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REMARKS ON THE DEDEKIND COMPLETION OF A NONSTANDARD MODEL OF THE REALS

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In 1980 Wattenberg constructed the Dedekind completion of a nonstandard model of the real numbers and applied the construction to obtain certain kinds of special measures on the set of integers. We feel that the Dedekind completion is a structure of interest for its own sake and we establish further properties here. Of particular interest is the connection with [1]. Specifically, the main concept we introduce is that of the absorption number of an element a which, roughly speaking, measures the degree to which the cancellation law $a + b = a + c \to b = c$ fails for a. The absorption number may be regarded as an element in the Dedekind completion of the value group of the valuation ring of finite numbers as discussed in [1].

Preliminaries. Let R be the set of real numbers and R^* a nonstandard model of R. Also let $R^\#$ be the class of all lower subsets α of R^* which are non-empty, with non-empty complement and with no greatest element. $R^\#$ is the Dedekind completion of R^* according to [**2**, page 227]. We identify $a \in R^*$ with $\alpha = \{x: x < a\}$.

DEFINITION. $\alpha + \beta = [a + b: a \in \alpha \wedge b \in \beta]$.

Addition is commutative and associative. Furthermore, $\alpha + 0 = \alpha$. Also, the embedding $R^* \to R^\#$ preserves sums.

DEFINITION. $-\alpha = (a \in R^*: \exists b[b > a \wedge -b \notin \alpha])$[1].

As we already noted the embedding $R^* \to R^\#$ preserves negation.

DEFINITION. $\alpha \leq \beta$ iff $\alpha \subset \beta$.

Then clearly $\beta \leq \gamma \to \alpha + \beta \leq \alpha + \gamma$.

[1]Note that the definition in [**2**] is technically incorrect. The latter defines $-\alpha$ as $(a \in R^*: -a \notin \alpha)$. If $\alpha \notin R^*$ i.e. α' has no mininum, then the definitions are equivalent. However if $\alpha \in R^*$ i.e. $\alpha = (x: x < a)$ for some $a \in R^*$ then according to the latter definition $-\alpha = (x: x \leq -a)$ whereas the definition of $R^\#$ requires that $-\alpha = (x: x < -a)$. In the usual treatment of Dedekind cuts for the ordinary real numbers both of the latter sets are regarded as equivalent so that no serious problem arises.

Furthermore $(-\alpha) + \alpha \leq 0$ and $(-\alpha) + (-\beta) \leq -(\alpha + \beta)$ with equality if either α or β is in R^*. $\alpha \leq \beta \to -\beta \leq -\alpha$. Counterexamples show that the inequalities cannot be replaced by equalities. In fact, an element which is especially useful in obtaining many counterexamples is μ which is the union of the set of non-positive numbers and positive infinitesimals. Then $\mu \neq 0$ but $\mu + \mu = \mu$. Furthermore $(-\mu) + \mu = -\mu$. Thus $R^{\#}$ is certainly not a group.

A product can also be defined. This is first done for positive numbers. Suppose $\alpha, \beta \in R^{\#}$ and $\alpha, \beta > 0$. Then

$$\alpha \cdot \beta = \left[(a \cdot b) \colon 0 < a \in \alpha, 0 < b \in \beta\right] \cup (a \colon a \leq 0).$$

If either α or β is 0 the product is defined to be 0 and the usual rule of signs is used if either α or β is negative.

Multiplication is commutative and associative. The distributivity with respect to addition holds for non-negative numbers.

We conclude this section by mentioning two other minor errors in [**2**]. The formula for inf A given on the top of page 229 is not quite correct; in fact the problem is similar to the one for negation. The formula says $\inf_{\alpha \in A} A = \bigcap_{\alpha \in A} \alpha$. This should read

$$\inf_{\alpha \in A} A = \left[a \colon (\exists d > 0)\left(a + d \in \bigcap_{d\alpha \in A} \alpha\right)\right].$$

For example let $A = \{a + d\}$ where d runs through the set of all positive numbers in R^*. Then $\inf A = a = \{x \colon x < a\}$. However, $\bigcap_{\alpha \in A} \alpha = (x \colon x \leq a)$.

Finally, another problem arises on page 236 because of this "pathological" nature of μ. The statement $\alpha \in [x - \mu, x + \mu]$ is not equivalent to $x \in [\alpha - \mu, \alpha + \mu]$.

In fact, if $\alpha = -\mu$ then the interval $[\alpha - \mu, \alpha + \mu]$ reduces to the single point $-\mu$! Hence $0 \notin [\alpha - u, \alpha + u]$ but $-\mu \in [0 - \mu, 0 + \mu]$.

Absorption Numbers. One way of defining the completion of R^* involves restricting oneself to subsets α which have the following property $(\forall \varepsilon > 0)$ $[(\exists x \in \alpha), (\exists y \in \alpha)(y - x < \varepsilon)]$. It is known that in this case we obtain a field. In fact the proof is essentially the same as the one used in the case of ordinary Dedekind cuts in the development of the standard real numbers. μ, of course, does not have the above property; no infinitesimal works.

This suggests the introduction of the concept of absorption number for an element α of $R^{\#}$ which, roughly speaking, measures how much α departs from having the above property.

DEFINITION. $\mathrm{ab}(\alpha) = (d \geq 0\colon (\forall x \in \alpha)[x + d \in \alpha])$.

EXAMPLES. $\mathrm{ab}(\alpha) = \{0\}$ iff α has the above property. $\mathrm{ab}(\mu) = \mu$. If $a \in R^*$ then $\mathrm{ab}(a + \mu) = \mu$.

LEMMA 1. (a) $c < \mathrm{ab}(\alpha)$ *and* $0 \leq d < c \to d \in \mathrm{ab}(\alpha)$.
(b) $c \in \mathrm{ab}(\alpha)$ *and* $d \in \mathrm{ab}(\alpha) \to c + d \in \mathrm{ab}(\alpha)$.

Proof. This is immediate from the definition.

