

Hidden $U_q(\mathfrak{sl}(2)) \otimes U_q(\mathfrak{sl}(2))$ Quantum Group Symmetry in Two Dimensional Gravity*

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Abstract: In a previous paper, the quantum-group-covariant chiral vertex operators in the spin 1/2 representation were shown to act, by braiding with the other covariant primaries, as generators of the well known $U_q(\mathfrak{sl}(2))$ quantum group symmetry (for a single screening charge). Here, this structure is transformed to the Bloch wave/Coulomb gas operator basis, thereby establishing for the first time its quantum group symmetry properties. A $U_q(\mathfrak{sl}(2)) \otimes U_q(\mathfrak{sl}(2))$ symmetry of a novel type emerges: The two Cartan-generator eigenvalues are specified by the choice of matrix element (bra/ket Verma-modules); the two Casimir eigenvalues are equal and specified by the Virasoro weight of the vertex operator considered; the co-product is defined with a matching condition dictated by the Hilbert space structure of the operator product. This hidden symmetry possesses a novel Hopf-like structure compatible with these conditions. At roots of unity it gives the right truncation. Its (non-linear) connection with the $U_q(\mathfrak{sl}(2))$ previously discussed is disentangled.

1. Introduction

Quantum integrability as we know it is essentially synonymous to the concept of R matrix and Yang-Baxter relations. While it is not known whether the latter always possess a group-theoretical interpretation, it is widely believed that this is true at least for the subclass of conformal integrable systems; well-known examples are given by the minimal models, the WZW models and Liouville/Toda theory, where the underlying symmetries are indeed known to be given by quantum groups [1–3]. However, in spite of extensive studies [4–6] our understanding of the quantum group symmetry in these

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theories is still somewhat incomplete as we lack an explicit realization of the symmetry generators as operators on the Hilbert space, similar to the representation of classical symmetry groups in conventional field theory. In a first paper [7], we have analyzed this question within the context of 2d gravity, and proposed a novel approach involving position-dependent symmetry generators. Perhaps the most striking feature of these generators is the fact that they are given in terms of the same operators that form irreducible representations of the quantum group $U_q(sl(2))$; more precisely, the basic generators J_{\pm}, J_3 were seen to be related to the fundamental representation of spin $\frac{1}{2}$, while the full enveloping algebra arises from tensor products of the latter and thus involves all the higher spins.

The covariant operator basis [3] used in ref. [7] has the appealing property of consisting of conformal primaries only, while in previous work it was found necessary to introduce vertex operators that are not fully covariant under the Virasoro symmetry [13]. The latter, on the other hand, have a somewhat simpler structure as they are given directly in terms of the familiar Coulomb gas vertex operators used for the free field description of rational conformal field theories. Remarkably, a conformally covariant version of the latter is known to constitute an alternative, equivalent operator basis for the description of 2d gravity, which we will call the Bloch wave basis¹; however, "conformal covariantization" does not conserve the transformation behaviour of these vertex operators under the quantum group. The question therefore arises naturally how the quantum group symmetry is realized in terms of the Bloch wave operators, and what is the relation between the symmetry generators in both pictures; this is the basic theme of the present paper. From a Hamiltonian point of view, one would expect that the symmetry generators are simply the same, as both sets of fields are living in the same Hilbert space. However, it turns out that there is a more natural realization of the quantum group symmetry on the Bloch waves, where the generators are also given by Bloch wave fields. Here the free field zero mode will play the role of the generator J_3 in the Bloch wave basis, just as in the conventional description, while J_{\pm} will be given in terms of the Bloch wave operators of spin $\frac{1}{2}$.

The structure we find in the Bloch wave basis proves to be much more intricate than in the covariant one: On the one hand, the commutation relations of the generators turn out to be essentially (up to central terms) those of $U_{\sqrt{q}}(sl(2))$, rather than $U_q(sl(2))$. On the other hand, these generators induce a symmetry of the operator algebra of the Bloch waves which is larger than the one of their commutation relations. This "internal" symmetry has a natural description in terms of $U_q(sl(2)) \otimes U_q(sl(2))$ ². We carefully show, using our explicit constructions, how these three algebras are related, both $U_q(sl(2))$ and $U_{\sqrt{q}}(sl(2))$ being non linearly "embedded" in our $U_q(sl(2)) \otimes U_q(sl(2))$, which is of a novel type. The fact that the Bloch waves are intertwining operators from the conformal point of view leads to two additional constraints: There is a matching condition for the magnetic indices of the two factors, and the Casimir eigenvalues of the two representations involved must be equal. Remarkably, there exists a new Hopf like structure, different from the standard one of $U_q(sl(2)) \otimes U_q(sl(2))$, which is consistent with these conditions.

¹ This name is motivated by the fact that these fields have well-defined monodromy properties.

² Note that this structure has nothing to do with the $U_q(sl(2)) \odot U_{\bar{q}}(sl(2))$ discussed in ref. [7], which arises from considering both semiclassical and non-semiclassical deformations of $sl(2)$, or in the language of 2d gravity, both screening charges. In the present paper, we will restrict ourselves to a single screening charge throughout.

The paper is organized as follows. In Sect. 2, we recall the essential points of the analysis of the first paper, and anticipate the general form of the relation between the generators in the two pictures. In Sect. 3, we work out explicitly the passage from the covariant to the Bloch wave generators by means of a nonlinear redefinition, and establish the commutation relations of the latter. In Sect. 4, we show that the action of the new generators on the Bloch waves can be described in terms of two commuting sets of matrices, both of which fulfill the commutation relations of $U_q(sl(2))$. We work out the new coproduct structure induced by the intertwining constraints, which makes this matrix algebra different from $U_q(sl(2)) \otimes U_q(sl(2))$ as a Hopf algebra. We characterize the symmetries of the operator algebra generated by these matrices and explain that the q 6j symbols describing the fusion of the Bloch wave fields can be viewed as Clebsch-Gordan coefficients for the new symmetry structure. In Sect. 5, we will show how this new symmetry directly connected with the Liouville zero mode can be used to classify the spectrum of primary fields and associated Verma modules. We will also discuss partially the case of q root of unity and show that the corresponding representation of $U_q(sl(2)) \otimes U_q(sl(2))$ gives the right truncation for the spectrum of zero modes. As an application of the formalism, we derive the transformation laws of the Coulomb gas operators heavily used in previous work on the quantum Liouville theory. In Sect. 6 we depart from the Liouville system and consider a conformal theory, as yet unknown, where the full $U_q(sl(2)) \otimes U_q(sl(2))$ would be operatorially realized. Some reasonable hypothesis allow us to derive the algebra of the quantum generators, which is a central extension of this internal symmetry group. Finally, in Sect. 7 we derive the full Hopf like structure using the properties of the operator algebra as a guide. Although it does not strictly obey the usual axioms, a coproduct, a counit and an antipode may be defined which satisfy very natural counterparts of the standard relations. We close with some open questions and indications of possible further developments along the lines of the present analysis.

2. Quantum Group Action in the Covariant Basis

We begin with a short recapitulation of the main results of ref. [7]. The starting point is the operator algebra of the chiral primaries $\xi_M^{(J)}$, which were constructed in refs.[3] and shown to form representations of spin J of $U_q(sl(2))$. The basic observation now is that the special operators $\xi_{\pm\frac{1}{2}}^{(\frac{1}{2})}$ can be viewed not only as covariant fields, but at the same time as generators. We define the following set of operators:

$$\mathcal{O}[J_+]_{\sigma_+} \equiv \kappa_+^{(+)} \xi_{\frac{1}{2}}^{(\frac{1}{2})}(\sigma_+), \quad \mathcal{O}[q^{J_3}]_{\sigma_+} \equiv \kappa_3^{(+)} \xi_{-\frac{1}{2}}^{(\frac{1}{2})}(\sigma_+), \quad (2.1)$$

where $\kappa_+^{(+)}$ and $\kappa_3^{(+)}$ are suitable normalization constants³, related by $\kappa_+^{(+)}/\kappa_3^{(+)} = q^{\frac{1}{2}}/(1 - q^2)$ and similarly

$$\mathcal{O}[J_-]_{\sigma_-} \equiv \kappa_-^{(-)} \xi_{-\frac{1}{2}}^{(\frac{1}{2})}(\sigma_-), \quad \mathcal{O}[q^{J_3}]_{\sigma_-} \equiv \kappa_3^{(-)} \xi_{\frac{1}{2}}^{(\frac{1}{2})}(\sigma_-). \quad (2.2)$$

From the braiding matrix of the $\xi_M^{(J)}$ fields, namely the universal R -matrix of $U_q(sl(2))$, it follows immediately that

³ In the previous paper, they have been denoted $\kappa_+^{(R+)}$, $\kappa_3^{(R+)}$. Since we will consider only the action of the generators to the right in this paper, the index R is dropped here and below.

$$\mathcal{O}[q^{J_3}]_{\sigma_{\pm}} \xi_M^{(J)}(\sigma) = \xi_N^{(J)}(\sigma) [q^{J_3}]_{NM} \mathcal{O}[q^{J_3}]_{\sigma_{\pm}}, \quad (2.3)$$

$$\mathcal{O}[J_{\pm}]_{\sigma_{\pm}} \xi_M^{(J)}(\sigma) = \xi_N^{(J)}(\sigma) [q^{-J_3}]_{NM} \mathcal{O}[J_{\pm}]_{\sigma_{\pm}} + \xi_N^{(J)}(\sigma) [J_{\pm}]_{NM} \mathcal{O}[q^{J_3}]_{\sigma_{\pm}}, \quad (2.4)$$

whenever $\sigma_+ > \sigma$, resp. $\sigma_- < \sigma$. Here $[J_{\pm}]_{NM}$ and $[q^{J_3}]_{NM}$ denote the usual representation matrices of the $U_q(sl(2))$ generators. Equations (2.3–2.4) describe the action of $U_q(sl(2))$ by coproduct, in accord with the general framework exposed in ref. [19]. Indeed, the $U_q(sl(2))$ coproduct $\Lambda(q^{\pm J_3}) = q^{\pm J_3} \otimes q^{\pm J_3}$, $\Lambda(J_{\pm}) = J_{\pm} \otimes q^{J_3} + q^{-J_3} \otimes J_{\pm}$, is immediately recognized to appear in Eqs. (2.3, 2.4), one of the generators being realized as a matrix, and the other one as an operator in the Hilbert space. Similarly, the action of the generators on products of fields is given by repeated application of the coproduct. On the other hand, the commutation relations of these generators were found to differ somewhat from the standard $U_q(sl(2))$ ones, essentially by central charges. They take the general form

$$q\mathcal{O}[J_+] \mathcal{O}[q^{J_3}] - \mathcal{O}[q^{J_3}] \mathcal{O}[J_+] = C_+, \quad (2.5)$$

$$\mathcal{O}[q^{J_3}] \mathcal{O}[J_-] - q^{-1} \mathcal{O}[J_-] \mathcal{O}[q^{J_3}] = C_-, \quad (2.6)$$

$$\mathcal{O}[J_+] \mathcal{O}[J_-] - \mathcal{O}[J_-] \mathcal{O}[J_+] = \mathcal{A}D + \frac{(\mathcal{O}[q^{J_3}])^2}{q - q^{-1}}, \quad (2.7)$$

where $\mathcal{A}D$ satisfies

$$C_+ \mathcal{O}[J_-] + C_- \mathcal{O}[J_+] = \mathcal{A}D \mathcal{O}[q^{J_3}] \quad (2.8)$$

and C_+, C_- are central charges. It was shown that this implies in particular that we can reexpress $\mathcal{O}[J_{\pm}]$ in terms of $\mathcal{A}D$ and $\mathcal{O}[q^{J_3}]$ and that there exist some relations between $\mathcal{A}D$ and $\mathcal{O}[q^{J_3}]$,

$$\mathcal{O}[q^{J_3}] \mathcal{A}D - q^{\mp 1} \mathcal{A}D \mathcal{O}[q^{J_3}] = \pm C_{\mp} (q - q^{-1}) \mathcal{O}[J_{\pm}], \quad (2.9)$$

$$\mathcal{O}[q^{J_3}] \mathcal{A}D^2 - (q + q^{-1}) \mathcal{A}D \mathcal{O}[q^{J_3}] \mathcal{A}D + \mathcal{A}D^2 \mathcal{O}[q^{J_3}] = C_+ C_- \mathcal{O}[q^{J_3}], \quad (2.10)$$

$$\mathcal{A}D \mathcal{O}[q^{J_3}]^2 - (q + q^{-1}) \mathcal{O}[q^{J_3}] \mathcal{A}D \mathcal{O}[q^{J_3}] + \mathcal{O}[q^{J_3}]^2 \mathcal{A}D = -C_+ C_- (q - q^{-1}). \quad (2.11)$$

