{-\nu} \\
& - [(5/3)(v_{11} d - \eta_{0k}) + d^2(3 v_{22} + d v_{33})](6c)^{-3} D^3 \\
& + O(D^2 x^{-\nu}) + O(D x^{-2\nu}) \\
& + \sum_{j=4}^{n+2} \sum_{s=1}^{j} O(D^s x^{-(j-s)\nu}) + O(D x^{-(n+2)\nu}) \\
& + O(x^{-(n+1)\nu}) + O(x^{-(n+2)\nu}) \Big\}.
\end{aligned} \tag{79}
$$

It is true that

$$
v_{11} = \eta_{0k}^2 [h_0'(\eta_{0k})]^{-1}, \tag{80}
$$

$$
v_{11} d - \eta_{0k} = 3[h_0'(\eta_{0k})]^{-1}, \tag{81}
$$

$$
v_{10} = \eta_{0k}[h_0'(\eta_{0k})]^{-1}, \tag{82}
$$

(83) $v_{21} = \eta_{0k}^2[3h_0'(\eta_{0k}) - \eta_{0k}h_0''(\eta_{0k})][h_0'(\eta_{0k})]^{-3}$,

(84) $v_{22} = 2^{-1}\eta_{0k}^3[4h_0'(\eta_{0k}) - \eta_{0k}h_0''(\eta_{0k})][h_0'(\eta_{0k})]^{-3}$,

(85) $v_{33} = 2^{-1}\eta_{0k}^4\{[3h_0'(\eta_{0k}) - \eta_{0k}h_0''(\eta_{0k})]^2 + [h_0'(\eta_{0k})]^2\}[h_0'(\eta_{0k})]^{-2}$.

Proof. From (5) and (71)–(74) we see that

$$\tau(x) = a_0^{1/3}(x)t(x), \tag{86}$$

$$t(x) = [1 + (2c)^{-1}(iD)]^{1/3}. \tag{87}$$

As $x \to \infty$,

$$t(x) = 1 + (6c)^{-1}(iD) - (6c)^{-2}(iD)^2 + (5/3)(6c)^{-3}(iD)^3 + O(D^4). \tag{88}$$

The functions $f_1(x)$ and $f_2(x)$ of (16)–(18) are given for $x \to \infty$ by

$$f_1(x) = d[-(6c)^{-1}(iD) + 2(6c)^{-2}(iD)^2 - (14/3)(6c)^{-3}(iD)^3 + O(D^4)], \tag{89}$$

$$\begin{aligned} f_2(x) = ac^{-2/3}x^{-\nu}[1 &- 2(6c)^{-1}(iD) \\ &+ 5(6c)^{-2}(iD)^2 - (40/3)(6c)^{-3}(iD)^3 + O(D^4)]. \end{aligned} \tag{90}$$

Now, $h^{-1} = h_0^{-1}[1 - (\eta/h_0)(\eta f_1 + f_2)]^{-1}$. Let n be a positive integer. For $\eta \in C_k$ and for $x \geqq x_0$,

$$\begin{aligned} h^{-1} &= h_0^{-1}\left\{1 + \sum_{j=1}^{n} (\eta/h_0)^j(\eta f_1 + f_2)^j + (\eta/h_0)^{n+1}(\eta f_1 + f_2)^{n+1}\right. \\ &\qquad\qquad \left.\times [1 - (\eta/h_0)(\eta f_1 + f_2)]^{-1}\right\} \\ &= h_0^{-1} + \sum_{j=1}^{n}\left(\sum_{s=0}^{j} a_{js}(\eta)f_1^s f_2^{j-s}\right) \\ &\quad + \sum_{s=0}^{n+1} a_{n+1,s}(\eta)f_1^s f_2^{n+1-s}[1 - (\eta/h_0)(\eta f_1 + f_2)]^{-1}. \end{aligned} \tag{91}$$

Hence,

$$\begin{aligned} \eta h_\eta h^{-1} &= \eta h_0' h_0^{-1} + \sum_{j=1}^{n+1}\sum_{s=0}^{j} b_{js}(\eta)f_1^s f_2^{j-s} \\ &\quad + \left[\sum_{s=0}^{n+1} c_{n+1,s}(\eta, x)f_1^s f_2^{n+1-s} + \sum_{s=0}^{n+2} c_{n+2,s}(\eta, x)f_1^s f_2^{n+2-s}\right] \\ &\quad \times [1 - (\eta/h_0)(\eta f_1 + f_2)]^{-1}. \end{aligned} \tag{92}$$