By Lemma 1 $\mathrm{ab}(\alpha)$ may be regarded as an element of $R^{\#}$ by adding on all negative elements of $R^{\#}$ to $\mathrm{ab}(\alpha)$. Of course if the condition $d \geq 0$ in the definition of $\mathrm{ab}(\alpha)$ is deleted we automatically get all the negative elements to be in $\mathrm{ab}(\alpha)$ since $x < y \in \alpha \to x \in \alpha$. The reason for our definition is that the real interest lies in the non-negative numbers. A technicality occurs if $\mathrm{ab}(\alpha) = \{0\}$. We then identify $\mathrm{ab}(\alpha)$ with 0. [$\mathrm{ab}(\alpha)$ becomes $\{x\colon x \leq 0\}$ which by our early convention is not in $R^{\#}$].

By Lemma 1(b), $\mathrm{ab}(\alpha)$ is idempotent.

LEMMA 2. (a) $\mathrm{ab}(\alpha)$ *is the maximum element* $\beta \in R^{\#}$ *such that* $\alpha + \beta = \alpha$.
(b) $\mathrm{ab}(\alpha) \leq \alpha$ *for* $\alpha > 0$.
(c) *If* α *is positive and idempotent then* $\mathrm{ab}(\alpha) = \alpha$.

Proof. All parts are immediate from the definition.

It follows from Lemma 2 that the elements of the form $\mathrm{ab}(\alpha)$ are precisely the non-negative idempotents.

Idempotents may be characterized in various ways. For convenience we state the following elementary result.

LEMMA 3. *Let* $\alpha \in R^{\#}$ *satsify* $\alpha > 0$. *Then the following are equivalent.* [*In what follows assume* $a, b > 0$]
(a) α *is idempotent*,
(b) $a, b \in \alpha \to a + b \in \alpha$,
(c) $a \in \alpha \to 2a \in \alpha$,
(d) $a \in \alpha \to n\alpha \in \alpha$ *for all standard integers* n,
(e) $a \in \alpha \to ra \in \alpha$ *for all finite* $r \in R^*$.

Proof. All parts are immediate.

Condition (e) is of special importance since this makes the connection with [**1**] apparent.

Connection with the value group. We define an equivalence relation on the positive elements of R^* as follows

$$a \sim b \quad \text{iff} \quad a/b \quad \text{and} \quad b/a \text{ are finite.}$$

Then the equivalence classes from a linear ordered set. We denote the order relation by $\ll$. The classes may be regarded as orders of infinity. According to [**1**] the subring of R^* consisting of the finite elements is a valuation ring, and the equivalence classes may also be regarded as elements of the value group.

Condition (e) in Lemma 3 essentially says that $a \in \alpha$ and $b \sim a \to b \in \alpha$, i.e. α may be regarded as a Dedekind cut in the value group.

Properties of the Absorption Function.

THEOREM 1. $(-\alpha) + \alpha = -\mathrm{ab}(\alpha)$.

Proof. This is clear if $\alpha \in R^*$ since in that case both sides are 0. If $\alpha \notin R^*$ then the definition in [**2**] is valid.

$$(-\alpha) + \alpha = [(-b + a) \colon a \in \alpha \wedge b \notin \alpha]$$
$$= [-(b - a) \colon b \notin \alpha \wedge a \in \alpha] = (-d \colon d \notin \mathrm{ab}(\alpha)).$$

The last equality is essentially a restatement of the definition of $\mathrm{ab}(\alpha)$.

This in turn is $-\mathrm{ab}(\alpha)$.

Note that $\mathrm{ab}(\alpha)$ is not a monotonic function of α. In fact since $\alpha \in R^* \to \mathrm{ab}(\alpha) = 0$ there are arbitrarily high α such that $\mathrm{ab}(\alpha) = 0$. However, we have the following

THEOREM 2. $\mathrm{ab}(\alpha + \beta) \geq \mathrm{ab}(\alpha)$.

Proof. This is immediate.

THEOREM 3. $\alpha + \beta \leq \alpha + \gamma \to [-\mathrm{ab}(\alpha)] + \beta \leq \gamma$.

Proof. $\alpha + \beta \leq \alpha + \gamma \to (-\alpha) + (\alpha + \beta) \leq (-\alpha) + (\alpha + \gamma)$.

Hence

$$[(-\alpha) + \alpha] + \beta \leq [(-\alpha) + \alpha] + \gamma,$$
$$[-\mathrm{ab}(\alpha)] + \beta \leq [-\mathrm{ab}(\alpha)] + \gamma \leq \gamma.$$

COROLLARY. $\alpha + \beta = \alpha + \gamma \to [-\mathrm{ab}(\alpha)] + \beta = [-\mathrm{ab}(\alpha)] + \gamma$.

The above results illustrate, heuristically speaking, that ab(α) measures to what extent α spoils the system from being a group.

Note. Theorem 3 cannot be replaced by the following

$$\alpha + \beta \leq \alpha + \gamma \rightarrow \beta \leq \gamma + \mathrm{ab}(\alpha).$$

In fact $(-\mu) + \mu \leq (-\mu) + (-\mu)$ since both sides are $-\mu$. However, $\mu \nleq (-\mu) + \mu$ since the right-hand side is $-\mu$.

THEOREM 4. (1) $\mathrm{ab}(-\alpha) = \mathrm{ab}(\alpha)$. (2) $\mathrm{ab}(\alpha + \beta) = \max[\mathrm{ab}(\alpha), \mathrm{ab}(\beta)]$.

Proof. (1) This is easy and left to the reader.

(2) By Theorem 2 $\mathrm{ab}(\alpha + \beta) \geq \max[\mathrm{ab}(\alpha), \mathrm{ab}(\beta)]$. Suppose

$$\mathrm{ab}(\alpha + \beta) > \max[\mathrm{ab}(\alpha), \mathrm{ab}(\beta)].$$

Then $[\exists d \in \mathrm{ab}(\alpha + \beta)]\ [d \in \mathrm{ab}(\alpha) \wedge d \in \mathrm{ab}(\beta)]$. Then $\frac{1}{2}d \notin \mathrm{ab}(\alpha)$ and $\frac{1}{2}d \notin \mathrm{ab}(\beta)$. Choose $a \in \alpha$, $b \in \beta$ so that $a + \frac{1}{2}d \notin \alpha$ and $b + \frac{1}{2}d \notin \beta$. Then $(a + \frac{1}{2}d) + (b + \frac{1}{2}d) \notin \alpha + \beta$. i.e. $a + b + d \notin \alpha + \beta$ which contradicts the fact that $d \in \mathrm{ab}(\alpha + \beta)$.