While on a purely formal level Eqs. (2.5–2.7) can be transformed into the standard $U_q(sl(2))$ commutation relations by redefinitions of the generators that preserve the coproduct action, this turns out to be impossible in our field-theoretic realization where the central charges are actually nontrivial operators. Thus, the commutation relations are realized only in this weak sense. Another peculiarity of the field-theoretic realization is that ordinary commutators are to be replaced by what we call fixed point (FP) commutators. Indeed, the operatorial realization of Eq. (2.5) is given by

$$\mathcal{O}[q^{J_3}]_{\sigma_+} \mathcal{O}[J_+]_{\sigma'_+} - q \mathcal{O}[J_+]_{\sigma_+} \mathcal{O}[q^{J_3}]_{\sigma'_+} = -q^{\frac{1}{2}} \kappa_+^{(+)} \kappa_3^{(+)} \xi_0^{[\frac{1}{2}, \frac{1}{2}]^{(0)}}(\sigma_+, \sigma'_+), \quad (2.12)$$

where the central charge $\xi_0^{[\frac{1}{2}, \frac{1}{2}]^{(0)}}(\sigma_+, \sigma'_+) = \sum_M (\frac{1}{2}, M; \frac{1}{2}, -M | 0) \xi_M^{(\frac{1}{2})}(\sigma_+) \xi_{-M}^{(\frac{1}{2})}(\sigma'_+)$ commutes with all the ξ fields being the singlet formed from the product of two spin $\frac{1}{2}$ representations. Note that in Eq. (2.12), the arguments σ_+, σ'_+ of the operators are not exchanged, and this is precisely the meaning of the FP prescription. The number of positions that appears in a given product of generators can be thought of as some kind

of additive gradation of the formal algebra Eqs. (2.5–2.7), such that Eqs. (2.5, 2.6, 2.7) have grading 2 and Eq. (2.8) grading 3. ⁴ For $\sigma < \sigma_+, \sigma'_+$, it follows directly from the braiding relations, governed by the universal R matrix, that the quantity $\xi_0^{[\frac{1}{2}, \frac{1}{2}](0)}(\sigma_+, \sigma'_+)$ commutes with all $\xi_M^{(J)}(\sigma)$ and thus, comparing with Eq. (2.3), one identifies

$$C_+ = q^{\frac{1}{2}} \kappa_+^{(+)} \kappa_3^{(+)} \xi_0^{[\frac{1}{2}, \frac{1}{2}](0)}(\sigma_+, \sigma'_+). \tag{2.13}$$

The discussion for the other Borel subalgebra takes the same form with the obvious replacements, and one has $C_- = -C_+$. In order to write the FP commutator of $\mathcal{O}[J_+]$ and $\mathcal{O}[J_-]$, we have to define $\mathcal{O}[J_-]$ at points σ_+, σ'_+ (or $\mathcal{O}[J_+]$ at points σ_-, σ'_-). This is achieved by invoking the monodromy operation $\sigma \rightarrow \sigma + 2\pi$, which transforms the point $\sigma_- < \sigma$ into a point $\sigma_+ > \sigma$. The consistent definition is $\mathcal{O}[J_-]_{\sigma_+} = \kappa_-^{(+)} \xi_{-\frac{1}{2}}^{(\frac{1}{2})}(\sigma_+ - 2\pi)$, which gives, using the monodromy,

$$\mathcal{O}[J_-]_{\sigma_+} = \kappa_-^{(+)} \left[q^{-\frac{1}{2}}(q^\varpi + q^{-\varpi}) \xi_{-\frac{1}{2}}^{(\frac{1}{2})}(\sigma_+) - q^{-1} \xi_{\frac{1}{2}}^{(\frac{1}{2})}(\sigma_+) \right], \tag{2.14}$$

with $\kappa_-^{(+)} / \kappa_3^{(+)} = q^{-\frac{1}{2}} / (1 - q^{-2})$. ϖ is the zero mode of the free field underlying the construction of the $\xi_M^{(J)}$ fields. We remark that the zero mode ϖ does not enter the algebra of the ξ fields, which is given exclusively in terms of quantum group symbols, and thus the monodromy is the only place where it appears. Using Eq. (2.14) one then obtains a realization of Eqs. (2.8, 2.7):

$$C_+(\sigma'_+, \sigma''_+) (\mathcal{O}[J_-]_{\sigma_+} - \mathcal{O}[J_+]_{\sigma_+}) = \mathcal{O}[D]_{\sigma'_+, \sigma''_+} \mathcal{O}[q^{J_3}]_{\sigma_+} \tag{2.15}$$

and

$$\mathcal{O}[J_+]_{\sigma_+} \mathcal{O}[J_-]_{\sigma'_+} - \mathcal{O}[J_-]_{\sigma_+} \mathcal{O}[J_+]_{\sigma'_+} = \mathcal{O}[D]_{\sigma_+, \sigma'_+} + \frac{\mathcal{O}[q^{J_3}]_{\sigma_+} \mathcal{O}[q^{J_3}]_{\sigma'_+}}{q - q^{-1}}, \tag{2.16}$$

with

$$\mathcal{O}[D]_{\sigma_+, \sigma'_+} = \frac{1}{q - q^{-1}} 2 \cos(h\varpi) C_+(\sigma_+, \sigma'_+). \tag{2.17}$$

This shows that the free field zero mode ϖ which has no apparent relation to the generators of $U_q(sl(2))$ (and in particular doesn't seem to be connected with J_3 as one would expect from [13]), is in fact part of the enveloping algebra. In the ψ basis ϖ is shifted by the ψ fields (as J_3 by the ξ fields). This suggests that there may be another basis of generators T_\pm and T_3 , where T_3 could be realized by ϖ in the ψ basis. Let us show that this is in fact true at the formal level. We are looking for an algebra of the type

$$\begin{aligned} \mathcal{O}[T_\pm] \mathcal{O}[q^{T_3}] &= q^{\mp 1} \mathcal{O}[q^{T_3}] \mathcal{O}[T_\pm], \\ [\mathcal{O}[T_+], \mathcal{O}[T_-]] &= F(\mathcal{O}[q^{T_3}]), \end{aligned}$$

where F is a function to be determined. Equation(2.17) suggests to write

$$\mathcal{O}[D] = \alpha_D (\mathcal{O}[q^{\beta T_3}] + \mathcal{O}[q^{-\beta T_3}]), \tag{2.18}$$

⁴ Actually the relations of grading larger than two are not directly FP realized in our scheme, but in the special case $C_+ = -C_-$, $\mathcal{O}[D] \sim C_+$ we have here, cancellations occur which always allow to reduce the grading to less than or equal to 2.

for some β . Furthermore, we will make the ansatz that there exists a relation of the form

$$\mathcal{O}[q^{J_3}] = F_+ (\mathcal{O}[q^{T_3}]) \mathcal{O}[T_+] + F_- (\mathcal{O}[q^{T_3}]) \mathcal{O}[T_-]$$

with F_{\pm} to be determined. Let us insert the above expressions for $\mathcal{O}[D]$ and $\mathcal{O}[q^{J_3}]$ into Eqs. (2.10, 2.11). From Eq. (2.10) we infer

$$\alpha_D^2 = -\frac{C_+ C_-}{(q - q^{-1})^2}, \quad \beta = \pm 1.$$

Equation (2.11) implies⁵

$$F_+ (\mathcal{O}[q^{T_3}]) = F_- (\mathcal{O}[q^{T_3}]) = \frac{\alpha_3}{(\mathcal{O}[q^{T_3}] - \mathcal{O}[q^{-T_3}])}$$

and

$$F (\mathcal{O}[q^{T_3}]) = \frac{\alpha_D (q - q^{-1})^2}{\alpha_3^2} (\mathcal{O}[q^{T_3}] - \mathcal{O}[q^{-T_3}]). \quad (2.19)$$

$\mathcal{O}[J_{\pm}]$ are then computable from formula(2.9). We finally get

$$\begin{aligned} \mathcal{O}[J_{\pm}] &= \frac{\alpha_{\pm}}{\mathcal{O}[q^{T_3}] - \mathcal{O}[q^{-T_3}]} (\mathcal{O}[q^{\mp T_3}] \mathcal{O}[T_+] + \mathcal{O}[q^{\pm T_3}] \mathcal{O}[T_-]), \\ \mathcal{O}[q^{J_3}] &= \frac{\alpha_3}{(\mathcal{O}[q^{T_3}] - \mathcal{O}[q^{-T_3}])} (\mathcal{O}[T_+] + \mathcal{O}[T_-]) \end{aligned} \quad (2.20)$$

with

$$\alpha_{\pm} = \frac{\alpha_D \alpha_3}{C_{\mp}}.$$

Choosing⁶

$$\alpha_3^2 = \alpha_D (q - q^{-1})^2, (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \quad (2.21)$$

we obtain the standard commutation relation of $U_{\sqrt{q}}(sl(2))$,

$$\begin{aligned} \mathcal{O}[T_{\pm}] \mathcal{O}[q^{T_3}] &= q^{\mp 1} \mathcal{O}[q^{T_3}] \mathcal{O}[T_{\pm}], \\ [\mathcal{O}[T_+], \mathcal{O}[T_-]] &= \frac{\mathcal{O}[q^{T_3}] - \mathcal{O}[q^{-T_3}]}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}. \end{aligned} \quad (2.22)$$

We go from $U_q(sl(2))$ to $U_{\sqrt{q}}(sl(2))$ essentially because $F (\mathcal{O}[q^{T_3}])$, the commutator of $\mathcal{O}[T_+]$ and $\mathcal{O}[T_-]$, is a linear function of $\mathcal{O}[q^{\pm T_3}]$ according to Eqs. (2.18 and 2.19). The representation Eqs. (2.20) involves the inverse of the operator $\mathcal{O}[q^{T_3}] - \mathcal{O}[q^{-T_3}]$, which is not defined for eigenstates of T_3 with vanishing eigenvalue. However, it turns out that the normalization of the operators $\mathcal{O}[T_{\pm}]$ vanishes at this point as well, so that a well-defined limit exists⁷.

⁵ More precisely, we can always transform the solution to this form by a suitable redefinition $T_{\pm} \rightarrow H_{\pm}(T_3)T_{\pm}$ with $H_+(T_3)H_-(T_3 - 1) = 1$, which leaves the algebra of T_{\pm} invariant.

⁶ In the case where $C_+ C_- = 0$, one finds that $[\mathcal{O}[T_+], \mathcal{O}[T_-]] = 0$, which gives a contraction of $U_q(sl(2))$.

⁷ Subtleties at $\varpi = 0$ can actually be expected for general reasons, as it is the fixed point of the Weyl reflection symmetry of the theory [20, 21].

3. Transformation Laws of the Bloch Wave Operators

As alluded to already in the introduction, one a priori expects that the $\mathcal{O}[J^a]$ operators generate the symmetries of the Bloch wave basis as well. We will therefore start by considering explicitly the action by braiding of the $\mathcal{O}[J^a]$ on the Bloch wave fields.

3.1 Action of the generators $\mathcal{O}[J^a]$ on the Bloch waves In previous works, the Bloch wave operators have been handled with various (ϖ dependent) normalizations⁸. It will be simplest to make use of the ψ fields of refs. [29, 8]; one may easily change to different normalizations afterwards. It is convenient to write the relation between ψ and ξ fields as

$$\xi_M^{(J)}(\sigma) = \sum_m \psi_m^{(J)}(\sigma) \mathcal{U}_{mM}^{(J)}(\varpi), \quad \psi_m^{(J)}(\sigma) = \sum_M \xi_M^{(J)}(\sigma) \mathcal{V}_{Mm}^{(J)}(\varpi). \quad (3.1)$$

On the other hand, for spin $\frac{1}{2}$ it is better to let

$$\xi_\alpha^{(\frac{1}{2})} = \sum_\beta u_{\alpha\beta}(\varpi) \psi_\beta^{(\frac{1}{2})}, \quad \psi_\alpha^{(\frac{1}{2})} = \sum_\beta v_{\alpha\beta}(\varpi) \xi_\beta^{(\frac{1}{2})}, \quad (3.2)$$

so that $u_{\alpha\beta}(\varpi) = \mathcal{U}_{\beta\alpha}^{(\frac{1}{2})}(\varpi + 2\beta)$. Explicitly one has

$$u_{-\frac{1}{2}-\frac{1}{2}} = q^{(\varpi-1)/2}, \quad u_{-\frac{1}{2}\frac{1}{2}} = q^{-\varpi/2}, \quad u_{\frac{1}{2}-\frac{1}{2}} = q^{-\varpi/2}, \quad u_{\frac{1}{2}\frac{1}{2}} = q^{(\varpi+1)/2}. \quad (3.3)$$

Greek indices take the values $\pm\frac{1}{2}$. No summation over repeated indices is assumed unless explicitly indicated, in order to avoid confusion. Since the fields $\xi_{\pm\frac{1}{2}}^{(\frac{1}{2})}(\sigma_+)$ generate the quantum group transformations of the fields $\xi_M^{(J)}(\sigma)$, it is natural to study their action on the fields $\psi_m^{(J)}(\sigma)$. In order to relate new and old structures, let us start from the transformation laws of the ξ fields Eqs. (2.3 and 2.4), derived in ref. [7], and transform $\xi_M^{(J)}$ into $\psi_m^{(J)}$ using Eq. (3.1). One finds at first

$$\begin{aligned} \mathcal{O}[q^{J_3}]_{\sigma_+} \psi_m^{(J)}(\sigma) &= \sum_{n, M} \psi_n^{(J)}(\sigma) \mathcal{U}_{nM}^{(J)}(\varpi) q^M \mathcal{O}[q^{J_3}]_{\sigma_+} \mathcal{V}_{Mm}^{(J)}(\varpi), \\ \mathcal{O}[J_\pm]_{\sigma_+} \psi_m^{(J)}(\sigma) &= \sum_{n, N, M} \psi_n^{(J)}(\sigma) \mathcal{U}_{nN}^{(J)}(\varpi) (J_\pm)_{NM} \mathcal{O}[q^{J_3}]_{\sigma_+} \mathcal{V}_{Mm}^{(J)}(\varpi) + \\ &\quad \sum_{n, M} \psi_n^{(J)}(\sigma) \mathcal{U}_{nM}^{(J)}(\varpi) q^{-M} \mathcal{O}[J_\pm]_{\sigma_+} \mathcal{V}_{Mm}^{(J)}(\varpi). \end{aligned} \quad (3.4)$$

Next, we have to braid the generators $\mathcal{O}[q^{J_3}]_{\sigma_+}$ and $\mathcal{O}[J_\pm]_{\sigma_+}$ with the $\mathcal{V}_{Mm}^{(J)}(\varpi)$ coefficients. This is done most simply in terms of the $\psi^{(\frac{1}{2})}$ fields, since they shift ϖ in a simple way. Indeed, one has in general

$$\psi_m^{(J)}(\sigma) \varpi = (\varpi + 2m) \psi_m^{(J)}(\sigma). \quad (3.5)$$

There are two cases. First, it follows from Eq. (2.1) that $\mathcal{O}[q^{J_3}]_{\sigma_+}$ and $\mathcal{O}[J_+]_{\sigma_+}$ are proportional to the $\xi^{(\frac{1}{2})}$ fields, so that they can be immediately reexpressed in terms of

⁸ denoted by the symbols ψ, V, \tilde{V} and U .