Substituting (92) into (20),

$$
(93)\qquad
\begin{aligned}
\eta_k(x) = \eta_{0k} &+ \sum_{j=1}^{n+1} \sum_{s=0}^{j} v_{js} f_1^s f_2^{j-s} \\
&+ \sum_{s=0}^{n+1} w_{n+1,s}(x) f_1^s f_2^{n+1-s} \\
&+ \sum_{s=0}^{n+2} w_{n+2,s}(x) f_1^s f_2^{n+2-s},
\end{aligned}
$$

where the v_{js} are constants which are real if η_{0k} is real, and the functions $w_{n+1,s}(x)$ and $w_{n+2,s}(x)$ are bounded as $x \to +\infty$. If we substitute (93) into (22), we obtain for $x \geqq x_0$,

$$
(94)\qquad
\begin{aligned}
\lambda_k(x) = i a_0^{1/3} \Big\{ & t\eta_{0k} + \sum_{j=1}^{3} \sum_{s=0}^{j} v_{js} t f_1^s f_2^{j-s} \\
&+ \sum_{j=4}^{n+1} \Big[v_{j0} t f_2^j + \sum_{s=1}^{j} v_{js} t f_1^s f_2^{j-s} \Big] \\
&+ w_{n+1,0} t f_2^{n+1} + \sum_{s=1}^{n+1} w_{n+1,s} t f_1^s f_2^{n+1-s} \\
&+ w_{n+2,0} t f_2^{n+2} + \sum_{s=1}^{n+2} w_{n+2,s} t f_1^s f_2^{n+2-s} \Big\}.
\end{aligned}
$$

We now use (88), (89), (90) to calculate asymptotic expansions for each of the terms $t\eta_{0k}$, $tf_1^s f_2^{j-s}$. We obtain

$$
\begin{aligned}
t\eta_{0k} &= \eta_{0k} + \eta_{0k}(6c)^{-1}(iD) - \eta_{0k}(6c)^{-2}(iD)^2 + (5/3)\eta_{0k}(6c)^{-3}(iD)^3 \\
&\quad + O(D^4), \\
tf_1 &= d[-(6c)^{-1}(iD) + (6c)^{-2}(iD)^2 - (5/3)(6c)^{-3}(iD)^3 + O(D^4)], \\
tf_2 &= ac^{-2/3}x^{-\nu}[1 - (6c)^{-1}(iD) + O(D^2)], \quad \text{etc.}
\end{aligned}
$$

Substituting into (94), we obtain (76). (79) follows immediately from (76). From the way in which (93) was derived, we see that $v_{11} = (2\pi i)^{-1} \int_{C_k} [\eta^3 h_0' - 2\eta^2 h_0] h_0^{-2}\, d\eta$. Hence,

$$
\begin{aligned}
v_{11} &= (2\pi i)^{-1} \int_{C_k} [\eta^2 h_0^{-1} - (d/d\eta)(\eta^3 h_0^{-1})]\, d\eta \\
&= (2\pi i)^{-1} \int_{C_k} \eta^2 h_0^{-1}\, d\eta = \eta_{0k}[h_0'(\eta_{0k})]^{-1}.
\end{aligned}
$$

This proves (80). (82)–(85) are proved similarly. (81) follows from (80) and the fact that $d = (\eta_{0k}^3 + 1)\eta_{0k}^{-2}$. This proves Lemma 23.

Let

$$\mu = \min\{\nu + 1 + \gamma/3, \gamma\} \quad \text{if} \quad a\alpha \neq 0$$
$$= \gamma \quad \text{if} \quad a\alpha = 0. \tag{95}$$

Then, as $x \to \infty$,

$$D = O(x^{-\mu}). \tag{96}$$

In the following we shall consider three cases. Case 1 is the case that $\nu = 2\mu$, which occurs if $\alpha = -4\gamma/3$. Case 2 is the case $\nu > 2\mu$, which occurs if $\alpha < -4\gamma/3$. Case 3 is the case $\nu < 2\mu$, which occurs if $-4\gamma/3 < \alpha < 2\gamma/3$.