We now classify the elements β such that $\alpha + \beta = \alpha$. For positive β we know by Lemma 2(a) that $\alpha + \beta = \alpha$ iff $\beta \leq \mathrm{ab}(\alpha)$.

THEOREM 5. *Assume $\beta > 0$. If α absorbs $-\beta$ then α abosrbs β.*

Proof. Suppose $\alpha + (-\beta) = \alpha$. Then $\alpha + \beta = \alpha + (-\beta) + \beta$. Hence $\alpha \leq \alpha + \beta \leq \alpha + [(-\beta) + \beta] \leq \alpha + 0 = \alpha$. Thus $\alpha = \alpha + \beta$.

This shows that if α absorbs $-\beta$ then $\beta \leq \mathrm{ab}(\alpha)$. On the other hand suppose $\beta < \mathrm{ab}(\alpha)$. Then $(\exists d, d' \in R^*)$ $(\beta < d < d' < \mathrm{ab}(\alpha))$. Hence $d' \in \mathrm{ab}(\alpha)$ i.e. $a \in \alpha \rightarrow a + d' \in \alpha$. Consider $\alpha + (-d)$. Clearly $\alpha + (-d) \leq \alpha$. Suppose $a \in \alpha$. Then $a + d' \in \alpha$. Hence $a = (a + d') + (-d') \in \alpha + (-d)$ since $-d' < -d$. Therefore $\alpha = \alpha + (-d)$. However $\alpha + (-d) \leq \alpha + (-\beta) \leq \alpha$. Hence $\alpha + (-\beta) = \alpha$.

We now know that for $\beta < \mathrm{ab}(\alpha)$, $\alpha + (-\beta) = \alpha$ and for $\beta > \mathrm{ab}(\alpha)$, $\alpha + (-\beta) \neq \alpha$. For $\beta = \mathrm{ab}(\alpha)$ the result is indeterminate. In fact if $\alpha = \mu$ we have $\mathrm{ab}(\alpha) = \mu$. $\mu + (-\mu) = -\mu \neq \mu$ thus μ does not absorb $-\mu$. On the other hand if $\alpha = -\mu$ then $\mathrm{ab}(\alpha) = \mu$ and $(-\mu) + (-\mu) = -\mu$ thus $-\mu$ does absorb $-\mu$.

THEOREM 6. *Let $0 < \alpha \in R^{\#}$. Then the following are equivalent*

(a) *α is an idempotent,*

(b) $(-\alpha) + (-\alpha) = -\alpha$,

(c) $(-\alpha) + \alpha = -\alpha$.

Proof. (a) $\to$ (b). In general $(-\alpha) + (-\alpha) \leq -\alpha$, so it suffices to show that $-\alpha \leq (-\alpha) + (-\alpha)$. Let $-a \in -\alpha$. Then $a \notin \alpha$. Hence $\frac{1}{2}a \notin \alpha$ [otherwise $a = 2(\frac{1}{2}a) \in \alpha$]. Hence $-\frac{1}{2}a \in -\alpha$. Therefore $-a = (-\frac{1}{2}a) + (-\frac{1}{2}a) \in (-\alpha) + (-\alpha)$.

(b) $\to$ (a). Assume $(-\alpha) + (-\alpha) = -\alpha$. Suppose $a \in \alpha$. It suffices to show that $2a \in \alpha$. Otherwise $2a \notin \alpha$ hence $-2a \in -\alpha$. Let $-2a = b + c$ where $b, c \in -\alpha$. Then either $b \geq -a$ or $c \geq -a$. Now $-a \notin -\alpha$ since $a \in \alpha$. Hence either $b \notin -\alpha$ or $c \notin -\alpha$ contradicting the above.

(a) $\to$ (c). In general $(-\alpha) + \alpha \geq -\alpha$. Suppose $a \in (-\alpha) + \alpha$. We must show that $a \in -\alpha$. Now a has the form $b + c$ with $b \in -\alpha$ and $c \in \alpha$. Then $-b \notin \alpha$. Since $c \in \alpha$ and α is closed with respect to addition $-b - c \notin \alpha$. Hence $b + c \in -\alpha$.

(c) $\to$ (a). Suppose α is not idempotent. Then $(\exists a, b \in \alpha)(a + b \notin \alpha)$. Therefore $-a - b \in -\alpha$. Hence $-a = (-a - b) + b \in (-\alpha) + \alpha$. However $-a \notin -\alpha$ since $a \in \alpha$. Thus $(-\alpha) + \alpha \neq -\alpha$.

REMARK. (c) is a strange characterization of idempotence. Such a result gives the subject a rather unusual flavor.

Equivalence Relations on $\mathbf{R}^{\#}$. Let Δ be a positive idempotent. We define three equivalence relations R, S and T on $R^{\#}$.

(1) $\alpha\ R\ \beta \bmod \Delta$ iff $\alpha + \Delta = \beta + \Delta$,

(2) $\alpha\ S\ \beta \bmod \Delta$ iff $\alpha + (-\Delta) = \beta + (-\Delta)$,

(3) $\alpha\ T\ \beta \bmod \Delta$ iff $(\exists d \in \Delta)(a \subset \beta + d$ and $\beta \subset \alpha + d)$.

To simplify the notation mod Δ is omitted when we are dealing with only one Δ. R and S are obviously equivalence relations. T is an equivalnce relation since Δ is idempotent. (Note incidentally that two different ways of regarding an expression such as $\alpha + d$ are equivalent. First as the sum $\alpha + d$ where d is regarded as an element of $R^{\#}$ and secondly, as $\{a + d: a \in \alpha\}$. Thus we can use whichever form is more convenient at the moment.)