$\psi^{(\frac{1}{2})}$ fields using Eq. (refundef). Second, the expression of $\mathcal{O}[J_-]_{\sigma_+}$ in terms of the $\xi^{(\frac{1}{2})}$ fields is given by Eq. (2.14), which makes use of the monodromy. Using Eqs. (3.2), (3.3), one then derives the equation

$$\mathcal{O}[J_-]_{\sigma_+} = \kappa_-^{(+)} \sum_{\lambda} u_{-\frac{1}{2}, \lambda} q^{-(2\lambda\varpi + \frac{1}{2})} \psi_{\lambda}^{(\frac{1}{2})}. \tag{3.6}$$

Using Eq. (3.5), one deduces from Eq. 3.4) that

$$\mathcal{O}[q^{J_3}]_{\sigma_+} \psi_m^{(J)}(\sigma) = \kappa_3^{(+)} \sum_{\lambda, n} \psi_n^{(J)}(\sigma) [q^{\mathcal{J}_3, \lambda} J_{nm}^{(J)} u_{-\frac{1}{2}, \lambda}(\varpi) \psi_{\lambda}^{(\frac{1}{2})}(\sigma_+), \tag{3.7}$$

$$\mathcal{O}[J_{\pm}]_{\sigma_+} \psi_m^{(J)}(\sigma) = \kappa_{\pm}^{(+)} \sum_{\lambda, n} \psi_n^{(J)}(\sigma) [\mathcal{X}_{\pm, \varpi}^{\lambda} J_{nm}^{(J)} u_{-\frac{1}{2}, \lambda}(\varpi) \psi_{\lambda}^{(\frac{1}{2})}(\sigma_+), \tag{3.8}$$

$$[\mathcal{X}_{\pm, \varpi}^{\lambda} J_{nm}^{(J)} = \frac{1 - q^{\pm 2}}{q^{\pm \frac{1}{2}}} [J_{\pm, \varpi}^{\lambda} J_{nm}^{(J)} + q^{\pm(2\lambda\varpi + \frac{1}{2})} [q^{-\mathcal{J}_3, \lambda} J_{nm}^{(J)}. \tag{3.9}$$

These formulae, as well as the equations below, involve the following transforms of the $U_q(sl(2))$ matrices

$$[q^{\pm \mathcal{J}_3, \lambda} J_{nm}^{(J)} = \sum_M \mathcal{U}_{nM}^{(J)}(\varpi) q^{\pm M} \mathcal{V}_{Mm}^{(J)}(\varpi + 2\lambda), \tag{3.10}$$

$$[J_{\pm, \varpi}^{\lambda} J_{nm}^{(J)} = \sum_{N, M} \mathcal{U}_{nN}^{(J)}(\varpi) [J_{\pm}]_{NM} \mathcal{V}_{Mm}^{(J)}(\varpi + 2\lambda), \tag{3.11}$$

which will play a key role. At this point, the co-action of our generators is not yet in a satisfactory form. We have to reexpress the right hand sides in terms of the generators. This is straightforward, in principle using Eq. (3.2). After that one may reexpress the r.h.s. in terms of the generators using Eqs. (2.1), and/or Eq. (2.14). Now we meet two difficulties. First there is an ambiguity, since there are two $\xi^{(\frac{1}{2})}$ fields for three generators⁹. Note that, although the braiding-algebra of the ξ fields does not involve ϖ , Eq. (2.14) does contain $\cos(h\varpi)$. Thus ϖ belongs to the algebra in some way, and indeed it is part of the operator $\mathcal{O}[D]_{\sigma_+, \sigma_-}$, as shown in Eq. (2.17). Thus from this viewpoint, the transformation from $\xi_M^{(J)}$ to $\psi_m^{(J)}$ (Eq. (3.1)) involves functions of the generators, and the $\psi_m^{(J)}$ fields appear as complicated members of the enveloping algebra. This explains the second difficulty, namely that the formulae just derived are not of the usual co-action type. Of course one may nevertheless rederive the consequences of the algebra of the generators, following the line of ref[7]. However, the resulting formulae are rather involved, and the underlying symmetry structure of the Bloch wave operator algebra is difficult to extract in this way. Therefore we will pass to a different form of the generators in the next subsection, where this structure will become much more transparent, while still equivalent to the one generated by the $\mathcal{O}[J^a]$ operators. On the way, it will be useful to collect some further formulae characterizing the action of the $\mathcal{O}[J^a]$ on the Bloch waves, to which we will come back in section 4.

By performing a (ϖ dependent) change of basis from the structure derived in ref. [7], we have arrived at formulae where the actions of $\mathcal{O}[q^{J_3}]_{\sigma_+}$ and of $\mathcal{O}[J_{\pm}]_{\sigma_+}$ look

⁹ There was no such difficulty for the action on the ξ fields because there, the transformation matrices did not contain ϖ , so that ϖ -dependent linear combinations of ξ fields as in Eq. (2.14) could be consistently viewed as separate operators.

different. Indeed, the former involves one matrix $[q^{\mathcal{J}_3, \lambda}]_{nm}^{(J)}$ while the latter involves two, that is $[\mathcal{J}_{\pm, \varpi}]_{nm}^{(J)}$ and $[q^{-\mathcal{J}_3, \lambda}]_{nm}^{(J)}$. Our next point is that this structure is redundant, and that the above actions may actually all be described in terms of the matrix $[q^{\mathcal{J}_3, \lambda}]_{nm}^{(J)}$ alone. As a first step, let us analyze our equations in terms of the possible shifts of ϖ between bras and kets, using Eqs. (3.2) and (3.5). On the l.h.s. of Eqs. (3.8) and (3.7), the possible shifts are $2m$ and $2m \pm 1$. Thus the matrices $[q^{\mathcal{J}_3, \lambda}]_{nm}^{(J)}$ and $[\mathcal{X}_{\pm, \varpi}]_{nm}^{(J)}$ are zero unless $n = m$ or $n = m \pm 1$. However, it can be seen easily that the two matrices $[\mathcal{J}_{\pm, \varpi}]_{nm}^{(J)}$ and $[q^{-\mathcal{J}_3, \lambda}]_{nm}^{(J)}$, involved in Eq. (3.9) do not share this property. Thus many cancellations between them take place, and one should not consider them separately. This, and the explicit computation of the matrices, is achieved by re-doing the calculation otherwise. We start from the braiding matrix of $\psi^{(\frac{1}{2})}$ with $\psi_m^{(J)}$ for general J and m derived in ref. [8]. One has in general,

$$\psi_{\alpha}^{(\frac{1}{2})}(\sigma) \psi_m^{(J)}(\sigma') = \sum_{\beta} \sum_n S_{\epsilon}^{(J) n \beta}(\varpi) \psi_n^{(J)}(\sigma') \psi_{\beta}^{(\frac{1}{2})}(\sigma), \tag{3.12}$$

where the non-vanishing matrix elements are given by (ϵ is the sign of $\sigma - \sigma'$)¹⁰

$$S_{\epsilon}^{(J) m - \frac{1}{2}}(\varpi) = S_{\epsilon}^{(J) -m, \frac{1}{2}}(-\varpi) = \frac{[\varpi + J + m]_q}{[\varpi]_q} e^{i h m \epsilon},$$

$$S_{\epsilon}^{(J) m - 1, \frac{1}{2}}(\varpi) = \frac{[J + m]_q}{[\varpi]_q} e^{i h \epsilon (1 - m - \varpi)} = S_{\epsilon}^{(J) -m + 1, -\frac{1}{2}}(-\varpi). \tag{3.13}$$

Here,

$$[x]_q := \frac{q^x - q^{-x}}{q - q^{-1}}$$

denotes q -numbers. Making use of Eqs. (3.2), one deduces from Eq. (3.12) that

$$\xi_{\gamma}^{(\frac{1}{2})}(\sigma_{\pm}) \psi_m^{(J)}(\sigma) = \sum_{\lambda, \alpha, n} \psi_n^{(J)}(\sigma) u_{\gamma \alpha}(\varpi - 2n) S_{\pm}^{(J) n \lambda}(\varpi - 2n) \psi_{\lambda}^{(\frac{1}{2})}(\sigma_{\pm}).$$

Comparing with Eq. (3.7), 3.8, one infers that

$$[q^{\mathcal{J}_3, \lambda}]_{nm}^{(J)} = \frac{u_{\mp \frac{1}{2}, n - m + \lambda}(\varpi - 2n)}{u_{\mp \frac{1}{2}, \lambda}(\varpi)} S_{\pm}^{(J) n \lambda}(\varpi - 2n),$$

$$[\mathcal{X}_{\pm, \varpi}^{\lambda}]_{nm}^{(J)} = \frac{u_{\pm \frac{1}{2}, n - m + \lambda}(\varpi - 2n)}{u_{\mp \frac{1}{2}, \lambda}(\varpi)} S_{\pm}^{(J) n \lambda}(\varpi - 2n). \tag{3.14}$$

Notice that both choices of the signs on the right hand sides lead to the same result. The equalities for the lower signs are easily verified using the monodromy properties of the $\xi_{-\frac{1}{2}}^{(\frac{1}{2})}$ field [7]:

$$\xi_{-\frac{1}{2}}^{(\frac{1}{2})}(\sigma + 2\pi) = \xi_{\frac{1}{2}}^{(\frac{1}{2})}(\sigma),$$

$$\xi_{-\frac{1}{2}}^{(\frac{1}{2})}(\sigma - 2\pi) = 2q^{-\frac{1}{2}} \cos(h\varpi) \xi_{-\frac{1}{2}}^{(\frac{1}{2})}(\sigma) - q^{-1} \xi_{\frac{1}{2}}^{(\frac{1}{2})}(\sigma), \tag{3.15}$$

¹⁰ for $\sigma - \sigma' \in [-2\pi, 2\pi]$, which we consider here

which allow to identify $\xi_{\frac{1}{2}}^{(\frac{1}{2})}(\sigma_-)$ with $\xi_{-\frac{1}{2}}^{(\frac{1}{2})}(\sigma_+)$ for the first equation 3.14, and $\mathcal{O}[J_-]_{\sigma_+}$ with $\xi_{-\frac{1}{2}}^{(\frac{1}{2})}(\sigma_-)$ for the second. Equations (3.15) are derived from the simple monodromy behaviour of the Bloch waves:

$$\psi_{\alpha}^{(\frac{1}{2})}(\sigma + 2\pi) = q^{2\alpha\varpi + \frac{1}{2}} \psi_{\alpha}^{(\frac{1}{2})}(\sigma). \tag{3.16}$$

It follows from the formulae just given that the only nonzero matrix elements of $[q^{\mathcal{J}_3, \lambda}]_{nm}^{(J)}$ are

$$[q^{\mathcal{J}_3, -\frac{1}{2}}]_{nn}^{(J)} = \frac{[\varpi + J - n]_q}{[\varpi - 2n]_q}, \quad [q^{\mathcal{J}_3, -\frac{1}{2}}]_{nn-1}^{(J)} = -q^{-\frac{1}{2}} \frac{[J - n + 1]_q}{[\varpi - 2n]_q}, \tag{3.17}$$

$$[q^{\mathcal{J}_3, \frac{1}{2}}]_{nn}^{(J)} = \frac{[\varpi - J - n]_q}{[\varpi - 2n]_q}, \quad [q^{\mathcal{J}_3, \frac{1}{2}}]_{nn+1}^{(J)} = q^{\frac{1}{2}} \frac{[J + n + 1]_q}{[\varpi - 2n]_q}. \tag{3.18}$$

