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NOT EVERY LODATO PROXIMITY IS COVERED

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In a recent paper Reed wrote, "In fact it may be that all Lodato proximities are covered. I was unable to find a counterexample". (Remark 1.10)

The purpose of this note is to show that, in general, Lodato proximities are not covered.

1. Preliminaries. A *closed filter* $\mathscr{F}$ on a topological space (X, c) is a proper filter (that is, a filter which does not contain the empty set) which has a base consisting of only closed sets. Maximal (with respect to set inclusion) closed filters are all called *ultraclosed filters*. For more information on the concept of ultraclosed filters see Thron [**3**].

Ultrafilters are maximal proper filters on a set and *grills* are exactly the unions of ultrafilters. For a detailed discussion on ultrafilters and grills, see Thron [**2**].

A *basic proximity* π on a set X is a symmetric binary relation on the power set $\mathscr{P}(X)$ of X satisfying the conditions:

$$(A, B \cup C) \in \pi \Leftrightarrow (A, B) \in \pi \quad \text{or } (A, C) \in \pi,$$
$$A \cap B \neq \varnothing \Rightarrow (A, B) \in \pi,$$
$$(A, \varnothing) \notin \pi, \quad \forall A \subset X.$$

The pair (X, π) is called a *basic proximity space* provided π is a basic proximity on X.

For a basic proximity π on X, we define

$$c_\pi(A) = \{x \in X: (\{x\}, A) \in \pi\} \quad \text{for all } A \subset X.$$

It is easily verified that c_π is a symmetric (Čech) closure operator. For a basic proximity π, c_π need not be a Kuratowski closure operator.

A basic proximity π on X is called a *Lodato proximity* if the following condition is saitsfied:

$$(c_\pi(A), c_\pi(B)) \in \pi \Rightarrow (A, B) \in \pi.$$

If π is a Lodato proximity on X then c_π is a Kuratowski closure operator on X and hence (X, c_π) is a topological space.

Let (X, π) be a basic proximity space and $\mathscr{G}$ be a grill on X. Then $\mathscr{G}$ is called a *π-clan* if

$$(A, B) \in \pi \quad \text{for all } A, B \text{ in } \mathscr{G}.$$

For more detailed information on the concepts discussed above, see Thron [**2**].

Let π be a Lodato proximity on X. Following Reed [**1**] we define the following concepts:

A *Wallman π-clan* is a π-clan which contains some ultraclosed filter. The proximity π is said to be *covered* if for each $(A, B) \in \pi$ there exists a Wallman π-clan $\mathcal{G}$ such that $\{A, B\} \subset \mathcal{G}$.

We conclude this section by proving the following results which will be used to make the final conclusion.

1.1. PROPOSITION. *Let $\mathcal{U}$ be an ultraclosed filter on (X, c) and $\mathcal{A}$ a base of $\mathcal{U}$ consisting of closed sets. If F is a closed set and $F \cap A \neq \varnothing$ for all A in $\mathcal{A}$ then $F \in \mathcal{U}$.*

Proof. Let $\mathcal{B}$ be the collection of all finite intersections of members of the family $\mathcal{A} \cup \{F\}$. Then $\mathcal{B}$ is a filter base consisting of closed sets. Let $\mathcal{U}_0$ be the filter generated by $\mathcal{B}$ as a base. Then $\mathcal{U}_0$ is a closed filter and $\mathcal{U}_0 \supset \mathcal{U} \cup \{F\}$. By the maximality of $\mathcal{U}$ it follows that $F \in \mathcal{U}$.

1.2. COROLLARY. *Let $\mathcal{U}$ be an ultraclosed filter on (X, c) and V an open set such that $V \cap F \neq \varnothing$ for all F in $\mathcal{U}$. Then $V \in \mathcal{U}$.*

Proof. If possible suppose that $V \notin \mathcal{U}$. Let $\mathcal{A}$ be a base of $\mathcal{U}$ consisting of closed sets. Then $V \not\supset A$ for all $A \in \mathcal{A}$. Thus $(X - V) \cap A \neq \varnothing$ for all $A \in \mathcal{A}$. Since $X - V$ is closed, by the above result it follows that $X - V \in \mathcal{U}$ and hence $V \cap (X - V) \neq \varnothing$—a contradiction.