It is immediate that R, S and T are congruence relations with respect to addition. Also, if $\sim$ stands for either R, S or T then $\alpha < \beta < \gamma$ and $\alpha \sim \gamma \to \alpha \sim \beta$. To see this it is convenient to have the following simple lemmas whose proofs are immediate.

LEMMA 4. *Suppose* $\alpha < \beta$. *then* $\alpha\ R\ \beta \leftrightarrow \beta \leq \alpha + \Delta$.

LEMMA 5. *Suppose* $\alpha < \beta$ *then* $\alpha\ S\ \beta \leftrightarrow \beta + (-\Delta) \leq \alpha$.

We next want to classify which of the above relations are preserved by negations. For this we need a technical lemma.

LEMMA 6. *Let Δ be a positive idempotent. Then* $-[\alpha + (-\Delta)] + (-\Delta) \leq -\alpha$.

REMARK. This is not immediate since the inequality $(-\alpha) + (-\beta) \leq -(\alpha + \beta)$ goes the wrong way. In fact, this seems surprising at first since the first addend may be bigger than one intuitively expects, e.g. if $\alpha = \Delta = \mu$ then $-[\alpha + (-\Delta)] = -[\mu + (-\mu)] = \mu > 0$. However, $\mu + (-\mu) = -\mu$ so the inequality is valid after all.

Proof. Suppose $x \in -[\alpha + (-\Delta)]$. Then $-x \notin [\alpha + (-\Delta)]$ i.e. $-x$ does not have the form $a + e$ with $a \in \alpha$ and $e \in -\Delta$, i.e., $(\forall a \in \alpha)\,(-x - a \notin -\Delta)$. Hence $(\forall a \in \alpha)\,(x + a \in \Delta)$. [Note the technicality that this is valid since $\Delta \notin R^*$.]

Now suppose $y \in -[\alpha + (-\Delta)] + (-\Delta)$. Then y has the form $x + d$ with $x \in -[\alpha + (-\Delta)]$ and $d \in -\Delta$. Let $a \in \alpha$. Then $y + a = (x + d) + a = (x + a) + d \in \Delta + (-\Delta) = -\Delta$. Hence $y + a < 0$. So certainly $y + a \neq 0$ so $-y \neq a$ i.e. $-y \notin \alpha$. Therefore $y \in -\alpha$. [We must beware of a technicality in the definition of negation in order to justify the last step. One way of resolving this is to choose $z > y$ so that $z \in -[\alpha + (-\Delta)] + (-\Delta)$. Then from $-z \notin \alpha$ we can deduce what we desire.]

THEOREM 7. *S is a congruence relation with respect to negation.*

Proof. Suppose $\alpha\, S\, \beta$. Then $\alpha + (-\Delta) = \beta + (-\Delta)$. Now $\alpha + (-\Delta) \leq \beta$. Hence $-\beta \leq -[\alpha + (-\Delta)]$. So $-\beta + (-\Delta) \leq -[\alpha + (-\Delta)] + (-\Delta) \leq -\alpha$. Similarly $-\alpha + (-\Delta) \leq -\beta$. Hence the result follows by Lemma 5.

THEOREM 8. *T is a congruence relation with respect to negation.*

Proof. Let $\alpha < \beta$. Suppose $\beta \leq \alpha + d$ where $d \in \Delta$. Then $-(\alpha + d) \leq -\beta$. So $(-\alpha) + (-d) \leq -[\alpha + d] \leq -\beta$. Hence $-\alpha = (-\alpha) + (-d) + d \leq (-\beta) + d$. The equality is valid since $d \in R^*$. This proves the result.

THEOREM 9. *R is not a congruence relation with respect to negation.*

Proof. Since Δ is idempotent, $\Delta + \Delta = \Delta = 0 + \Delta$ hence $\Delta\ R\ 0$. However $(-\Delta) + \Delta = -\Delta$ and $0 + \Delta = \Delta$. So $(-\Delta) + \Delta \neq 0 + \Delta$; i.e. $-\Delta \not{R}\ 0$.

THEOREM 10. *$\alpha + \Delta$ is the maximum element β satisfying $\beta\ R\ \alpha$.*

Proof. Clearly $\alpha + \Delta\ R\ \Delta$ since Δ is idempotent. However $\beta\ R\ \alpha \to \beta \le \alpha + \Delta$. (Note that Theorem 10 does *not* say that Δ is the largest element such that $\alpha + \Delta\ R\ \alpha$. In fact $\mathrm{ab}(\alpha)$ may be bigger than Δ.)

Similarly, we have

THEOREM 1. *$\alpha + (-\Delta)$ is the minimum element β satisfying $\beta\ S\ \alpha$.*

We now compare R, S, and T.

THEOREM 12. *$T \subset R \subset S$. Both inclusions are proper.*

Proof. $d \in \Delta \to d < \Delta$. Hence $\alpha \le \beta + d \to \alpha \le \beta + \Delta$. It is thus clear that $T \subset R$. The inclusion is proper since $\Delta\ R\ 0$ but no $d \in \Delta$ satisfies $\Delta \le 0 + d = d$.

Now suppose $\alpha + \Delta = \beta + \Delta$.

Then

$$(\alpha + \Delta) + (-\Delta) = (\beta + \Delta) + (-\Delta),$$
$$\alpha + [\Delta + (-\Delta)] = \beta + [\Delta + (-\Delta)],$$
$$\alpha + (-\Delta) = \beta + (-\Delta).$$

Hence $R \subset S$. We already know that $-\Delta\ \not R\ 0$. Certainly $-\Delta\ S\ 0$ so the inclusion is proper.