Using the explicit expressions Eq. (3.3), one sees that

$$[\mathcal{X}_{\pm, \varpi}^{\lambda}]_{nm}^{(J)} = q^{\pm 2(n+\lambda-m)(\varpi-2n) \pm \frac{1}{2}} [q^{\mathcal{J}_3, \lambda}]_{nm}^{(J)}, \tag{3.19}$$

from which it follows, since $n - m + \lambda = \pm \frac{1}{2}$

$$q^{-\frac{1}{2}} [\mathcal{X}_{+, \varpi}^{\mu}]_{np}^{(J)} + q^{\frac{1}{2}} [\mathcal{X}_{-, \varpi}^{\mu}]_{np}^{(J)} = \sum_m D(\varpi)_{nm} [q^{\mathcal{J}_3, \mu}]_{mp}^{(J)}, \tag{3.20}$$

$$q^{\frac{1}{2}} [\mathcal{X}_{+, \varpi}^{\mu}]_{np}^{(J)} + q^{-\frac{1}{2}} [\mathcal{X}_{-, \varpi}^{\mu}]_{np}^{(J)} = \sum_m [q^{\mathcal{J}_3, \mu}]_{nm}^{(J)} D(\varpi + 2\mu)_{mp}, \tag{3.21}$$

where

$$D(\varpi)_{nm} = \delta_{n,m} (q^{\varpi-2n} + q^{-\varpi+2n}). \tag{3.22}$$

Returning to the transformation laws of the $\psi^{(J)}$ fields, one sees that we may finally rewrite them under the form

$$\frac{1}{\kappa_3^{(+)}} \mathcal{O}[q^{\mathcal{J}_3}]_{\sigma_+} \psi_m^{(J)}(\sigma) = \sum_{\lambda, n} \psi_n^{(J)}(\sigma) [q^{\mathcal{J}_3, m-n+\lambda}]_{nm}^{(J)} u_{-\frac{1}{2}, m-n+\lambda} \psi_{m-n+\lambda}^{(\frac{1}{2})}(\sigma_+),$$

$$\frac{1}{\kappa_{\pm}^{(+)}} \mathcal{O}[J_{\pm}]_{\sigma_+} \psi_m^{(J)}(\sigma) = \sum_{\lambda, n} q^{\pm(2\lambda\varpi + \frac{1}{2})} \psi_n^{(J)}(\sigma) [q^{\mathcal{J}_3, m-n+\lambda}]_{nm}^{(J)} u_{-\frac{1}{2}, m-n+\lambda} \psi_{m-n+\lambda}^{(\frac{1}{2})}(\sigma_+). \tag{3.23}$$

Up to the factors $q^{\pm(\frac{1}{2}+2\lambda\varpi)}$, the right-hand sides involve the same matrices $[q^{\mathcal{J}_3, m-n+\lambda}]_{nm}^{(J)}$ and operators $u_{-\frac{1}{2}, m-n+\lambda} \psi_{m-n+\lambda}^{(\frac{1}{2})}(\sigma_+)$.

3.2 Changing basis on the generators Up to now we have performed a change of basis only on the fields but not the generators. However, as announced above, the most natural realization of the quantum group on the Bloch wave fields turns out to be given in terms of new generators which are represented by Bloch waves rather than ξ fields.

More precisely, the raising and lowering generators will be given in terms of the $\psi_{\alpha}^{(\frac{1}{2})}$, while the Cartan generator will be represented by the zero mode ϖ . Compared to the last subsection, where it appeared to be merely a parameter appearing in the basis transformation between the ξ and the ψ fields, the role of the zero mode thus changes

drastically. In this way, we will arrive at a FP realization of the transformation Eq. (2.20) by generators denoted $\mathcal{O}[q^{T_3}]$, and $\mathcal{O}[T_{\pm}]_{\sigma_{\pm}}$. In the new picture, functions of ϖ which appear in the coproduct action of the generators on the Bloch waves are much easier to handle because the commutation relations of functions of T_3 with the other generators are very simple. We first observe that it is just the transformation between $\xi^{(\frac{1}{2})}$ and $\psi^{(\frac{1}{2})}$ fields (Eqs. (3.2)) which may be used to realize a redefinition of generators similar to Eqs. (2.20) of section 2. The steps of the derivation are as follows. First according to Eq. (3.5) $\psi_m^{(J)}$ shift ϖ by $2m$, so that we will realize the first line of Eqs. (2.22), if we let

$$\mathcal{O}[q^{\pm T_3}] = q^{\mp \varpi}, \tag{3.24}$$

and choose $\mathcal{O}[T_{\pm}]_{\sigma_{\pm}}$ to be proportional to $\psi_{\pm \frac{1}{2}}^{(\frac{1}{2})}$, as suggested above. Second, the first relation of Eq. (3.2) with $\alpha = -\frac{1}{2}$ may be rewritten as

$$\mathcal{O}[q^{J_3}]_{\sigma_+} = \kappa_3^{(+)} (\mathcal{O}[q^{T_3}] - \mathcal{O}[q^{-T_3}])^{-1} [\mathcal{O}[T_+]_{\sigma_+} + \mathcal{O}[T_-]_{\sigma_+}], \tag{3.25}$$

if we let

$$\mathcal{O}[T_{\pm}]_{\sigma_{\pm}} = (q^{-\varpi} - q^{\varpi}) u_{-\frac{1}{2}, \pm \frac{1}{2}}(\varpi) \psi_{\pm \frac{1}{2}}^{(\frac{1}{2})}(\sigma_{\pm}). \tag{3.26}$$

Third, the first relation of Eq. (3.2) with $\alpha = \frac{1}{2}$ may be transformed into

$$\mathcal{O}[J_+]_{\sigma_+} = \kappa_+^{(+)} q^{\frac{1}{2}} (\mathcal{O}[q^{T_3}] - \mathcal{O}[q^{-T_3}])^{-1} [\mathcal{O}[q^{-T_3}] \mathcal{O}[T_+]_{\sigma_+} + \mathcal{O}[q^{T_3}] \mathcal{O}[T_-]_{\sigma_+}]. \tag{3.27}$$

Fourth, Eq. (3.6) gives

$$\mathcal{O}[J_-]_{\sigma_+} = \kappa_-^{(+)} q^{-\frac{1}{2}} (\mathcal{O}[q^{T_3}] - \mathcal{O}[q^{-T_3}])^{-1} [\mathcal{O}[q^{T_3}] \mathcal{O}[T_+]_{\sigma_+} + \mathcal{O}[q^{-T_3}] \mathcal{O}[T_-]_{\sigma_+}]. \tag{3.28}$$

Fifth, Eq. (2.17) may be rewritten as

$$\mathcal{O}[D]_{\sigma_+, \sigma'_+} = \frac{C_+(\sigma_+, \sigma'_+)}{q - q^{-1}} [\mathcal{O}[q^{T_3}] + \mathcal{O}[q^{-T_3}]]. \tag{3.29}$$

One sees that the FP version of the general Eqs. (2.20) is realized with

$$\begin{aligned} \alpha_{\pm} &= q^{\pm \frac{1}{2}} \kappa_{\pm}^{(+)}, \quad \alpha_3 = \kappa_3^{(+)}, \\ \alpha_D &= \frac{C_+(\sigma_+, \sigma'_+)}{q - q^{-1}}, \end{aligned} \tag{3.30}$$

and with this choice, we arrive at the FP relations

$$\begin{aligned} \mathcal{O}[T_{\pm}]_{\sigma_{\pm}} \mathcal{O}[q^{T_3}] &= q^{\mp 1} \mathcal{O}[q^{T_3}] \mathcal{O}[T_{\pm}]_{\sigma_{\pm}}, \\ \mathcal{O}[T_+]_{\sigma_+} \mathcal{O}[T_-]_{\sigma'_+} - \mathcal{O}[T_-]_{\sigma_+} \mathcal{O}[T_+]_{\sigma'_+} &= \frac{C_+(\sigma_+, \sigma'_+)}{q^{\frac{1}{2}} \kappa_+^{(+)} \kappa_3^{(+)}} [\mathcal{O}[q^{T_3}] - \mathcal{O}[q^{-T_3}]]. \end{aligned} \tag{3.31}$$

The second equation can be verified, for instance, using Eqs. (2.13), (3.26) and the explicit expression for $\xi_0^{(\frac{1}{2}, \frac{1}{2})^{(0)}}(\sigma_+, \sigma'_+)$ in terms of the ψ fields (see Eq. (5.3) of ref. [7]). Up to a multiplicative factor, this is the FP realization of the $U_{\sqrt{q}}(sl(2))$ commutation relation of Sect. 2. Note that this multiplicative factor is itself proportional to the field

$C_+ \propto \xi_0^{[\frac{1}{2}, \frac{1}{2}]^{(0)}}(\sigma_+, \sigma'_+)$, so that it is not really possible to get rid of it by changing the normalization of $\mathcal{O}[T_\pm]_{\sigma_+}$ as was done in Sect. 2 by imposing Eq. (2.21). Altogether the present FP realization of the commutation relation is such that $\mathcal{O}[q^{\pm T_3}]$, $\mathcal{O}[T_\pm]_{\sigma_+}$ have grading zero and one, respectively. The fact that $\mathcal{O}[q^{\pm T_3}]$ has grading zero is of course necessary to be able to define the right hand side of Eqs. (3.27) and (3.28). Next, one substitutes Eq. (3.25) into the first line of Eq. (3.23). Making use of Eq. (3.5), one sees that $\mathcal{O}[T_\pm]_{\sigma_+}$ give a total shift of ϖ of $m \pm 1/2$ respectively on the left hand side. Separating the corresponding contributions on the right hand side one obtains

$$\mathcal{O}[T_\pm]_{\sigma_+} \psi_m^{(J)}(\sigma) = \sum_n \psi_n^{(J)}(\sigma) [T_\varpi^{\pm \frac{1}{2}}]_{n,m}^{(J)} \mathcal{O}[T_{+2(m-n \pm \frac{1}{2})}]_{\sigma_+}, \quad (3.32)$$

$$[T_\varpi^\lambda]_{n,m}^{(J)} = \frac{\sin[h(\varpi - 2n)]}{\sin[h(\varpi)]} [q^{\mathcal{J}_{3,m-n+\lambda}}]_{nm}^{(J)}. \quad (3.33)$$

This is consistent with the definition Eq. (3.26) since by construction, $[T_\varpi^\lambda]_{n,m}^{(J)}$ vanishes unless $m - n + \lambda = \pm \frac{1}{2}$. On the right hand side of the first equation, the index of T is always ± 1 , and from now on we use both notations $\mathcal{O}[T_\pm]_{\sigma_+}$ and $\mathcal{O}[T_{\pm 1}]_{\sigma_+}$. Furthermore, let us note here that the FP braiding relations of our generators lead to the following relations on $[T_\varpi^\lambda]_{n,m}^{(J)}$:

$$\begin{aligned} [T_\varpi^\lambda]_{m,m}^{(J)} [T_\varpi^{-\lambda}]_{m,m}^{(J)} - [T_\varpi^{-\lambda}]_{m,m+2\lambda}^{(J)} [T_\varpi^\lambda]_{m+2\lambda,m}^{(J)} &= \frac{[\varpi - 2m]_q}{[\varpi]_q}, \\ [T_\varpi^\lambda]_{m+2\lambda,m}^{(J)} [T_\varpi^{-\lambda}]_{m,m}^{(J)} - [T_\varpi^{-\lambda}]_{m+2\lambda,m+2\lambda}^{(J)} [T_\varpi^\lambda]_{m+2\lambda,m}^{(J)} &= 0. \end{aligned} \quad (3.34)$$

Making use of the relation to the original matrix realization of $U_q(sl(2))$ displayed in this section, one may rederive these relations. This will be done later on. Note that these relations involve particular values of the indices m and n of $[T_\varpi^\lambda]_{m,n}^{(J)}$ **without summation over these indices**. So far, we have used ψ fields at point σ_+ to realize the operators $\mathcal{O}[T_\pm]_{\sigma_+}$. We can do exactly the same thing with ψ fields at point σ_- , they will realize operators $\mathcal{O}[T_\pm]_{\sigma_-}$. As in the case of the ξ fields both representations are related by monodromy which is diagonal for the ψ fields (see Eq. 3.16). More precisely we have

$$\begin{aligned} \mathcal{O}[T_\pm]_{\sigma_-} &= (q^{-\varpi} - q^\varpi) u_{\frac{1}{2}, \pm \frac{1}{2}}(\varpi) \psi_{\pm \frac{1}{2}}^{(\frac{1}{2})}(\sigma_-) \\ &= (q^{-\varpi} - q^\varpi) u_{-\frac{1}{2}, \pm \frac{1}{2}}(\varpi) \psi_{\pm \frac{1}{2}}^{(\frac{1}{2})}(\sigma_- + 2\pi) \\ &= \mathcal{O}[T_\pm]_{\sigma'_+} \text{ at } \sigma'_+ = \sigma_- + 2\pi. \end{aligned} \quad (3.35)$$

Next we turn to the action of $\mathcal{O}[T_{+2\lambda}]_{\sigma_+}$ on a product of ψ fields, which has a very simple structure in terms of the $[T_\varpi^\alpha]_{n,m}^{(J)}$ matrices. One has

$$\begin{aligned} \mathcal{O}[T_{+2\lambda}]_{\sigma_+} \psi_{m_1}^{(J_1)}(\sigma_1) \psi_{m_2}^{(J_2)}(\sigma_2) &= \\ \sum_{n_1, n_2} \psi_{n_1}^{(J_1)}(\sigma_1) \psi_{n_2}^{(J_2)}(\sigma_2) [A(T)_\varpi^\lambda]_{n_1 n_2 m_1 m_2}^{(J_1, J_2)} \mathcal{O}[T_{+2(m_1+m_2-n_1-n_2+\lambda)}]_{\sigma_+}, \end{aligned} \quad (3.36)$$

where

$$[\Lambda(\mathcal{T})^\lambda]_{n_1 n_2 m_1 m_2}^{(J_1, J_2)} = [\mathcal{T}_{\varpi}^{-2n_2}]_{n_1, m_1}^{(J_1)} [\mathcal{T}_{\varpi}^{m_1 - n_1 + \lambda}]_{n_2, m_2}^{(J_2)}. \quad (3.37)$$

Moreover, we may rewrite Eq. (3.32) under the following form similar to the general co-product action[19] discussed in ref. [10]:

$$\mathcal{O}[T_{+2\lambda}]_{\sigma_+} \psi_m^{(J)}(\sigma) = \sum_n \psi_n^{(J)}(\sigma) \Lambda(\mathcal{O}[T_{+2\lambda}]_{\sigma_+})_{nm}, \quad (3.38)$$

where

$$\Lambda(\mathcal{O}[T_{+2\lambda}]_{\sigma_+})_{nm} = [\mathcal{T}_{\varpi}^\lambda]_{n, m}^{(J)} \mathcal{O}[T_{+2(m-n+\lambda)}]_{\sigma_+}. \quad (3.39)$$

Clearly Eq. (3.39) is analogous to Eq. (3.37), the second matrix being replaced by the generator. Notice however that the second term depends upon the row/column indices of the first matrix, so that a priori it doesn't seem to have an interpretation in terms of a coproduct. We will see in Sect. 4 that such an interpretation in fact becomes available once we realize that the true underlying symmetry is not $U_{\sqrt{q}}(sl(2))$ but $U_q(sl(2)) \otimes U_q(sl(2))$.