1.3. PROPOSITION. *On a compact topological space (X, c) every ultraclosed filter converges.*

Proof. Let $\mathcal{U}$ be an ultraclosed filter on (X, c). Since the space is compact if follows that there exists an x in X such that $x \in c(F)$ for all $F \in \mathcal{U}$. Let V be an open neighbourhood of x. Then $V \cap F \neq \varnothing$ for all $F \in \mathcal{U}$. Thus by the above corollary, $V \in \mathcal{U}$. Hence $\mathcal{U}$ converges to x.

1.4. PROPOSITION. *On a T_1-space (X, c), every convergent ultraclosed filter has the form $\mathcal{U}(x)$, for some $x \in X$, where $\mathcal{U}(x) = \{A \subset X : x \in A\}$.*

Proof. Let $\mathcal{U}$ be an ultraclosed filter on (X, c) such that it converges to a point $x \in X$. Obviously $x \in c(F)$ for all $F \in \mathcal{U}$. Hence, in particular,

x belongs to each member of a base of $\mathscr{U}$ consisting of closed sets. Since $\{x\}$ is a closed set it follows by Proposition 1.1, that $\{x\} \in \mathscr{U}$. Thus $\mathscr{U} = \mathscr{U}(x)$.

1.5. THEOREM. *Let (X, c) be a compact T_1-space such that it has two infinite components. Then*

$$\pi = \{(E, F) \colon c(E) \cap c(F) \neq \varnothing \text{ or } E \text{ and } F \text{ are both infinite}\}$$

is a Lodato proximity on X such that $c_\pi = c$ and π is not covered.

Proof. It is easy to verify that π is indeed a Lodato proximity on X such that $c_\pi = c$.

Let A, B be two infinite components of (X, c). Obviously $(A, B) \in \pi$. However, no Wallman π-clan can contain both A and B. For suppose $\mathscr{G}$ is such a Wallman π-clan. Let $\mathscr{U}$ be an ultraclosed filter such that $\mathscr{U} \subset \mathscr{G}$. Then since (X, c) is a compact T_1-space it follows, by Propositions 1.3 and 1.4, that $\mathscr{U} = \mathscr{U}(x)$ for some $x \in X$. Thus $\{x\}$, A and B are all in $\mathscr{G}$. From this it follows that

$$x \in c_\pi(A) \cap c_\pi(B) = c(A) \cap c(B) = A \cap B = \varnothing.$$

Clearly this is impossible.

2. Many examples of compact T_1-spaces with two infinite components can easily be constructed. Two such examples are given below.

2.1. EXAMPLE. Let X be the union of closed intervals $[1, 2]$ and $[3, 4]$. Then X with the topology induced by the usual topology of real line is an example of a compact T_1-space with two infinite components.

2.2. EXAMPLE. Let $X = A \cup B$ such that A, B are both infinite sets and $A \cap B = \varnothing$. Define $c \colon \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\begin{aligned} c(D) &= D && \text{if } D \text{ is a finite subset of } X, \\ &= A \cup D && \text{if } A \cap D \text{ is infinite and } B \cap D \text{ is finite}, \\ &= B \cup D && \text{if } B \cap D \text{ is infinite and } A \cap D \text{ is finite}, \\ &= X && \text{otherwise.} \end{aligned}$$

Then (X, c) is a T_1-topological space with two infinite components A and B.

Set

$$\mathscr{A}_1 = \{A - F \colon F \text{ is a finite subset of } A\},$$

$$\mathscr{A}_2 = \{B - F \colon F \text{ is a finite subset of } B\},$$

$$\mathscr{A}_3 = \{X - F \colon F \text{ is a finite subset of } X\}.$$

Then $\mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3$ is the collection of all nonempty open sets in (X, c). Let $\mathscr{A}$ be an open cover of X. If $\mathscr{A} \cap \mathscr{A}_3 \neq \varnothing$ then obviously $\mathscr{A}$ has a finite subcover. If $\mathscr{A} \cap \mathscr{A}_3 = \varnothing$ then, since $\mathscr{A}$ covers X, $\mathscr{A} \cap \mathscr{A}_1 \neq \varnothing$ and $\mathscr{A} \cap \mathscr{A}_2 \neq \varnothing$ and hence in this case also $\mathscr{A}$ has a finite subcover. Thus the space is compact.

2.3. Remark. By Theorem 1.5 and Examples 2.1 and 2.2 it follows that there are Lodato proximities which are not covered.

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References

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