LEMMA 24. *Suppose the coefficients of L are given by* (71)–(74) *and that* $b/c^{1/3} \neq 3/2^{2/3}$. *If* η_{0k} *is real,* $\operatorname{Re}\lambda_k(x)$ *has the following asymptotic expansions*:

Case 1. $\nu = 2\mu$ (*i.e.*, $\alpha = -4\gamma/3$). *Then,*

$$\begin{aligned}\operatorname{Re}\lambda_k(x) = a_0^{1/3}\{&[h_0'(\eta_{0k})]^{-1}(2c)^{-1}D \\ &+ ac^{-2/3}[v_{10} + dv_{21}](6c)^{-1}Dx^{-2\mu} \\ &- [5(h_0'(\eta_{0k}))^{-1} + d^2(3v_{22} + dv_{33})](6c)^{-3}D^3 \\ &+ O(x^{-4\mu})\}.\end{aligned} \tag{97}$$

Case 2. $\nu > 2\mu$ (i.e., $\alpha < -4\gamma/3$). *Then,*

$$\begin{aligned}\operatorname{Re}\lambda_k(x) = a_0^{1/3}\{&[h_0'(\eta_{0k})]^{-1}(2c)^{-1}D \\ &- [5(h_0'(\eta_{0k}))^{-1} + d^2(3v_{22} + dv_{33})](6c)^{-3}D^3 + O(x^{-3\mu-\epsilon})\},\end{aligned} \tag{98}$$

where $\epsilon > 0$.

Case 3. $\nu < 2\mu$ (i.e., $-4\gamma/3 < \alpha < 2\gamma/3$). *Then,*

$$\begin{aligned}\operatorname{Re}\lambda_k(x) = a_0^{1/3}\{&[h_0'(\eta_{0k})]^{-1}(2c)^{-1}D \\ &+ ac^{-2/3}[v_{10} + dv_{21}](6c)^{-1}Dx^{-\nu} \\ &+ O(x^{-\mu-\nu-\epsilon})\},\end{aligned} \tag{99}$$

where $\epsilon > 0$.

Proof. (97) and (98) follow directly from (79). If we choose n so large that $n\nu > \mu$, then we also see that (99) follows from (79). This proves Lemma 24.

LEMMA 25. *If* $b/c^{1/3} > 3/2^{2/3}$, $b/c^{1/3} \neq 3/2^{1/3}$, *and* $\sigma \neq a\alpha/2$, *then the coefficients of* L *given by* (71)–(74) *satisfy Assumptions* I–VI.

Proof. Since $d = b/c^{1/3} > 3/2^{2/3}$, $h_0(\eta) = 0$ has three real roots. Because $d \neq 3/2^{1/3}$, $h_0'(\eta_{01})$, $h_0'(\eta_{02})$, $h_0'(\eta_{03})$ are all distinct. From (78) and (95) we see that $D = C_1 x^{-\mu}[1 + o(1)]$, where $C_1 \neq 0$ because $\sigma \neq a\alpha/2$. By Lemma 24,

$$\begin{aligned} \operatorname{Re}[\lambda_j(x) - \lambda_k(x)] = C_1(2c)^{-1}a_0^{1/3}\{[h_0'(\eta_{0j})]^{-1} - [h_0'(\eta_{0k})]^{-1}\} \\ \times x^{-\mu}[1 + o(1)]. \end{aligned} \tag{100}$$

From (100) and (74) it follows that Assumption VI is satisfied. This proves Lemma 25.

LEMMA 26. *Suppose the coefficients of* L *are given by* (71)–(74) *and that* $b/c^{1/3} = 3/2^{1/3}$. *Then the roots of* $h_0(\eta) = 0$ *are* $\eta_{01} = 2^{-1/3}(1 - 3^{1/2})$, $\eta_{02} = 2^{-1/3}$, $\eta_{03} = 2^{-1/3}(1 + 3^{1/2})$, *and*

$$h_0'(\eta_{01}) = h_0'(\eta_{03}) \neq h_0'(\eta_{02}), \tag{101}$$

$$v_{10}(\eta_{01}) + dv_{21}(\eta_{01}) = 3^{-1}2^{-2/3}(-2 + 3^{1/2}), \tag{102}$$

$$v_{10}(\eta_{03}) + dv_{21}(\eta_{03}) = 3^{-1}2^{-2/3}(-2 - 3^{1/2}), \tag{103}$$

$$3v_{22}(\eta_{01}) + dv_{33}(\eta_{01}) = 3^{-1}2^{-2/3}[250 - (143)3^{1/2}], \tag{104}$$

$$3v_{22}(\eta_{03}) + dv_{33}(\eta_{03}) = 3^{-1}2^{-2/3}[250 + (143)3^{1/2}]. \tag{105}$$

The proof follows immediately from (80)–(85) and the fact that $h_0(\eta) = \eta^3 - (3/2^{1/3})\eta^2 + 1$.