4. Hidden $U_q(sl(2)) \otimes U_q(sl(2))$ Structure

4.1 The matrix algebra In order to retransform Eqs. (3.34) into matrix relations, it is convenient to introduce two sets of matrices A^λ and B^λ defined by

$$[A_{\varpi}^\lambda]_{n, m}^{(J)} = [\mathcal{T}_{\varpi}^\lambda]_{m, m}^{(J)} \delta_{m, n} \quad [B_{\varpi}^\lambda]_{n, m}^{(J)} = [\mathcal{T}_{\varpi}^{-\lambda}]_{m-2\lambda, m}^{(J)} \delta_{n, m-2\lambda}. \quad (4.1)$$

Equations (3.34) become

$$\sum_p ([B_{\varpi}^\lambda]_{n, p}^{(J)} [A_{\varpi+2\lambda}^\lambda]_{p, m}^{(J)} - [A_{\varpi}^\lambda]_{n, p}^{(J)} [B_{\varpi+2\lambda}^\lambda]_{p, m}^{(J)}) = 0, \quad (4.2)$$

$$\sum_p ([A_{\varpi}^\lambda]_{n, p}^{(J)} [A_{\varpi+2\lambda}^{-\lambda}]_{p, m}^{(J)} - [B_{\varpi}^\lambda]_{n, p}^{(J)} [B_{\varpi+2\lambda}^{-\lambda}]_{p, m}^{(J)}) = \frac{[\omega - 2m]_q}{[q]} \delta_{mn}. \quad (4.2)$$

Equations (3.32) takes the form

$$\mathcal{O}[T_{+2\lambda}]_{\sigma_+} \psi_m^{(J)}(\sigma) = \sum_n \psi_n^{(J)}(\sigma) \left\{ [A_{\varpi}^\lambda]_{n, m}^{(J)} \mathcal{O}[T_{+2\lambda}]_{\sigma_+} + [B_{\varpi}^{-\lambda}]_{n, m}^{(J)} \mathcal{O}[T_{-2\lambda}]_{\sigma_+} \right\}. \quad (4.3)$$

There remains to handle the ϖ dependence which prevents Eqs. (4.2) from being ordinary commutation relations. For this, we interpret ϖ as an additional index, by writing

$$\psi_{\rho, \varpi}^{(J)} = \psi_m^{(J)} P_{\varpi}, \quad \rho = \varpi - 2m \quad (4.4)$$

where P_{ϖ} is the projector onto the Verma module characterized by¹¹ ϖ . Since

$$\mathcal{O}[T_{+2\lambda}]_{\sigma_+} P_{\varpi} = P_{\varpi-2\lambda} \mathcal{O}[T_{+2\lambda}]_{\sigma_+}$$

Eqs. (4.3) become

¹¹ Of course, this is nothing but a slightly different notation for the standard chiral vertex operators V_{JK}^J of conformal field theory -cf. ref. [10].

$$\begin{aligned} & \mathcal{O}[T_{+2\lambda}]_{\sigma_+} \psi_{\rho, \varpi}^{(J)}(\sigma) \\ &= \sum_{\rho', \varpi'} \psi_{\rho', \varpi'}^{(J)}(\sigma) \left\{ [A^\lambda]_{\rho', \varpi', \rho \varpi}^{(J)} \mathcal{O}[T_{+2\lambda}]_{\sigma_+} + [B^{-\lambda}]_{\rho', \varpi', \rho \varpi}^{(J)} \mathcal{O}[T_{-2\lambda}]_{\sigma_+} \right\}, \quad (4.5) \end{aligned}$$

where we have defined

$$\begin{aligned} [A^\lambda]_{\rho', \varpi', \rho \varpi}^{(J)} &= [A_{\varpi'}^\lambda]_{\frac{\varpi' - \rho'}{2}, \frac{\varpi - \rho}{2}}^{(J)} \delta_{\varpi', \varpi - 2\lambda}, \\ [B^\lambda]_{\rho', \varpi', \rho \varpi}^{(J)} &= [B_{\varpi'}^\lambda]_{\frac{\varpi' - \rho'}{2}, \frac{\varpi - \rho}{2}}^{(J)} \delta_{\varpi', \varpi - 2\lambda}. \end{aligned} \quad (4.6)$$

At this point it is convenient to introduce

$$[q^{\Omega_R}]_{\rho', \varpi', \rho \varpi} = q^\varpi \delta_{\rho', \rho} \delta_{\varpi', \varpi}, \quad [q^{\Omega_L}]_{\rho', \varpi', \rho \varpi} = q^\rho \delta_{\rho', \rho} \delta_{\varpi', \varpi}. \quad (4.7)$$

Equations (4.2) become true matrix relations

$$\begin{aligned} A^\lambda A^{-\lambda} - B^\lambda B^{-\lambda} &= \frac{[\Omega_L]_q}{[\Omega_R]_q}, \\ [A^\lambda, B^\lambda] &= 0. \end{aligned} \quad (4.8)$$

Making use of Eqs. (3.17), (3.18), (3.33), (4.1) one derives the explicit expressions

$$\begin{aligned} [A^\lambda]_{\rho', \varpi', \rho \varpi}^{(J)} &= \frac{[(\rho' + \varpi')/2 - 2\lambda J]_q}{[\varpi']_q} \delta_{(\rho' + \varpi')/2, (\rho + \varpi)/2 - 2\lambda} \delta_{(\rho' - \varpi')/2, (\rho - \varpi)/2}, \\ [B^\lambda]_{\rho', \varpi', \rho \varpi}^{(J)} &= \frac{q^\lambda [(-\rho + \varpi)/2 + 2\lambda J]_q}{[\varpi']_q} \delta_{(-\rho' + \varpi')/2, (-\rho + \varpi)/2 - 2\lambda} \delta_{(\rho' + \varpi')/2, (\rho + \varpi)/2}. \end{aligned} \quad (4.9)$$

These last formulae suggest to define

$$\begin{aligned} [A_{2\lambda}]_{\rho', \varpi', \rho, \varpi}^{(J)} &= 2\lambda [A^{-\lambda}]_{\rho', \varpi', \rho, \varpi}^{(J)} [\varpi']_q & [q^{A_3}]_{\rho', \varpi', \rho, \varpi}^{(J)} &= q^{\frac{\varpi + \rho}{2}} \delta_{\rho \rho'} \delta_{\varpi \varpi'} \\ [B_{2\lambda}]_{\rho', \varpi', \rho, \varpi}^{(J)} &= 2\lambda [B^{-\lambda}]_{\rho', \varpi', \rho, \varpi}^{(J)} [\varpi']_q & [q^{B_3}]_{\rho', \varpi', \rho, \varpi}^{(J)} &= q^{\frac{\varpi - \rho}{2}} \delta_{\rho \rho'} \delta_{\varpi \varpi'} \end{aligned} \quad (4.10)$$

We use the shorthand notation $\mathcal{A}_\pm, \mathcal{B}_\pm$ for the matrices $[A_{2\lambda}]_{\rho', \varpi', \rho, \varpi}^{(J)}, [B_{2\lambda}]_{\rho', \varpi', \rho, \varpi}^{(J)}$ and similarly for the diagonal generators. Writing down the explicit expressions derived from Eqs. (3.17), (3.18), (3.33), (4.6), one sees that, in the spaces where they act non-trivially, these matrices are in fact equal to the ones of the standard realization of $U_q(sl(2))$, up to simple changes of normalization. Accordingly, the following matrix algebra holds:

$$[\mathcal{A}_+, \mathcal{A}_-] = [2\mathcal{A}_3]_q, \quad \mathcal{A}_\pm q^{\mathcal{A}_3} = q^{\mp 1} q^{\mathcal{A}_3} \mathcal{A}_\pm, \quad (4.11)$$

$$[\mathcal{B}_+, \mathcal{B}_-] = [2\mathcal{B}_3]_q, \quad \mathcal{B}_\pm q^{\mathcal{B}_3} = q^{\mp 1} q^{\mathcal{B}_3} \mathcal{B}_\pm, \quad (4.12)$$

$$\mathcal{B}_a \mathcal{A}_b - \mathcal{A}_b \mathcal{B}_a = 0, \quad (4.13)$$

$$\mathcal{A}_- \mathcal{A}_+ + ([\mathcal{A}_3 + 1/2]_q)^2 = \mathcal{B}_- \mathcal{B}_+ + ([\mathcal{B}_3 + 1/2]_q)^2. \quad (4.14)$$

The fact that the matrices \mathcal{A} and \mathcal{B} commute is obvious since they act non-trivially in spaces which are orthogonal. The last equation just imposes that the Casimir operators

associated to the two $U_q(sl(2))$ be equal for this matrix representation, and be given by $(|J + 1/2|_q)^2$. Effectively, this means that what we are dealing with is not truly $U_q(sl(2)) \otimes U_q(sl(2))$ but a reduction of it to five generators.

Finally, let us return to the action on a product of fields already displayed on Eqs. (3.36), (3.37), in order to understand better the co-product. For this we rewrite the product of ψ fields using the notation introduced by Eq. (4.4). Clearly

$$\psi_{m_1}^{(J_1)} \psi_{m_2}^{(J_2)} P_{\varpi_2} = \psi_{\rho_1, \varpi_1}^{(J_1)} \psi_{\varpi_1, \varpi_2}^{(J_2)}, \quad \rho_1 = \varpi_2 - 2(m_1 + m_2), \quad \varpi_1 = \varpi_2 - m_2. \quad (4.15)$$

The existence of a single value of the zero mode intermediate state is what restricts us to only consider the product just written instead of the true zero-mode tensor product $\psi_{\rho_1, \varpi_1}^{(J_1)} \psi_{\rho_2, \varpi_2}^{(J_2)}$. The corresponding matching condition,

$$\varpi_1 = \rho_2, \quad (4.16)$$

will play a key role. Now we apply Eq. (4.5) twice obtaining

$$\begin{aligned} \mathcal{O}[T_{+2\lambda}]_{\sigma_+} \psi_{\rho_1, \varpi_1}^{(J_1)}(\sigma_1) \psi_{\rho_2, \varpi_2}^{(J_2)}(\sigma_2) \delta_{\varpi_1, \rho_2} = \\ \sum_{\rho'_1, \rho'_2, \varpi'_1, \varpi'_2} \psi_{\rho'_1, \varpi'_1}^{(J_1)}(\sigma_1) \psi_{\rho'_2, \varpi'_2}^{(J_2)}(\sigma_2) \delta_{\varpi_1, \rho_2} \times \\ \times \left\{ \tilde{A}(A^\lambda)_{\rho'_1, \varpi'_1, \rho'_2, \varpi'_2; \rho_1, \varpi_1, \rho_2, \varpi_2}^{(J_1, J_2)} \mathcal{O}[T_{+2\lambda}]_{\sigma_+} + \right. \\ \left. + \tilde{A}(B^{-\lambda})_{\rho'_1, \varpi'_1, \rho'_2, \varpi'_2; \rho_1, \varpi_1, \rho_2, \varpi_2}^{(J_1, J_2)} \mathcal{O}[T_{-2\lambda}]_{\sigma_+} \right\}, \quad (4.17) \end{aligned}$$

where

$$\begin{aligned} \tilde{A}(A^\lambda)_{\rho'_1, \varpi'_1, \rho'_2, \varpi'_2; \rho_1, \varpi_1, \rho_2, \varpi_2}^{(J_1, J_2)} = \\ [A^\lambda]_{\rho'_1, \varpi'_1, \rho_1, \varpi_1}^{(J_1)} [A^\lambda]_{\rho'_2, \varpi'_2, \rho_2, \varpi_2}^{(J_2)} + [B^{-\lambda}]_{\rho'_1, \varpi'_1, \rho_1, \varpi_1}^{(J_1)} [B^\lambda]_{\rho'_2, \varpi'_2, \rho_2, \varpi_2}^{(J_2)}, \quad (4.18) \end{aligned}$$

$$\begin{aligned} \tilde{A}(B^\lambda)_{\rho'_1, \varpi'_1, \rho'_2, \varpi'_2; \rho_1, \varpi_1, \rho_2, \varpi_2}^{(J_1, J_2)} = \\ [A^{-\lambda}]_{\rho'_1, \varpi'_1, \rho_1, \varpi_1}^{(J_1)} [B^\lambda]_{\rho'_2, \varpi'_2, \rho_2, \varpi_2}^{(J_2)} + [B^\lambda]_{\rho'_1, \varpi'_1, \rho_1, \varpi_1}^{(J_1)} [A^\lambda]_{\rho'_2, \varpi'_2, \rho_2, \varpi_2}^{(J_2)}. \quad (4.19) \end{aligned}$$