For convenience we mention elementwise characterizations of R, S, and T. Suppose $\alpha < \beta$. Then

$$\alpha\ R\ \beta \leftrightarrow (\forall b \in \beta)(\exists d \in \Delta)(b - d \in \alpha),$$
$$\alpha\ S\ \beta \leftrightarrow (\forall b \in \beta)(\forall d \notin \Delta)(b - d \in \alpha),$$
$$\alpha\ T\ \beta \leftrightarrow (\exists d \in \Delta)(\forall b \in \beta)(b - d \in \alpha).$$

We next compare different Δ's. We already know that $-\Delta$ absorbs Δ. However, we do have the following

THEOREM 13. *Let Δ_1 and Δ_2 be two positive idempotents such that $\Delta_2 > \Delta_1$. Then $\Delta_2 + (-\Delta_1) = \Delta_2$.*

Note that it is obvious that Δ_2 absorbs Δ_1 but this is not our concern here. Let $a \in \Delta_2$ with $a \notin \Delta_1$. Then $2a \in \Delta_2$ and $-a \in -\Delta_1$. Hence $a = 2a + (-a) \in \Delta_2 + (-\Delta_1)$. This is almost enough to show that $\Delta_2 \in \Delta_2 + (-\Delta_1)$. To complete the argument suppose that $b \in \Delta_2$ is arbitrary. If $b \notin \Delta_1$ then $b \in \Delta_2 + (-\Delta_1)$ by the above. If $b \in \Delta_1$ we can choose an a as above. Necessarily $b < a$. Now by the above $a \in \Delta_2 + (-\Delta_1)$. Hence $b \in \Delta_2 + (-\Delta_1)$. Since $\Delta_2 + (-\Delta_1) \le \Delta_2$ this completes the proof.

COROLLARY. *Let Δ_1 and Δ_2 be two positive idempotents such that $\Delta_2 > \Delta_1$ then $\alpha \; S \; \beta \bmod \Delta_1 \to \alpha \; R \; \beta \bmod \Delta_2$.*

Proof. This is similar to the proof that $R \subset S$.

Actually, we can prove something stronger.

THEOREM 14. *Let Δ_1 and Δ_2 be two positive idempotents such that $\Delta_2 > \Delta_1$. Then $\alpha \; S \; \beta \bmod \Delta_1 \to \alpha \; T \; \beta \bmod \Delta_2$ but not conversely.*

Proof. Without loss of generality suppose $\alpha < \beta$. Then $(\forall b \in \beta)$ $(\forall d \notin \Delta_1)$ $(b - d \in \alpha)$. Choose $d \in \Delta_2$ such that $d \notin \Delta_1$. Then $(\forall b \in \beta)$ $(b - d) \in \alpha$ which is just what we need. Finally $d \; T \; 0 \bmod \Delta_2$ but $d \not S$ $0 \bmod \Delta_1$ since $d + (-\Delta_1) > 0$ because $0 \in d + (-\Delta_1)$ whereas $0 + (-\Delta_1)$ $= -\Delta_1 < 0$.

Once more we fix Δ.

THEOREM 15. *S is the smallest congruence relation with respect to addition and negation containing R.*

Proof. We know that S is a congruence relation containing R.

Let $\sim$ be any congruence relation containing R. Then $\Delta \sim 0$.

Since $\sim$ is a congruence relation $-\Delta \sim -0 = 0$. Now suppose $\alpha \; S \; \beta$. Then $\alpha + (-\Delta) = \beta + (-\Delta)$. Hence $\alpha = \alpha + 0 \sim \alpha + (-\Delta) = \beta + (-\Delta)$ $\sim \beta + 0 = \beta$.

THEOREM 16. *Any convex congruence relation $\sim$ containing T properly must contain S.*

Proof. By Theorem 15 it suffices to show that it contains R. Assume $\alpha < \beta$. Suppose $\alpha \sim \beta$ but $\alpha \not T \beta$. Then $(\forall d \in \Delta)$ $(\alpha + d \subset \beta)$. Hence $\alpha + \Delta \leq \beta$. Since $\alpha \leq \alpha + \Delta \leq \beta$ it follows by convexity that $\alpha + \Delta \sim \alpha$.

Case 1. $\alpha + \Delta > \alpha$. Then add $-\alpha$ to both sides of the congruence $\alpha + \Delta \sim \alpha$ to obtain $-\alpha + (\alpha + \Delta) \sim (-\alpha) + \alpha \leq 0$. Choose $x \in \alpha + \Delta$ such that $x \notin \alpha$ and $d \in \Delta$. Then $x + d \in (\alpha + \Delta) + \Delta = \alpha + \Delta$. Also $-x \in -\alpha$. [As we did earlier we can use the technicality of using $y > x$ to justify the last statement. As an alternative we can note that the implication $x \notin \alpha \to -x \in -\alpha$ breaks down only when $\alpha \in R^*$ in which case what we are aiming for is trivial anyway]. Hence $d = (x + d) + (-x) \in$ $(\alpha + \Delta) + (-\alpha)$. Thus $\Delta \leq (\alpha + \Delta) + (-\alpha) \sim \alpha + (-\alpha) \leq 0$. So by convexity $\Delta \sim 0$.

Case 2. $\alpha + \Delta = \alpha$. Then $\Delta \leq \text{ab}(\alpha) = \text{ab}(-\alpha)$. Hence $(-\alpha) + \Delta = -\alpha$. Now $\beta \sim \alpha$. Hence $\beta + (-\alpha) \sim \alpha + (-\alpha) \leq 0$. Let $x \in \beta$ such that $x \notin \alpha$. Then $-x \in -\alpha$ so $0 = x + (-x) \in \beta + (-\alpha)$. Therefore $\beta + (-\alpha) > 0$. Also $\beta + (-\alpha) = \beta + (-\alpha) + \Delta \geq 0 + \Delta = \Delta$. By convexity $\Delta \sim 0$.

So in either case we obtain $\Delta \sim 0$. If $\alpha \; R \; \beta$ then $\alpha + \Delta = \beta + \Delta$. Hence $\alpha \sim \alpha + \Delta = \beta + \Delta \sim \beta$. Therefore the congruence relation contains R.

We shall now classify all convex congruence relation on $R^{\#}$. Given any such relation $\sim$ let $\Delta = (\alpha \in R^*\colon \alpha \geq 0 \text{ and } \alpha \sim 0)$. (We use the embedding $R^* \to R^{\#}$ to make the natural identifications. Now $\alpha \sim 0$ and $\beta \sim 0$ imply $\alpha + \beta \sim 0$ hence Δ generates an idempotent if we annex all negative numbers in R^* to Δ.