LEMMA 27. *Suppose that* $b/c^{1/3} = 3/2^{1/3}$, $\alpha < -4\gamma/3$. *Then, the coefficients of* L *given by* (71)–(74) *satisfy Assumptions* I–VI.

Proof. Since $\alpha < -4\gamma/3$, $\nu + 1 + \gamma/3 > \gamma$. By (95), $\mu = \gamma$. By (78), $D = -2\sigma x^{-\gamma}(1 + o(1))$. From (101) and (98) it follows that $\operatorname{Re}[\lambda_2(x) - \lambda_1(x)]$ and $\operatorname{Re}[\lambda_2(x) - \lambda_3(x)]$ satisfy (a), (b) or (c) of Assumption VI. From (98), (101), (104), (105),

$$\operatorname{Re}[\lambda_3(x) - \lambda_1(x)] = C_1 x^{-8\gamma/3}(1 + o(1)),$$

where $C_1 \neq 0$. Thus, $\operatorname{Re}[\lambda_3(x) - \lambda_1(x)]$ also satisfies (a), (b) or (c) of Assumption VI. This proves Lemma 27.

LEMMA 28. *Suppose that* $b/c^{1/3} = 3/2^{1/3}$, $-4\gamma/3 < \alpha < 2\gamma/3$, $\sigma \neq a\alpha/2$, $a \neq 0$. *Then, the coefficients of* L *given by* (71)–(74) *satisfy Assumptions* I–VI.

Proof. It follows from (78) that $D = C_1 x^{-\mu}(1 + o(1))$, where $C_1 \neq 0$ because $\sigma \neq a\alpha/2$. By (101) and (99), $\operatorname{Re}[\lambda_2(x) - \lambda_1(x)]$ and $\operatorname{Re}[\lambda_2(x) - \lambda_3(x)]$ satisfy (a), (b) or (c) of Assumption VI. From (99) and (101)–(103), $\operatorname{Re}[\lambda_3(x) - \lambda_1(x)] = C_2 x^{-\mu-\nu+\gamma/3}(1 + o(1))$, where $C_2 \neq 0$ because $a \neq 0$. Hence, $\operatorname{Re}[\lambda_3(x) - \lambda_1(x)]$ satisfies (a), (b) or (c) of Assumption VI. This proves Lemma 28.

LEMMA 29. *Suppose that* $b/c^{1/3} = 3/2^{1/3}$, $\alpha = -4\gamma/3$, $\sigma^2 \neq -2^{2/3} a c^{4/3}/143$. *Then, the coefficients of* L *given by* (71)–(74) *satisfy Assumptions* I–VI.

Proof. Since $\alpha = -4\gamma/3$, $\mu = \gamma$. Hence, $D = -2\sigma x^{-\gamma}(1 + o(1))$ by (78). From (101) and (97) it follows that $\operatorname{Re}[\lambda_2(x) - \lambda_1(x)]$ and $\operatorname{Re}[\lambda_2(x) - \lambda_3(x)]$ satisfy (a), (b) or (c) of Assumption VI. From (97) and (101)–(105), $\operatorname{Re}[\lambda_3(x) - \lambda_1(x)] = C_1 x^{-8\gamma/3}(1 + o(1))$, where $C_1 \neq 0$ because $\sigma^2 \neq -2^{2/3} a c^{4/3}/143$. Hence, $\operatorname{Re}[\lambda_3(x) - \lambda_1(x)]$ satisfies (a), (b) or (c) of Assumption VI. This proves Lemma 29.