The interpretation of Ω_L and Ω_R in terms of Verma modules suggests to define

$$\tilde{A}(q^{\Omega_L}) = q^{\Omega_L} \otimes \mathbf{1}, \quad \tilde{A}(q^{\Omega_R}) = \mathbf{1} \otimes q^{\Omega_R}, \quad (4.20)$$

where the obvious index structure has been suppressed. The mapping \tilde{A} cannot yet be interpreted as a coproduct. This is because the algebra Eqs. (4.11)–(4.13) turns out to be preserved by \tilde{A} only on a subspace of the tensor product of the representation spaces. The necessary projection is defined by the matching condition Eq. (4.16). The important point here is that the structure of the $[A^\lambda]$ and $[B^\lambda]$ is precisely such that \tilde{A} respects the matching condition, i.e.

$$\tilde{A}(A^\lambda)_{\rho'_1, \varpi'_1, \rho'_2, \varpi'_2; \rho_1, \varpi_1, \rho_2, \varpi_2}^{(J_1, J_2)} \delta_{\varpi_1, \rho_2} = \delta_{\varpi'_1, \rho'_2} \tilde{A}(A^\lambda)_{\rho'_1, \varpi'_1, \rho'_2, \varpi'_2; \rho_1, \varpi_1, \rho_2, \varpi_2},$$

and likewise for $\tilde{A}(B^\lambda)$. Therefore we introduce the restriction

$$\Lambda(A^\lambda) \equiv \tilde{A}(A^\lambda) P = P \tilde{A}(A^\lambda)$$

of \tilde{A} to the subspace defined by Eq. (4.16), with P the corresponding projector. The above equations are then written compactly as

$$\begin{aligned} \Lambda(A^\lambda) &= (A^\lambda \otimes A^\lambda + B^{-\lambda} \otimes B^\lambda)P \equiv A^\lambda \underbrace{\otimes}_{\Lambda} A^\lambda + B^{-\lambda} \underbrace{\otimes}_{\Lambda} B^\lambda, \\ \Lambda(B^\lambda) &= (A^{-\lambda} \otimes B^\lambda + B^\lambda \otimes A^\lambda)P \equiv A^{-\lambda} \underbrace{\otimes}_{\Lambda} B^\lambda + B^\lambda \underbrace{\otimes}_{\Lambda} A^\lambda, \end{aligned} \tag{4.21}$$

and

$$\Lambda(q^{\Omega_R}) = \mathbf{1} \underbrace{\otimes}_{\Lambda} q^{\Omega_R} \quad \Lambda(q_L^\Omega) = q_L^\Omega \underbrace{\otimes}_{\Lambda} \mathbf{1}, \tag{4.22}$$

where the "braced" tensor product serves as a convenient shorthand notation for the projection. At the abstract level, without referring to a matrix realization, the present definition of $\underbrace{\otimes}_{\Lambda}$ is supposed to satisfy

$$q^{\Omega_R} \underbrace{\otimes}_{\Lambda} \mathbf{1} = \mathbf{1} \underbrace{\otimes}_{\Lambda} q^{\Omega_L}. \tag{4.23}$$

In order to show that the coproduct just defined respects the matrix relations Eqs. (4.11–4.14), one rewrites it in terms of the generators of $U_q(sl(2)) \otimes U_q(sl(2))$:

$$\begin{aligned} \Lambda(A_\pm) &= \frac{\pm 1}{[\mathcal{A}_3 + \mathcal{B}_3]_q} A_\pm \underbrace{\otimes}_{\Lambda} A_\pm + \frac{\mp 1}{[\mathcal{A}_3 + \mathcal{B}_3]_q} B_\mp \underbrace{\otimes}_{\Lambda} B_\pm, \\ \Lambda(B_\pm) &= \frac{\mp 1}{[\mathcal{A}_3 + \mathcal{B}_3]_q} A_\mp \underbrace{\otimes}_{\Lambda} B_\pm + \frac{\pm 1}{[\mathcal{A}_3 + \mathcal{B}_3]_q} B_\pm \underbrace{\otimes}_{\Lambda} A_\pm. \end{aligned} \tag{4.24}$$

Note that, due to the property 4.23, the factor $1/[\mathcal{A}_3 + \mathcal{B}_3]_q$ simply becomes $1/[\mathcal{A}_3 - \mathcal{B}_3]_q$ if it is carried over to the other side of the tensor product. One further defines

$$\begin{aligned} \Lambda(q^{\mathcal{A}_3}) &= q^{\frac{1}{2}\mathcal{A}_3} q^{-\frac{1}{2}\mathcal{B}_3} \underbrace{\otimes}_{\Lambda} q^{\frac{1}{2}\mathcal{A}_3} q^{\frac{1}{2}\mathcal{B}_3}, \\ \Lambda(q^{\mathcal{B}_3}) &= q^{\frac{1}{2}\mathcal{B}_3} q^{-\frac{1}{2}\mathcal{A}_3} \underbrace{\otimes}_{\Lambda} q^{\frac{1}{2}\mathcal{A}_3} q^{\frac{1}{2}\mathcal{B}_3} \end{aligned} \tag{4.25}$$

It now follows immediately that the algebra Eqs. (4.11–4.13) is preserved by the above definition of the coproduct. Since the matching condition is automatically respected by the matrices under consideration, one can formally calculate without it, taking into account only Eq. (4.23). On the other hand, the Casimir constraint Eq. (4.14) enters explicitly. We remark that the coproduct takes the simpler form Eq. (4.21) in terms of the matrices A^λ and B^λ , while their algebra is more complicated than that of the A_\pm, B_\pm . Note that the constraint of equality between the Casimir eigenvalues as well as the matching condition are not compatible with the ordinary coproduct for $U_q(sl(2)) \otimes U_q(sl(2))$. Once we have defined a coproduct, the question naturally arises whether we have a full Hopf algebra. We will show, in Sect. 7 that this is not strictly the case, but that an interesting Hopf like structure of a novel type arises which is a natural generalization of the usual one.

4.2 Relation with the ξ transformation laws Finally, we return to our beginning (Sect. 3.1) to connect the present $U_q(sl(2)) \otimes U_q(sl(2))$ with the original $U_q(sl(2))$ that appeared in the ξ transformation laws. For this it is convenient to define matrices with four indices by formulae similar to Eqs. (4.6). Consider Eqs. (3.10) and (3.11). It is convenient to let

$$\begin{aligned} \mathcal{J}_{\pm}^{\lambda(J)}{}_{\rho'\varpi',\rho\varpi} &= [\mathcal{J}_{\pm}^{\lambda}{}_{\varpi'}]_{\frac{\varpi'-\rho'}{2},\frac{\varpi-\rho}{2}}^{(J)} \delta_{\varpi',\varpi-2\lambda}, [q^{\pm\mathcal{J}_3\lambda}]_{\rho'\varpi',\rho\varpi}^{(J)} \\ &= [q^{\pm\mathcal{J}_3,\lambda}]_{\frac{\varpi'-\rho'}{2},\frac{\varpi-\rho}{2}}^{(J)} \delta_{\varpi',\varpi-2\lambda}, \end{aligned}$$

$$[\mathcal{X}_{\pm}^{\lambda}]_{\rho'\varpi',\rho\varpi}^{(J)} = [\mathcal{X}_{\pm}^{\lambda}{}_{\varpi'}]_{\frac{\varpi'-\rho'}{2},\frac{\varpi-\rho}{2}}^{(J)} \delta_{\varpi',\varpi-2\lambda}, \quad (4.26)$$

Then the relations Eqs. (3.14), (3.19), (3.33), (4.1), (4.9), (4.10) may be summarized by the matrix equalities

$$\begin{aligned} q^{\mathcal{J}_3\lambda} &= \frac{2\lambda}{[\mathcal{A}_3 - \mathcal{B}_3]_q} [\mathcal{A}_{-2\lambda} + \mathcal{B}_{-2\lambda}], \\ \mathcal{X}_{\pm}^{\lambda} &= \frac{2\lambda q^{\pm 1/2}}{[\mathcal{A}_3 - \mathcal{B}_3]_q} [q^{\pm 2\lambda(\mathcal{A}_3 - \mathcal{B}_3)} \mathcal{A}_{-2\lambda} + q^{\mp 2\lambda(\mathcal{A}_3 - \mathcal{B}_3)} \mathcal{B}_{-2\lambda}], \\ \mathcal{J}_{\pm}^{\lambda} &= \frac{\pm q^{\mp 1/2}}{q - q^{-1}} \left[\mathcal{X}_{\pm}^{\lambda} - q^{\pm 2\lambda(\mathcal{A}_3 + \mathcal{B}_3) \pm \frac{1}{2}} q^{-\mathcal{J}_3\lambda} \right]. \end{aligned} \quad (4.27)$$

These relations imply some identities between these matrices, i.e.

$$\begin{aligned} q^{-\frac{1}{2}} \mathcal{X}_+^{\lambda} + q^{\frac{1}{2}} \mathcal{X}_-^{\lambda} &= (q^{\Omega_L} + q^{-\Omega_L}) q^{\mathcal{J}_3\lambda}. \\ &= (q - q^{-1})(\mathcal{J}_-^{\lambda} - \mathcal{J}_+^{\lambda}) + (q^{\Omega_R} + q^{-\Omega_R}) q^{-\mathcal{J}_3\lambda} \end{aligned} \quad (4.28)$$

For consistency we finally show how the algebra of the matrices on the left hand sides, which follow from the definitions Eqs. (3.10), (3.11), (3.9), may be derived from Eqs. (4.11)–(4.14). For this we will use the matrix equivalent of Eqs. (3.10), (3.11). Let us define

$$\begin{aligned} [J_{\pm}^{\lambda}]_{\rho'\varpi',\rho\varpi}^{(J)} &= [J_{\pm}^{\lambda}]_{\frac{\varpi'-\rho'}{2},\frac{\varpi-\rho}{2}} \delta_{\varpi',\varpi-2\lambda}, \\ [q^{\pm\mathcal{J}_3\lambda}]_{\rho'\varpi',\rho\varpi}^{(J)} &= [q^{\pm\mathcal{J}_3}]_{\frac{\varpi'-\rho'}{2},\frac{\varpi-\rho}{2}} \delta_{\varpi',\varpi-2\lambda}, \\ [U]_{\rho'\varpi',\rho\varpi}^{(J)} &= \mathcal{U}_{\frac{\varpi'-\rho'}{2},\frac{\varpi-\rho}{2}}^{(J)}(\varpi) \delta_{\varpi',\varpi}, \\ [\mathcal{V}]_{\rho'\varpi',\rho\varpi}^{(J)} &= \mathcal{V}_{\frac{\varpi'-\rho'}{2},\frac{\varpi-\rho}{2}}^{(J)}(\varpi) \delta_{\varpi',\varpi}. \end{aligned} \quad (4.29)$$

On the r.h.s. of the first line, we use the $U_q(sl(2))$ matrix representation that appeared in Sect. 2. It follows from the definition Eqs. (3.1) that the last two matrices are inverses of one another,

$$\sum_{\rho_2, \varpi_2} [U^{\lambda}]_{\rho_3 \varpi_3, \rho_2 \varpi_2}^{(J)} [\mathcal{V}^{\lambda}]_{\rho_2 \varpi_2, \rho_1 \varpi_1}^{(J)} = \delta_{\varpi_3, \varpi_1} \delta_{\rho_3, \rho_1}, \quad (4.30)$$

so that Eqs. (3.10), (3.11) take the matrix forms

$$q^{\pm\mathcal{J}_3\lambda} = \mathcal{U} q^{\pm\mathcal{J}_3\lambda} \mathcal{U}^{-1}, \quad \mathcal{J}_{\pm}^{\lambda} = \mathcal{U} J_{\pm}^{\lambda} \mathcal{U}^{-1}. \quad (4.31)$$

So far we defined our matrices with indices $\lambda = \pm 1/2$. The definitions given immediately extend to arbitrary $m/2$ with m integer. Then it is easy to see that the matrices $J_{\pm}^{m/2}$, and $q^{\pm\mathcal{J}_3, m/2}$, q^{Ω_R} generate a loop extension of $U_q(sl(2))$, that is

$$\begin{aligned}
\left[J_+^{\frac{m}{2}}, J_-^{\frac{n}{2}} \right] &= \frac{q^{2J_3, \frac{m+n}{2}} - q^{-2J_3, \frac{m+n}{2}}}{q - q^{-1}}, \\
q^{2J_3, \frac{m+n}{2}} &= q^{J_3, \frac{m}{2}} q^{J_3, \frac{n}{2}}, \quad q^{J_3, \frac{m}{2}} q^{-J_3, -\frac{m}{2}} = 1, \\
q^{J_3, \frac{m}{2}} J_{\pm}^{\frac{n}{2}} &= q^{\pm 1} J_{\pm}^{\frac{n}{2}} q^{J_3, \frac{m}{2}}, \\
q^{\Omega_R} J_{\pm}^{\frac{m}{2}} &= J_{\pm}^{\frac{m}{2}} q^{\Omega_R - m}, \quad q^{\Omega_R} q^{\pm J_3, \frac{m}{2}} = q^{\pm J_3, \frac{m}{2}} q^{\Omega_R - m}. \tag{4.32}
\end{aligned}$$