Assume $\alpha < \beta$. If $\alpha \; T \; \beta$ then $\beta \leq \alpha + d$ for some $d \in \Delta$. Since $d \sim 0$ we have $\alpha \sim \alpha + d$ and $\alpha \leq \beta \leq \alpha + d$. Hence $\alpha \sim \beta$. This shows that $\sim$ contains T.

We now show that S contains $\sim$. Otherwise suppose there exist α, β such that $\alpha \sim \beta$ but $\alpha \not{S} \beta$.

Then $\alpha < \beta + (-\Delta)$. Let $x \in \beta + (-\Delta)$ such that $x \notin \alpha$. Then $x = b + (-e)$, with $b \in \beta$, $-e \in -\Delta$. $\alpha \leq x < b \leq \beta$ (this follows since $x \notin \alpha \to \alpha \leq x$ and $b \in \beta \to b < \beta$).

Since $\alpha \sim \beta$ it follows by convexity that $x \sim b$ i.e. $b - x \sim 0$. However, $b - x = e$. Since $-e \in -\Delta$, $e \notin \Delta$. We have deduced that $e \notin \Delta$ but $e \sim 0$ which contradicts the definition of Δ.

We have shown that $T \leq \sim \leq S$. By Theorem 16 $\sim$ must be T or S. Conversely if $\alpha \; T \; \beta \bmod \Delta$ or $\alpha \; S \; \beta \bmod \Delta$ then the above process gives rise to Δ itself; i.e. $[a \geq 0\colon a \in R^* \text{ and } a \; T \; 0] = [a \geq 0\colon a \in R^* \text{ and } a \; S \; 0] = \Delta$ (strictly speaking the non-negative elements in Δ).

Special Kinds of Idempotents. Let $a \in R^*$ such that $a > 0$. Then a gives rise to two idempotents in a natural way.

Let $A_a = [x\colon (\exists \text{ standard integer } n)\, (x \leq na)]$.

Let $B_a = [x\colon (\forall \text{ positive standard reals } r)\, (x \leq ra)]$.

Then it is immediate that A_a and B_a are idempotents. (The usual "$\varepsilon/2$ argument" shows this for B_a.) It is also clear that A_a is the smallest idempotent containing a and B_a is the largest idempotent not containing a. It follows that B_a and A_a are consecutive idempotents. Note that $B_1 = \mu$ which we have already considered. A_1, which is the set of finite numbers (plus all negative numbers) is what is called ϕ in **[2]**.

THEOREM 17. (a) *No idempotent of the form A_a has an immediate successor.*

(b) *All consecutive pairs of idempotents have the form* B_a *and* A_a *for some* $a \in R^*$.

Proof. (a) Let $A_a \subset \Delta$ but $A_a \neq \Delta$. Suppose $x \in \Delta$ but $x \notin A_a$. Then $x \geq na$ for all positive standard integers n. Let $y = \sqrt{xa}$ which is defined since R^* is a nonstandard model of R. Then $y \geq \sqrt{n}\ a$ for all positive standard integers n so that $y \notin A_a$. So $A_y > A_a$. Similarly $x \geq \sqrt{n}\,y$ so $x \notin A_y$. Hence $A_y < \Delta$. Thus A_a and Δ are not consecutive.

(b) Let C and D be consecutive idempotents such that $C < D$. Let $a \in D$ with $a \notin C$. Then $C \leq B_a < A_a \leq D$. Hence $C = B_a$ and $D = A_a$.

Types of α for Given ab(α). Among elements of α such that ab(α) = Δ we can distinguish two types. (Assume $\Delta > 0$).

(1) α has type 1 if $(\exists x \in \alpha)\,(\forall y)\,[x + y \in \alpha \to y \in \Delta]$,

(2) α has type 2 if $(\forall x \in \alpha)\,(\exists y \notin \Delta)\,(x + y \in \alpha)$, i.e. α has type 2 iff α does not have type 1.

A similar classification exists from above.

(1) α has type 1A if $(\exists x \notin \alpha)\,(\forall y)\,(x - y \notin \alpha \to y \in \Delta)$,

(2) α has type 2A if $(\forall x \notin \alpha)\,(\exists y \notin \Delta)\,(x - y \notin \alpha)$.

Again α has type 2A iff α does not have type 1A.

THEOREM 18. *α has type* 1 *iff* $-\alpha$ *has type* 1A.

Proof. This is straightforward and left to the reader.

EXAMPLES. μ has type 1 thus $-\mu$ has type 1A. Also $-\mu$ has type 2, thus μ has type 2A. One can think of type 2 as corresponding to a slightly stronger absorption power than type 1.

THEOREM 19. *α cannot have type* 1 *and type* 1A *simultaneously.*

Proof. Let α have both types. Choose $a \in \alpha$, $b \notin \alpha$ satisfying the conditions for both types respectively. Consider $(a + b)/2$. Since ab(α) = Δ, $b - a \notin \Delta$. Hence $(b - a)/2 \notin \Delta$; i.e. $(a + b)/2 - a \notin \Delta$ and $b - (a + b)/2 \notin \Delta$. By choice of a the first statement gives $(a + b)/2 \notin \alpha$ and by the choice of b the second statement gives $(a + b)/2 \in \alpha$. Thus we obtain a contradiction.

THEOREM 20. *Suppose* ab(α) $= \Delta > 0$. *Then α has type* 1 *iff α has the form* $a + \Delta$ *for some* $a \in R^*$.

Proof. Let $\alpha = a + \Delta$. Then $\mathrm{ab}(\alpha) = \Delta$ by Theorem 4, (2). Since $\Delta > 0$, $a \in a + \Delta$ [we chose $d \in \Delta$ such that $0 < d$ and write a as $(a - d) + d$]. It is clear that a works to show that α has type 1.

Conversely, suppose α has type 1 and choose $a \in \alpha$ such that $a + y \in \alpha \to y \in \Delta$. Then we claim that $\alpha = a + \Delta$. By definition of $\mathrm{ab}(\alpha)$ certainly $a + \Delta \leq \alpha$. On the other hand by choice of a, every element of α has the form $a + d$ with $d \in \Delta$. Choose $d' \in \Delta$ such that $d' > d$, then $a + d = [a - (d' - d)] + d' \in a + \Delta$. Hence $\alpha \leq a + \Delta$. Therefore $\alpha = a + \Delta$.