LEMMA 30. *Suppose the coefficients of* L *are given by* (71)–(74) *and that* $b/c^{1/3} \neq 3/2^{2/3}$. *If* η_{0k} *is real,* $\operatorname{Re}\lambda_k(x)$ *has the following asymptotic expansions:*

Case A. *Suppose* $a = 0$. *Then,*

$$\operatorname{Re}\lambda_k(x) = -\sigma[h_0'(\eta_{0k})]^{-1} c^{-2/3} x^{-2\gamma/3}(1 + o(1)). \tag{106}$$

Case B. *Suppose* $a \neq 0$.
(i) *Suppose* $1 < 2\gamma/3$.
(a) *If* $1 < \alpha < 2\gamma/3$, *then*

$$\operatorname{Re}\lambda_k(x) = a\alpha c^{-2/3}[2h_0'(\eta_{0k})]^{-1} x^{\alpha-1-2\gamma/3}(1 + o(1)). \tag{107}$$

(b) *If* $\alpha = 1$ *and* $\sigma \neq a/2$,

$$\operatorname{Re}\lambda_k(x) = (a - 2\sigma)[2h_0'(\eta_{0k})]^{-1} c^{-2/3} x^{-2\gamma/3}(1 + o(1)). \tag{108}$$

(c) *If* $\alpha < 1$, (106) *is valid.*
(ii) *If* $\alpha < 2\gamma/3 \leqq 1$, (106) *is valid.*

The proof follows directly from Lemma 24 with calculation of μ and D in the various cases.

4. The deficiency index of the operator L.

In the following, L_2 will denote the space $L_2[1,\infty)$, i.e., the space of complex-valued functions on $[1,\infty)$ which have Lebesgue square integrable absolute values.

LEMMA 31. *Suppose the coefficients of L are given by* (71)–(74) *and that* $b/c^{1/3} < 3/2^{2/3}$, *so that* $\eta_{0k} = u_k + iv_k$, *where* $v_2 > 0$ *and* $v_3 < 0$. *Then the function* $f_k(x) = a_0^{-1/3}(x)\exp\int_{x_0}^{x}\lambda_k(t)\,dt$, $x \geqq x_0$, *has the following properties*:

(i) *If* $k = 2$ *and* $c > 0$ *or if* $k = 3$ *and* $c < 0$, *then* $f_k \in L_2$ *for* $\sigma > 0$ *and for* $\sigma < 0$.

(ii) *If* $k = 2$ *and* $c < 0$ *or if* $k = 3$ *and* $c > 0$, *then* $f_k \notin L_2$ *for* $\sigma > 0$ *and for* $\sigma < 0$.

Proof. We shall give an intuitive proof which can be made precise as in Naimark [**9**, §23]. We have by (23) that

$$\begin{aligned}|f_k(x)| &\approx |c|^{-1/3}x^{-\gamma/3}\exp\left[-v_k c^{1/3}\int_{x_0}^{x} t^{\gamma/3}\,dt\right]\\ &= |c|^{-1/3}x^{-\gamma/3}\exp[-v_k c^{1/3}(\gamma/3+1)^{-1}(x^{\gamma/3+1}-x_0^{\gamma/3+1})]\\ &\to +\infty \quad \text{if} \quad v_k c^{1/3} < 0.\end{aligned}$$

This proves (ii). Also,

$$\begin{aligned}|f_k(x)|^2 &\approx |c|^{-2/3}x^{-2\gamma/3}\exp\left[-2v_k c^{1/3}\int_{x_0}^{x} t^{\gamma/3}\,dt\right]\\ &\leqq |c|^{-2/3}x^{\gamma/3}\exp\left[-2v_k c^{1/3}\int_{x_0}^{x} t^{\gamma/3}\,dt\right]\\ &= (-2v_k c)^{-1}(d/dx)\exp\left[-2v_k c^{1/3}\int_{x_0}^{x} t^{\gamma/3}\,dt\right].\end{aligned}$$

This proves (i).

LEMMA 32. *Suppose the coefficients of L are given by* (71)–(74) *and that* $b/c^{1/3} \neq 3/2^{2/3}$. *If* η_{0k} *is real, the function* $f(x) = a_0^{-1/3}(x)$

$\exp \int_{x_0}^{x} \lambda_k(t)\, dt$, $x \geqq x_0$, *has the following properties*:

(I) *If* $2\gamma/3 > 1$ *and* $\sigma \neq a/2$, *then* $f \in L_2$ *for* $\sigma > 0$ *and for* $\sigma < 0$.

(II) *If* $2\gamma/3 \leqq 1$, *then* $f \in L_2$ *for* $\sigma / h_0'(\eta_{0k}) > 0$, *and* $f \notin L_2$ *for* $\sigma / h_0'(\eta_{0k}) < 0$.