Thus q^{Ω_R} is the grading operator. Moreover, if we introduce the matrix

$$[I]_{\rho' \omega', \rho \omega} = \delta_{\frac{\omega' - \rho'}{2}, \frac{\omega - \rho}{2}} \delta_{\omega', \omega - 1}, \tag{4.33}$$

we obviously have

$$\begin{aligned}
J_+^{\frac{m}{2}} &= J_+^0 I^m, \quad q^{J_3, \frac{m}{2}} = q^{J_3, 0} I^m, \\
J_+^{\frac{m}{2}} I &= I J_+^{\frac{m}{2}}, \quad q^{J_3, \frac{m}{2}} I = I q^{J_3, \frac{m}{2}}, \tag{4.34}
\end{aligned}$$

where J_+^0 , and $q^{J_3, 0}$ are the usual matrices of $U_q(sl(2))$. They commute with q^{Ω_R} and I , which satisfy

$$q^{\Omega_R} I = I q^{\Omega_R + 1}. \tag{4.35}$$

Clearly, in view of the similarity relation Eq. (4.31), if we let

$$\mathcal{I} = U I U^{-1}, \tag{4.36}$$

the matrices $J_{\pm}^{\frac{m}{2}} q^{J_3, \frac{m}{2}} q^{\Omega_R} \mathcal{I}$ also satisfy the algebra¹² just displayed. Next we connect its restriction for $\lambda = \pm 1/2$ to the $U_q(sl(2)) \otimes U_q(sl(2))$ algebra Eqs. (4.11–4.14) by making use of Eqs. (4.27). Note that the first two equalities of (4.27) are simple while the last one involves $q^{-J_3 - \lambda}$ which, being the inverse of $q^{J_3, \lambda}$, is to be computed by taking the inverse of both sides of the first equation. Consistently with that, and by a mechanism which resembles the use of Eq. (2.8) to eliminate $\mathcal{O} [q^{-J_3}]$ (recalled in Sect. 2), we will see that if we consider the algebra of $q^{J_3, \mu}$ and χ_{\pm}^{λ} (instead of $q^{J_3, \mu}$ and J_{\pm}^{λ}), the complicated operator $q^{-J_3, \lambda}$ will disappear from the algebra. For this, one first deduces the following relations satisfied by the χ matrices

$$\begin{aligned}
q^{J_3, \lambda} \chi_{\pm}^{\mu} - q^{\pm 1} \chi_{\pm}^{\mu} q^{J_3, \lambda} &= \mp (q - q^{-1}) q^{\mp(2\lambda\Omega_R - \frac{1}{2})} \delta_{\lambda+\mu, 0}, \\
q^{J_3, \lambda} \chi_{\pm}^{\mu} - q^{\pm 1} \chi_{\pm}^{\lambda} q^{J_3, \mu} &= (q - q^{-1}) \left[q^{\mp(2\lambda\Omega_R - \frac{1}{2})} - q^{\pm(2\lambda\Omega_R + \frac{1}{2})} \right] \delta_{\lambda+\mu, 0}, \tag{4.37}
\end{aligned}$$

$$\begin{aligned}
\left[\chi_+^{\mu}, \chi_-^{\lambda} \right] &= -(q - q^{-1}) q^{J_3, \mu} q^{J_3, \lambda} \\
&+ (q - q^{-1}) q^{-2\lambda\Omega_R} (q^{-\frac{1}{2}} \chi_+^{\lambda} + q^{\frac{1}{2}} \chi_-^{\lambda}) q^{-J_3, \lambda} \delta_{\lambda+\mu, 0}, \\
\chi_+^{\mu} \chi_-^{\lambda} - \chi_+^{\lambda} \chi_-^{\mu} &= -(q - q^{-1}) q^{J_3, \mu} q^{J_3, \lambda} \\
- [q^{2\lambda\Omega_R - 1} - q^{-2\lambda\Omega_R + 1}] & (q^{-\frac{1}{2}} \chi_+^{\lambda} + q^{\frac{1}{2}} \chi_-^{\lambda}) q^{-J_3, \lambda} \delta_{\lambda+\mu, 0}, \tag{4.38}
\end{aligned}$$

$$\left[\chi_+^{\mu}, \chi_+^{\lambda} \right] = \left[\chi_-^{\mu}, \chi_-^{\lambda} \right] = 0. \tag{4.39}$$

Using Eqs. (4.28), Eqs. (4.38) become

¹² q^{Ω_R} is invariant under the transformation.

$$\begin{aligned} [\chi_+^\mu, \chi_-^\lambda] &= -(q - q^{-1})q^{J_3, \mu} q^{J_3, \lambda} + (q - q^{-1})[q^{-2\lambda\Omega_R}(q^{\Omega_L} + q^{-\Omega_L})]\delta_{\lambda+\nu, 0}, \\ \chi_+^\mu \chi_-^\lambda - \chi_-^\mu \chi_+^\lambda &= -(q - q^{-1})q^{J_3, \mu} q^{J_3, \lambda} - [q^{2\lambda\Omega_R-1} - q^{-2\lambda\Omega_R+1}][q^{\Omega_L} + q^{-\Omega_L}]\delta_{\lambda+\nu, 0}. \end{aligned} \tag{4.40}$$

It is now straightforward to verify that Eqs. (4.37), 4.39, 4.40, as well as the relations

$$[q^{J_3, \lambda}, q^{J_3, \mu}] = 0 \tag{4.41}$$

follow from the $U_q(sl(2)) \otimes U_q(sl(2))$ relations displayed by Eqs. (4.11–4.14) if one uses the first two relations of Eqs. (4.27). Moreover, Eqs. (4.27) can be used to rewrite the \mathcal{A} and \mathcal{B} matrices as

$$\begin{aligned} \mathcal{A}_{2\lambda} &= \frac{1}{2} \{ [\Omega_L]_q q^{J_3, -\lambda} + [\Omega_R]_q q^{-J_3, -\lambda} - 2\lambda(\mathcal{J}_+^{-\lambda} + \mathcal{J}_-^{-\lambda}) \}, \\ \mathcal{B}_{2\lambda} &= \frac{1}{2} \{ [\Omega_L]_q q^{J_3, -\lambda} - [\Omega_R]_q q^{-J_3, -\lambda} + 2\lambda(\mathcal{J}_+^{-\lambda} + \mathcal{J}_-^{-\lambda}) \}. \end{aligned} \tag{4.42}$$

Thus the six operators $q^{A_3}, q^{B_3}, \mathcal{A}_\pm, \mathcal{B}_\pm$ are functions of five operators, e.g. $\Omega_L, \Omega_R, q^{J_3, \pm\frac{1}{2}}, \mathcal{J}_+^{\frac{1}{2}} + \mathcal{J}_-^{\frac{1}{2}}, \mathcal{I}$ (the latter is defined by Eq. (4.36)). So there must be one constraint relating them, which is precisely the equality of the Casimir operators discussed above.

4.3 Internal invariance. The aim of this subsection is to show that the matrices $A^\lambda, B^\lambda, q^{\Omega_L}, q^{\Omega_R}$ introduced above, which describe the operatorial actions Eqs. (4.5), (3.5), (3.24), give rise to symmetries of the operator algebra of the Bloch waves. The reasoning we apply is closely parallel to the one for the covariant basis, which was laid out in ref. [7]. Let us therefore recall briefly the situation for the covariant fields. Consider Eqs. (2.4), (2.3). For each of the first two operator actions, two matricial actions, $[q^{-J_3}]_{NM}, [J_+]_{NM}$ and $[q^{-J_3}]_{NM}, [J_-]_{NM}$ appear. They are multiplied by different operators, and thus lead to independent symmetries of the operator algebra. This is easily seen upon combining the operatorial actions with the commutativity of fusion and braiding, and the Yang-Baxter equation, two special cases of the general Moore-Seiberg consistency conditions [23]. Since $[q^{-J_3}]_{NM}$ appears twice, we have three symmetries $[J_\pm]_{NM}, [J_3]_{NM}$ altogether, in correspondence with the number of generators. More precisely, the following matricial, or ‘internal’ transformations:

$$\xi_M^{(J)}(\sigma) \rightarrow \sum_N \xi_N^{(J)}(\sigma) [J^a]_{NM}, \tag{4.43}$$

$$\xi_{M_1}^{(J_1)}(\sigma_1)\xi_{M_2}^{(J_2)}(\sigma_2) \rightarrow \sum_{N_1, N_2, b, c} \xi_{N_1}^{(J_1)}(\sigma_1)\xi_{N_2}^{(J_2)}(\sigma_2)A_{bc}^a [J^b]_{N_1 M_1} [J^c]_{N_2 M_2}, \tag{4.44}$$

preserve the fusion and the braiding of the ξ fields. The fusion of the ξ fields is given essentially in terms of $q - 3j$ symbols,

$$\xi_{M_1}^{(J_1)}(\sigma_1)\xi_{M_2}^{(J_2)}(\sigma_2) = \sum_{J_{12}=|J_1-J_2}^{J_1+J_2} g_{J_1 J_2}^{J_{12}}(J_1, M_1; J_2, M_2 | J_{12})(\xi_{M_1+M_2}^{(J_{12})}(\sigma_2) + \text{desc.}), \tag{4.45}$$

where $g_{J_1 J_2}^{J_{12}}$ are coupling constants (reduced matrix elements) and desc. denotes Virasoro descendants. The fusion and braiding properties of the latter are the same as those of the primaries [11], so we will not consider them explicitly in the following. The quantum

group invariance of the fusion is equivalent to the standard recursion relations for the q -Clebsch-Gordan coefficients,

$$\sum_{N_1+N_2=N_{12}} (J_1, N_1; J_2, N_2 | J_{12}) \Lambda_{de}^b [J^d]_{N_1 M_1} [J^e]_{N_2 M_2} = (J_1, M_1; J_2, M_2 | J_{12}) [J^b]_{N_{12} M_{12}}. \tag{4.46}$$

Similarly, as the braiding of two ξ fields is given by the universal R -matrix of $U_q(sl(2))$, the invariance of the braiding is tantamount to the defining relation between co-product and R -matrix, viz.

$$(J_1, J_2)_{N_1 N_2}^{P_2 P_1} \Lambda_{de}^b [J^d]_{N_1 M_1} [J^e]_{N_2 M_2} = \Lambda_{de}^b [J^d]_{P_2 N_2} [J^e]_{P_1 N_1} (J_1, J_2)_{M_1 M_2}^{N_2 N_1}. \tag{4.47}$$

Equation (4.47) arises in our operator formalism as a direct consequence of the Yang-Baxter equation.

Let us now examine the case of the ψ fields and look for an internal type invariance of the fusion and the braiding. We start from the fusion relation for the ψ fields ($m_{12} = m_1 + m_2$),

$$\psi_{m_1}^{(J_1)}(\sigma_1) \psi_{m_2}^{(J_2)}(\sigma_2) = \sum_{J_{12}=-m_{12}}^{J_1+J_2} g_{J_1 J_2}^{J_{12}} N \left| \begin{matrix} J_1 & J_2 & J_{12} \\ m_1 & m_2 & m_1+m_2 \end{matrix} \right| \varpi \left(\psi_{m_{12}}^{(J_{12})}(\sigma_2) + \text{desc.} \right), \tag{4.48}$$

where $g_{J_1 J_2}^{J_{12}}$ are the same coupling constants as in Eq. (4.45), and the fusion coefficient N is given in terms of a $q - 6j$ symbol¹³,

$$N \left| \begin{matrix} J_1 & J_2 & J_{12} \\ m_1 & m_2 & m_{12} \end{matrix} \right| \varpi = \frac{\tilde{g}_{J_{12} x+m_{12}}^x}{\tilde{g}_{J_1 x+m_1}^x \tilde{g}_{J_2 x+m_{12}}^x} \left\{ \begin{matrix} J_1 & J_2 & J_{12} \\ x+m_{12} & x & x+m_1 \end{matrix} \right\}. \tag{4.49}$$

Here we have introduced the abbreviations $m_{12} = m_1 + m_2$, $n_{12} = n_1 + n_2$ and, as in previous references,

$$x := \frac{1}{2}(\varpi - 1 - \pi/h). \tag{4.50}$$

The constants \tilde{g} can be absorbed into the normalization of the $\psi_m^{(J)}$ and will not play any significant role. We recall that the $6j$ coefficient with a vertical bar used here and in previous references is related to the tetrahedron-symmetric $q - 6j$ symbol by

$$\left\{ \begin{matrix} a & b & e \\ d & c & f \end{matrix} \right\} = ((2e + 1)_q [2f + 1]_q)^{-\frac{1}{2}} (-1)^{a+b-c-d-2e} \left\{ \begin{matrix} a & b & e \\ d & c & f \end{matrix} \right\}. \tag{4.51}$$