COROLLARY. *α has type* 1A *iff α has the form $a + (-\Delta)$.*

Proof. α has type 1A

iff $-\alpha$ has type 1,
iff $-\alpha$ has the form $a + \Delta$,
iff α has the form $(-a + \Delta) = -a + (-\Delta)$.

THEOREM 21. (a) *If* $\mathrm{ab}(\alpha) > \mathrm{ab}(\beta)$ *then* $\alpha + \beta$ *has type* 1 *iff* α *has type* 1.

(b) *If* $\mathrm{ab}(\alpha) = \mathrm{ab}(\beta)$ *then* $\alpha + \beta$ *has type* 2 *iff either* α *or* β *has type* 2.

Proof. (a) Suppose α has type 2. Let $a + b \in \alpha + \beta$ where $a \in \alpha$ and $b \in \beta$. Since α has type 2 there exists $d \notin \mathrm{ab}(\alpha) = \mathrm{ab}(\alpha + \beta)$ such that $a + d \in \alpha$. Then $(a + b) + d = (a + d) + b \in \alpha + \beta$. Hence $\alpha + \beta$ has type 2.

Now suppose α has type 1. Using Theorem 20 choose $a \in \alpha$ such that $\alpha \subset a + \mathrm{ab}(\alpha)$. Choose $d \in \mathrm{ab}(\alpha)$ such that $d \notin \mathrm{ab}(\beta)$. $(\exists b \in \beta)\ (b + d \notin \beta)$. Hence $\beta \subset b + d \leq b + \mathrm{ab}(\alpha)$. We thus obtain $\alpha + \beta < [a + \mathrm{ab}(\alpha)] + [b + \mathrm{ab}(\alpha)] = (a + b) + \mathrm{ab}(\alpha) = a + b + \mathrm{ab}(\alpha + \beta)$. This completes the proof of (a).

(b) This is similar to the proof of (a). If α or β has type 2 then we can follow the proof of the fist part of (a) exactly. It both α and β have type 1 we obtain as in the above proof $\alpha \leq a + \mathrm{ab}(\alpha)$ and $\beta \leq b + \mathrm{ab}(\beta)$. Hence $\alpha + \beta \leq a + b + \mathrm{ab}(\alpha + \beta)$. [Since $\mathrm{ab}(\alpha) = \mathrm{ab}(\beta)$ both are equal to $\mathrm{ab}(\alpha + \beta)$.]

THEOREM 22. *If* $\mathrm{ab}(\alpha)$ *has the form* B_a *then* α *has type* 1 *or* 1A.

Proof. Incidentally, we already know that in general $\mathrm{ab}(\alpha)$ cannot have type 1 and 1A simultaneously. Now $a \in B_a$. Hence $(\exists b \in \alpha, c \notin \alpha)$ $(c - b = a)$.

We now define an *ordinary* Dedekind cut for the real numbers as follows. Let $r \in L$ (where L is the set of lower elements) iff $b + ra \in \alpha$.

It is immediate that $0 \in L$, $1 \notin L$, $x < y \in L \rightarrow x \in L$. So we have a Dedekind cut. Then L has a maximum or L' has a minimum.

Suppose first that L has a maximum r. Then $b + ra \in \alpha$ but for any real $s > r$, $b + sa \notin \alpha$. We now claim that $b + ra$ works to show that α has type 1. In fact, suppose $b + ra + x \in \alpha$. Let $s > r$. Since $b + sa \notin \alpha$, $b + sa > b + ra + x$. Therefore $x < (s - r)a$. Thus $x < \varepsilon a$ for every positive real ε; i.e. $x \in B_a$.

A similar argument shows that α has type 1A if L' has a minimum.

The result applies in particular to $B_1 = \mu$. It follows from Theorem 20 and its corollary that every α with $\mathrm{ab}(\alpha) = \mu$ must be either of the form $a + \mu$ or $a + (-\mu)$ with $a \in R^*$.

More General Functions. We first make some preliminary remarks on the closure operator.

Let $\mathrm{cl}\, S = \{x: (\exists y \in S)(x \leq y)\}$. Then $\mathrm{cl}\, s$ satisfies the usual axioms for a closure operation. If $S \neq \phi$, $S' \neq \phi$ and S has no maximum, then $\mathrm{cl}\, S \in R^{\#}$.

Let f be a continuous strictly increasing function in each variable from a subset of R^n into R. Specifically, we want the domain to be the cartesian product $\prod_{i=1} A_i$ where $A_i = (x: x > a_i)$ for some $a_i \in R$ and the range is $A = [x: x > a]$ for some $a \in R$. [Some of the a's may be $-\infty$; i.e. some of the A's may be R.] By transfer f extends to a function from the corresponding subset of R^{*n} into R^* which is also strictly increasing in each variable and continuous in the Q topology (i.e. ε and δ range over arbitrary positive elements in R^*).

We now extend f to $R^{\#}$. Let $\alpha_i > a_i$ then

$$\bar{f}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \mathrm{cl}[f(b_1, b_2, \ldots, b_n): a_i < b_i \in \alpha_i].$$

Note first that this really defines an element of $R^{\#}$. $f(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is clearly non-empty. If $c_i \notin \alpha_i$, then $f(b_1, b_2, \ldots, b_n) < f(c_1, c_2, \ldots, c_n)$ for all $b_i \in \alpha_i$ since f is strictly increasing, so $f(c_1, c_2, \ldots, c_n) \notin \bar{f}(\alpha_1, \alpha_2, \ldots, \alpha_n)$. Again since f is strictly increasing $\bar{f}(\alpha_1, \ldots, \alpha_n)$ has no maximum. Thus $\bar{f}(\alpha_1, \alpha_2, \ldots, \alpha_n) \in R^{\#}$.