Proof. Case A. Suppose $a = 0$. By (106),

$$\begin{aligned} |f(x)|^2 &\approx c^{-2/3} x^{-2\gamma/3} \exp\left\{ -2\sigma c^{-2/3} [h_0'(\eta_{0k})]^{-1} \int_{x_0}^{x} t^{-2\gamma/3}\, dt \right\} \\ &= (-2\sigma)^{-1} h_0'(\eta_{0k}) (d/dx) \exp\left\{ -2\sigma c^{-2/3} [h_0'(\eta_{0k})]^{-1} \int_{x_0}^{x} t^{-2\gamma/3}\, dt \right\}. \end{aligned}$$

From this last expression we see that (I) and (II) are true for Case A.

Case B. Suppose $a \neq 0$. If $1 < \alpha < 2\gamma/3$, then by (107),

$$\begin{aligned} |f(x)|^2 &\approx c^{-2/3} x^{-2\gamma/3} \exp\left\{ a\alpha c^{-2/3} [h_0'(\eta_{0k})]^{-1} \int_{x_0}^{x} t^{\alpha-1-2\gamma/3}\, dt \right\} \\ &\leqq c^{-2/3} x^{\alpha-1-2\gamma/3} \exp\left\{ a\alpha c^{-2/3} [h_0'(\eta_{0k})]^{-1} \int_{x_0}^{x} t^{\alpha-1-2\gamma/3}\, dt \right\} \\ &= (a\alpha)^{-1} h_0'(\eta_{0k}) (d/dx) \exp\left\{ a\alpha c^{-2/3} [h_0'(\eta_{0k})]^{-1} \int_{x_0}^{x} t^{\alpha-1-2\gamma/3}\, dt \right\}. \end{aligned}$$

Since $\int_{x_0}^{x} t^{\alpha-1-2\gamma/3}\, dt$ converges, we see that (I) is true if $1 < \alpha < 2\gamma/3$. If $\alpha = 1 < 2\gamma/3$ and $\sigma \neq a/2$, then by (108),

$$\begin{aligned} |f(x)|^2 &\approx c^{-2/3} x^{-2\gamma/3} \exp\left\{ (a-2\sigma) c^{-2/3} [h_0'(\eta_{0k})]^{-1} \int_{x_0}^{x} t^{-2\gamma/3}\, dt \right\} \\ &= (a-2\sigma)^{-1} h_0'(\eta_{0k}) \\ &\qquad \times (d/dx) \exp\left\{ (a-2\sigma) c^{-2/3} [h_0'(\eta_{0k})]^{-1} \int_{x_0}^{x} t^{-2\gamma/3}\, dt \right\}. \end{aligned}$$

Since $\int_{x_0}^{x} t^{-2\gamma/3}\, dt$ converges, we see that (I) is true for $\alpha = 1 < 2\gamma/3$ and $\sigma \neq a/2$. If $\alpha < 1 < 2\gamma/3$ or if $\alpha < 2\gamma/3 \leqq 1$, then by Lemma 30, (106) is valid and therefore (I) and (II) follow as in Case A. This proves Lemma 32.

Let n_+ denote the dimension of the space of solutions of $Ly = i\sigma y$, $x \geqq r$, which are in $L_2[r, \infty)$ for $\sigma > 0$. It is known that n_+ is independent of σ. Let n_- denote the same number for $\sigma < 0$. We shall call n_+ and

n_- the deficiency numbers of L, and we shall call the pair (n_+, n_-) the deficiency index.

THEOREM 2. *Suppose that the coefficients of L are given by* (71)–(74) *and that $b/c^{1/3} < 3/2^{2/3}$. If $2\gamma/3 > 1$, $n_+ = n_- = 2$. If $2\gamma/3 \leqq 1$, $n_+ = 2$, $n_- = 1$.*