EXAMPLES. Addition, in which case the domain is all of R^2. Multiplication in which case $a_1 = a_2 = a = 0$. In both cases the present definition agrees with the one given earlier. A new example of special interest is $\exp x$. In this case the domain is R and $a = 0$. More generally we have the binary function a^b for $a > 1$.

It is clear that $\bar{f}$ is strictly increasing. By continuity (specifically left continuity) $\bar{f}$ is an extension of f using the above embedding $R^* \to R^\#$.

THEOREM 23. *If f and g are functions of one variable $\overline{fg} = \bar{f}\bar{g}$.*

Proof.

$$\overline{fg}(\alpha) = \text{cl}\, fg(\alpha),$$
$$\bar{f}\bar{g}(\alpha) = \bar{f}\text{cl}\, g(\alpha) = \text{cl}\, f\text{cl}\, g(\alpha).$$

Since f is an increasing functoin both sets are the same.

The result can be generalized to more general compositions involving functions of several variables. The notation is cumbersome but the proof is essentially the same. Note also the obvious fact that the identity function extends to the identity function.

We state the next theorem for a function of two variables and one parameter in R^* for convenience in notation although the theorem can be obviously extended to functions of any number of variables and any set of parameters in R^*.

THEOREM 24. *Let f be a function of two variables. Then $\bar{f}(\alpha, a) =$ $\text{cl}[f(b, a)\colon b \in \alpha]$.*

Proof. Using the embedding $R^* \to R^\#$ we have

$$f(\alpha, a) = \text{cl}[f(b, c)\colon b \in \alpha, c < a].$$

Since f is increasing it is clear that $f(\alpha, a) \subset \text{cl}[f(b, a)\colon b \in \alpha]$. Now consider any element of the form $f(b, a)$ with $b \in \alpha$. Let $b' > b$. Then $f(b', a) > f(b, a)$. By continuity of f there exists $c < a$ such that $f(b', c)$ $> f(b, a)$. Hence $f(b, a) \in \text{cl}\, f(\alpha, a)$. Hence $\{f(b, a)\colon b \in \alpha\} \subset$ $\text{cl}\, f(\alpha, a)$.

Another convenient fact is the following which we state for functions of two variables again for simplicity of notation. $f(\alpha, \alpha) = \text{cl}[f(a, a)\colon$ $a \in \alpha]$. By definition $f(\alpha, \alpha) = \text{cl}[f(a, b)\colon a, b \in \alpha]$. It is clear that the former set is included in the latter. On the other hand $f(a, b) \leq$ $f[\max(a, b), \max(a, b)]$ since f is increasing. So the latter set is included in the former.

EXAMPLE. As a consequence of Theorem 23 and the above remark (generalized to any number of variables) we have

$$\alpha\beta + \alpha\gamma = (ab + ac\colon a \in \alpha,, b \in \beta, c \in \gamma).$$

We are now ready for a general transfer theorem.

THEOREM 25. *Let f and g be any two terms obtained by compositions of strictly increasing continuous functions possibly containing parameters in R^*. Then any relation $f = g$ or $f \leq g$ valid in R^* extends to $R^\#$*

Proof. This follows from the above discussion.

EXAMPLES. All identities for addition and multiplication such as the distributive law restricted to positive elements. Of special interest to us are the laws of exponents $\exp(\alpha + \beta) = \exp\alpha \exp\beta$, $(\exp\alpha)^\beta = \beta\exp\alpha$, and $(\alpha^\beta)^\gamma = \alpha^{\beta\gamma}$. [Of course, we must always beware of the restriction in the domain when it comes to multiplication]. Note that Theorem 24 justifies the equality $\exp x = e^x$. We also have $\ln\exp\alpha = \exp\ln\alpha = \alpha$.

Finally, we never used right continuity in our discussion so Theorem 25 can be obviously generalized. However, this added generality is of no special interest to us at this time.

As an application of transfer applied to the exponential and logarithmic function we can reduce the study of multiplicative idempotents to that of additive idempotents.

THEOREM 25. *The map $\alpha \to \exp\alpha$ maps the set of additive idempotents onto the set of all multiplicative idempotents other than* 0.

Proof. Suppose $\alpha + \alpha = \alpha$. Then $\exp\alpha\exp\alpha = \exp(\alpha + \alpha) = \exp\alpha$. Now suppose $\alpha > 0$ and $\alpha^2 = \alpha$. Then $\ln\alpha$ is defined $\ln(\alpha) + \ln(\alpha) = \ln(\alpha^2) = \mathrm{lm}\,\alpha$. Hence $\ln\alpha$ is an additive idempotent. The proof is completed by noting the identity $\exp(\ln\alpha) = \alpha$.

Similarly, multiplicative absorption can be defined and reduced to the study of additive absorption. Incidentally the map $\alpha \to \exp\alpha$ is essentially the same as the map in [**1**, Theorem 6] which is the map from the set of ideals onto the set of all prime ideals of the valuation ring consisting of the finite elements of R^*.

The existence of nonzero additive idempotents illustrate a limitation of transfer. Another example is the following. $\exp x > x$ for all $x \in R$ and hence by transfer for all $x \in R^*$. However $\exp\phi = \phi$ (recall that $\phi = A_1$). Also, if R^* is a comprehensive enlargement of R, then *any* strictly increasing continuous function f such that $f(x) \geq x$ for all $x \in R$ has a fixed point in $R^\#$; i.e. an α such that $\bar{f}(\alpha) = \alpha$. To see this, let $\beta \in R^\#$ be arbitrary. We define f_n inductively by $f_0(x) = x$ and $f_{n+1} = ff_n(x)$. Then let $\alpha = (\bigcup_{n=0}^{\infty} f_n(x);\, x \in \beta)$. We claim that $\alpha \in R^\#$. The only thing that is not immediately obvious is that $\alpha' \neq \varnothing$. Let $x \notin \beta$. Consider

$\{x, f_1(x), f_2(x) \cdots f_n(x) \cdots\}$. Since R^* is a comprehensive enlargement of R $(\exists y \in R^*)$ $(\forall n)$ $[y > f_n(x)]$. Then $y \notin \alpha$. It is easy to see that $f(\alpha) = \alpha$.

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