Proof. By Lemma 22, the coefficients of L satisfy Assumptions I–VI. By Theorem 1, $Ly = i\sigma y$, $x \geqq 1$, $\sigma \neq 0$, has three linearly independent solutions y_k given by (65). By Lemma 31, for $c > 0$, $y_2 \in L_2$ and $y_3 \notin L_2$ for $\sigma > 0$ and for $\sigma < 0$; for $c < 0$, $y_2 \notin L_2$ and $y_3 \in L_2$ for $\sigma > 0$ and for $\sigma < 0$. By Lemma 32, if $2\gamma/3 > 1$, $y_1 \in L_2$ for $\sigma > 0$ and for $\sigma < 0$, $\sigma \neq a/2$; if $2\gamma/3 \leqq 1$, $y_1 \in L_2$ for $\sigma > 0$, and $y_2 \notin L_2$ for $\sigma < 0$, because $h_0'(\eta_{01}) > 0$. It follows that if $2\gamma/3 > 1$, then $n_+ = n_- = 2$, and if $2\gamma/3 \leqq 1$, then $n_+ = 2$. It also follows that if $2\gamma/3 \leqq 1$, then $n_- = 1$, provided we can show that for $c > 0$ and $\sigma < 0$ no nontrivial linear combination of y_1 and y_3 is in L_2, and for $c < 0$ and $\sigma < 0$ no nontrivial linear combination of y_1 and y_2 is in L_2. We deal with the case $c > 0$, $\sigma < 0$; the case $c < 0$ and $\sigma < 0$ is similar. It is sufficient to show that $y_1 + By_3 \notin L_2$ if $B \neq 0$. By Theorem 1, (23), and Lemma 30,

$$\begin{aligned} |y_1/y_3| &= [1 + o(1)] \exp \int_{x_0}^{x} [\operatorname{Re} \lambda_1(t) - \operatorname{Re} \lambda_3(t)]\, dt \\ &= [1 + o(1)] \exp c^{1/3} v_3 \int_{x_0}^{x} t^{\gamma/3}[1 + o(1)]\, dt \to 0 \quad \text{as} \quad x \to +\infty. \end{aligned}$$

Hence, for $x \geqq x_1$, $|y_1/y_3 + B|^2 \geqq K$, where K is a constant. Thus

$$\int_{x_1}^{\infty} |y_1 + By_3|^2\, dx = \int_{x_1}^{\infty} |y_3|^2 |y_1/y_3 + B|^2\, dx \geqq K \int_{x_1}^{\infty} |y_3|^2\, dx.$$

It follows that $y_1 + By_3 \notin L_2$. This completes the proof of Theorem 2.

THEOREM 3. *Suppose that the coefficients of L are given by* (71)–(74) *and that $b/c^{1/3} > 3/2^{2/3}$.*

Case A. *Suppose $b/c^{1/3} \neq 3/2^{1/3}$. If $2\gamma/3 > 1$, $n_+ = n_- = 3$. If $2\gamma/3 \leqq 1$, $n_+ = 2$, $n_- = 1$.*

Case B. *Suppose $b/c^{1/3} = 3/2^{1/3}$ and $\alpha \leqq -4\gamma/3$. If $2\gamma/3 > 1$, $n_+ = n_- = 3$. If $2\gamma/3 \leqq 1/4$, $n_+ = 2$, $n_- = 1$.*

Case C. *Suppose $b/c^{1/3} = 3/2^{1/3}$, $-4\gamma/3 < \alpha < 2\gamma/3$, $a \neq 0$. If $2\gamma/3 > 1$, $n_+ = n_- = 3$. If $4\gamma/3 - 1 \leqq \alpha < 2\gamma/3 < 1$, then $n_+ = 2$, $n_- = 1$.*

Proof. By Lemmas 25–29, the coefficients of L satisfy Assumptions I–VI in all three cases, provided $\sigma \neq a\alpha/2$ and $\sigma^2 \neq -2^{2/3}ac^{4/3}/143$. Hence, if we avoid these values of σ, $Ly = i\sigma y$, $x \geqq 1$, $\sigma \neq 0$, has three linearly independent solutions y_k given by (65). By Lemma 32 we have the following: (I) If $2\gamma/3 > 1$ and $\sigma \neq a/2$, then $y_1, y_2, y_3 \in L_2$ for $\sigma > 0$ and for $\sigma < 0$; (II) if $2\gamma/3 \leqq 1$, then for $\sigma > 0$, $y_1, y_3 \in L_2$ and $y_2 \notin L_2$, while for $\sigma < 0$, $y_2 \in L_2$ and $y_1, y_3 \notin L_2$. By (I) we see that if $2\gamma/3 > 1$, then $n_+ = n_- = 3$ in all three cases. If $2\gamma/3 \leqq 1$, then $n_+ = 2$ and $n_- = 1$, provided we can show that no non-trivial linear combination of y_1 and y_3 is in L_2. Using (106), this can be proved for Case A as in the proof of Theorem 2. In Cases B and C it is necessary to use (97)–(99). The assumptions in Cases B and C enable one to do this as in the proof of Theorem 2. This completes the proof of Theorem 3.

THEOREM 4. *Suppose that the coefficients of L are given by* (71–74) (*without the requirements that* $\alpha < 2\gamma/3$, $\gamma > 0$). *Then the deficiency index of L is as follows for the indicated values of the parameters* γ, α:

I. $\gamma > 3/2$, $\alpha < 2\gamma/3$: $(2,2)$ *if* $b/c^{1/3} < 3/2^{2/3}$; $(3,3)$ *if* $b/c^{1/3} > 3/2^{2/3}$, $b/c^{1/3} \neq 3/2^{1/3}$.

II. $0 < \gamma \leqq 3/2$, $\alpha < 2\gamma/3$: $(2,1)$ *if* $b/c^{1/3} \neq 3/2^{2/3}$ *and* $b/c^{1/3} \neq 3/2^{1/3}$.

III. $\gamma \leqq 0$, $\alpha \leqq 0$: $(2,1)$.

IV. $0 < \alpha \leqq 1$, $\alpha > 2\gamma/3$: $(2,1)$.

V. $1 < \alpha$, $\alpha > 2\gamma/3$: $(3,3)$ *if* $a > 0$; $(2,2)$ *if* $a < 0$.

Proof. The statements for regions I and II follow from Theorems 2 and 3. III follows from the fact that $n_+ + n_- = 3$ by Dunford and Schwartz [**4**, XIII. 10. E.II(5)] and from the fact that $2 \leqq n_+$ and $1 \leqq n_-$ by Everitt [**5**] or Kogan and Rofe-Beketov [**8**]. IV and V follow from Unsworth [**12**]. This proves Theorem 4.

REMARK 1. Note that $\alpha = 2\gamma/3$, $\gamma > 0$, is the only portion of the (γ, α)-plane not included in Theorem 4.

REMARK 2. The results of §7 of Pfeiffer [**5**] are included in Theorem 4 except for the case $c = 0$.

REFERENCES

1. F. V. Atkinson, *Discrete and Continuous Boundary Problems*, Academic Press, New York, 1964.

2. R. Bellman, *Stability Theory of Differential Equations*, McGraw-Hill Book Company, New York, 1953.

3. A. Devinatz, *The deficiency index of certain fourth-order ordinary self-adjoint differential operators*, Quart. J. Math. (Oxford) (2), **23** (1972), 267–286.

4. N. Dunford and J. T. Schwartz, *Linear Operators. Part II: Spectral Theory*, Interscience Publishers, New York, 1963.
5. W. N. Everitt, *Integrable-square solutions of ordinary differential equations*, Quart. J. Math. (Oxford) (2), **10** (1959), 145–155.
6. M. V. Fedorjuk, *Asymptotic methods in the theory of one-dimensional singular differential operators*, Transactions of the Moscow Mathematical Society, **15** (1966), 333–386. English translation, Amer. Math. Soc.
7. V. I. Kogan and F. S. Rofe-Beketov, *The deficiency indices of differential operators of odd order*, Kharkov Ped. Inst. named after Lenin, Materials of the sci. and techn. conference on results of sci. work in 1969, publ. by Kharkov State Univ., 1970, No. 7, 93–95.
8. ———, *On the question of the deficiency indices of differential operators with complex coefficients*, Proc. Roy. Soc. Edinburgh Sect. A72, part 4, (1975), 281–298.
9. M. A. Naimark, *Linear Differential Operators*, English translation, Frederick Ungar Pub. Co., New York, 1968.
10. G. W. Pfeiffer, *Deficiency indices of a third order equation*, Differential Equations, **11** (1972), 474–490.
11. R. A. Shirikyan, *Asymptotic methods in the theory of odd-order one-dimensional singular differential operators*, Differentsial'nye Uravneniya, **3** (1967), 1942–1956.
12. K. Unsworth, *Asymptotic expansions and deficiency indices associated with third-order self-adjoint differential operators*, Quart. J. Math. (Oxford) (2), **24** (1973), 177–188.

Received September 4, 1974 and in revised form March 1, 1976. This work was supported in part by the National Science Foundation, Grants GP 36640 and GP 36640 A-1.

CALIFORNIA STATE UNIVERSITY — FULLERTON
FULLERTON, CA 92634