Let us introduce a $3j$ symbol with two magnetic quantum numbers by¹⁴

$$(J_1 \rho_1 \varpi_1; J_2 \rho_2 \varpi_2 | J_{12} \rho_{12} \varpi_{12}) := \delta_{\rho_{12}, \rho_1} \delta_{\varpi_{12}, \varpi_2} \delta_{\varpi_1, \rho_2} \times N \left| \begin{matrix} J_1 & J_2 & J_{12} \\ \frac{\varpi_1 - \rho_1}{2} & \frac{\varpi_2 - \rho_2}{2} & \frac{\varpi_{12} - \rho_{12}}{2} \end{matrix} \right|, \rho_1. \tag{4.52}$$

Then Eq. (4.48) can be rewritten as

¹³ The condition $m_{12} = m_1 + m_2$ is not implied by the property of the $6j$ symbols and should be added by hand
¹⁴ The conservation of m imposed earlier now follows from a combination of the three delta functions.

$$\psi_{\rho_1, \varpi_1}^{(J_1)} \psi_{\rho_2, \varpi_2}^{(J_2)} \delta_{\varpi_1, \rho_2} = \sum_{J_{12}=-m_{12}}^{J_1+J_2} g_{J_1 J_2}^{J_{12}}(J_1, \rho_1, \varpi_1; J_2, \rho_2, \varpi_2 | J_{12}, \rho_{12}, \varpi_{12}) \times (\psi_{\rho_{12}, \varpi_{12}}^{(J_{12})} + \text{desc.}). \tag{4.53}$$

Let us now braid both sides of Eq. (4.53) with $\mathcal{O}[T_{+2\lambda}]_{\sigma_+}$, and then fuse again the result on the left hand side to $\psi_{\rho_{12}, \varpi_{12}}^{(J_{12})}$. Using Eq. (4.5), we get

$$\begin{aligned} &\sum_{\rho'_1 \varpi'_1, \rho'_2 \varpi'_2} (J_1, \rho'_1, \varpi'_1; J_2, \rho'_2, \varpi'_2 | J_{12}, \rho'_{12}, \varpi'_{12}) \{ \mathcal{O}[T_{+2\lambda}]_{\sigma_+} ([A^\lambda]_{\rho'_1, \varpi'_1; \rho_1, \varpi_1}^{(J_1)} [A^\lambda]_{\rho'_2, \varpi'_2; \rho_2, \varpi_2}^{(J_2)}) \\ &+ [B^{-\lambda}]_{\rho'_1, \varpi'_1; \rho_1, \varpi_1}^{(J_1)} [B^\lambda]_{\rho'_2, \varpi'_2; \rho_2, \varpi_2}^{(J_2)} + \mathcal{O}[T_{-2\lambda}]_{\sigma_+} ([A^\lambda]_{\rho'_1, \varpi'_1; \rho_1, \varpi_1}^{(J_1)} [B^{-\lambda}]_{\rho'_2, \varpi'_2; \rho_2, \varpi_2}^{(J_2)}) \\ &+ [B^{-\lambda}]_{\rho'_1, \varpi'_1; \rho_1, \varpi_1}^{(J_1)} [A^{-\lambda}]_{\rho'_2, \varpi'_2; \rho_2, \varpi_2}^{(J_2)} \} = \\ &(J_1, \rho_1, \varpi_1; J_2, \rho_2, \varpi_2 | J_{12}, \rho_{12}, \varpi_{12}) \{ \mathcal{O}[T_{+2\lambda}]_{\sigma_+} [A^\lambda]_{\rho'_{12}, \varpi'_{12}; \rho_{12}, \varpi_{12}}^{(J_{12})} \\ &+ \mathcal{O}[T_{-2\lambda}]_{\sigma_+} [B^{-\lambda}]_{\rho'_{12}, \varpi'_{12}; \rho_{12}, \varpi_{12}}^{(J_{12})} \} \end{aligned} \tag{4.54}$$

Exactly as for the case of the covariant fields, we now compare the coefficients of like operators and obtain

$$\begin{aligned} &\sum_{\rho'_1 \varpi'_1, \rho'_2 \varpi'_2} (J_1, \rho'_1, \varpi'_1; J_2, \rho'_2, \varpi'_2 | J_{12}, \rho'_{12}, \varpi'_{12}) \Lambda(A^\lambda)_{\rho'_1 \varpi'_1, \rho'_2 \varpi'_2; \rho_1 \varpi_1, \rho_2 \varpi_2} = \\ &[A^\lambda]_{\rho'_{12}, \varpi'_{12}; \rho_{12}, \varpi_{12}}^{(J_{12})} (J_1, \rho_1, \varpi_1; J_2, \rho_2, \varpi_2 | J_{12}, \rho_{12}, \varpi_{12}), \end{aligned} \tag{4.55}$$

where Eq. (4.18) has been used, and similarly with A^λ replaced by B^λ . Recalling that the generalized $3j$ symbols above are given in terms of $q - 6j$ symbols, Eq. (4.55) demonstrates that for the ψ basis, the $6j$ symbols acquire an interpretation as Clebsch-Gordan coefficients for the new quantum group structure. We can use Eq. (4.55) to determine the new $3j$ symbols by recursion, in analogy to the standard case, except that here we need to know, for example, the coefficient $(J_1, \varpi_1 - 2J_1, \varpi_1; J_2, \varpi_2 - 2J_2, \varpi_2 | J_{12}, \rho_{12}, \varpi_{12})$ for all ϖ_2 as a starting point. Eq. (4.55) arises from the commutativity of fusion and braiding - one of the Moore-Seiberg consistency conditions - exactly as in the covariant basis. We have thus seen that the fusion is invariant under the internal transformations

$$\psi_{\rho, \varpi}^{(J)} \rightarrow \sum_{\rho', \varpi'} \psi_{\rho', \varpi'}^{(J)} [A^\lambda]_{\rho' \varpi'; \rho \varpi}^{(J)}, \tag{4.56}$$

and

$$\psi_{\rho, \varpi}^{(J)} \rightarrow \sum_{\rho', \varpi'} \psi_{\rho', \varpi'}^{(J)} [B^\lambda]_{\rho' \varpi'; \rho \varpi}^{(J)}, \tag{4.57}$$

together with the co-product defined by Eqs. (4.17). Concerning the zero modes, we have two additional internal invariances which leave the fusion invariant:

$$\psi_{\rho, \varpi}^{(J)} \rightarrow \psi_{\rho, \varpi}^{(J)} q^\rho, \quad \psi_{\rho, \varpi}^{(J)} \rightarrow \psi_{\rho, \varpi}^{(J)} q^\varpi, \tag{4.58}$$

corresponding to the matrices q^{Ω_L} and q^{Ω_R} respectively (see Eq. (4.7)). The action on $\psi \otimes \psi$ is given by their co-product displayed on Eq. (4.25). Including Ω_L and Ω_R gives

six generators for the internal invariance. The Casimir constraint Eq. (4.14) tells us that only five of them are independent.

A similar discussion applies to the braiding of the ψ fields. The general braiding relation reads [10]

$$\psi_{m_1}^{(J_1)}(\sigma_1)\psi_{m_2}^{(J_2)}(\sigma_2) = S(J_1, J_2; \varpi)_{m_1 m_2}^{m'_2 m'_1} \psi_{m'_2}^{(J_2)}(\sigma_2)\psi_{m'_1}^{(J_1)}(\sigma_1),$$

with

$$\begin{aligned} S(J_1, J_2; \varpi)_{m_1 m_2}^{m'_2 m'_1} &= \sum_{m'_1, m'_2} q^{\mp(2m_1 m_2 + m_2^2 - m_1^2 + \varpi(m_2 - m'_1))} \delta_{m'_1 + m'_2, m_1 + m_2} \frac{\tilde{g}_{J_1, x+m'_2}^{x+m'_1} \tilde{g}_{J_2, x+m'_1}^x}{\tilde{g}_{J_2, x+m_1}^{x+m_2} \tilde{g}_{J_1, x+m_1}^x} \\ &\times \left\{ \begin{matrix} J_1 & x+m_2 \\ J_2 & x \end{matrix} \middle| \begin{matrix} x+m'_2 \\ x+m_1 \end{matrix} \right\} \end{aligned} \tag{4.59}$$

and the upper sign is to be taken when $\sigma_1 > \sigma_2$. An explicit δ coefficient has been written for the conservation of m , because it is not automatically implied by the properties of the $6j$ symbol. We note that the braiding of descendants of the ψ fields is given by the same formula [10]. Again, we can rewrite this formula in terms of double index symbols:

$$\begin{aligned} \psi_{\rho_1 \varpi_1}^{(J_1)} \psi_{\varpi_2 \omega_2}^{(J_2)} &= \sum_{\varpi'_1 \rho'_2 \omega'_2} S(J_1, J_2)_{\rho_1 \varpi_1, \varpi'_1 \omega'_2}^{\rho'_2 \omega'_2, \varpi'_1 \omega'_1} \psi_{\rho'_2 \omega'_2}^{(J_2)} \psi_{\varpi'_1 \omega'_1}^{(J_1)}, \end{aligned} \tag{4.60}$$

The braiding matrix on the right hand side is given by

$$S(J_1, J_2)_{\rho_1 \varpi_1, \rho_2 \omega_2}^{\rho'_1 \omega'_1, \rho'_2 \omega'_2} := S(J_1, J_2, \varpi_2 - 2n_{12})_{m_1 m_2}^{m'_2 m'_1} \delta_{\varpi_1, \rho_2} \delta_{\omega'_2, \rho'_1} \delta_{\rho_1, \rho'_2} \delta_{\omega_2, \varpi'_1} \tag{4.61}$$

with

$$m_1 = \frac{\varpi_1 - \rho_1}{2}, m_2 = \frac{\varpi_2 - \rho_2}{2}, m'_1 = \frac{\varpi'_1 - \rho'_1}{2}, m'_2 = \frac{\varpi'_2 - \rho'_2}{2}.$$

The Kronecker symbols $\delta_{\varpi_1, \rho_2} \delta_{\omega'_1, \rho'_1}$ represent the matching conditions, while $\delta_{\rho_1, \rho'_2} \delta_{\omega_2, \varpi'_1}$ incorporate the conservation of m . The internal invariances of the braiding are now generated by comparing the action of $\mathcal{O}[T_{+2\lambda}]_{\sigma_+}$ on a product of two ψ fields before and after the braiding of the latter. One obtains, by a similar argument as for the case of fusion,

$$\begin{aligned} &\sum_{\rho_3 \varpi_3, \rho_4 \varpi_4} S(J_1, J_2)_{\rho_3 \varpi_3, \rho_4 \varpi_4}^{\rho_6 \varpi_6, \rho_5 \varpi_5} A(A^\lambda)_{\rho_3 \varpi_3, \rho_4 \varpi_4; \rho_1 \varpi_1, \rho_2 \omega_2} \\ &= \sum_{\rho_3 \varpi_3, \rho_4 \varpi_4} S(J_1, J_2)_{\rho_1 \varpi_1, \rho_2 \omega_2}^{\rho_4 \varpi_4, \rho_3 \varpi_3} \bar{A}(A^\lambda)_{\rho_6 \varpi_6, \rho_5 \varpi_5; \rho_4 \varpi_4, \rho_3 \varpi_3}, \end{aligned} \tag{4.62}$$

where \bar{A} is the co-product with factors 1 and 2 exchanged, and similarly for the case of B^λ . Of course, the commutativity of the two orders of the braiding is again one of the Moore-Seiberg conditions. Equation (4.62) relates the co-product to the corresponding R matrix, again just as in the standard case, so that Eq. (4.61) defines a kind of “universal R matrix” associated with our quantum group structure.

Returning briefly to the ξ fields, we remark that the action of $U_q(sl(2)) \otimes U_q(sl(2))$ on them may be defined by using Eqs. (4.42). It reduces to the action recalled earlier by the matrices J_\pm, q^{J_3} (see Eqs. (4.43), (4.44)), and the trivial left and right multiplications by q^ϖ .

5. Application to State/Operator Classifications

Since our internal symmetry group is directly connected with the Liouville zero mode, it should be a good tool to classify the spectrum of primary fields and associated Verma modules. We will discuss some aspects of this here, without going into details.

5.1 General aspects of the representations We will make use of the expressions Eqs. (4.9), (4.10) for the generators. It will be convenient to simplify notation by letting $\mu = (\varpi + \rho)/2, \nu = (\varpi - \rho)/2$, so that

$$\begin{aligned} [\mathcal{A}_{\pm}]_{\mu',\nu';\mu,\nu} &= \pm[\mu \pm (J + 1)]_q \delta_{\mu', \mu \pm 1} \delta_{\nu', \nu}, \\ [\mathcal{B}_{\pm}]_{\mu',\nu';\mu,\nu} &= \pm q^{\mp \frac{1}{2}} [J \pm \nu]_q \delta_{\mu', \mu} \delta_{\nu', \nu \pm 1}. \end{aligned} \tag{5.1}$$

Note that, since the matrix elements are proportional to q deformed numbers, one has

$$\begin{aligned} [\mathcal{A}_{\pm}]_{\mu',\nu';\mu,\nu} &= (-1)^\alpha [\mathcal{A}_{\pm}]_{\mu'+\alpha\frac{\pi}{h},\nu';\mu+\alpha\frac{\pi}{h},\nu}, \\ [\mathcal{B}_{\pm}]_{\mu',\nu';\mu,\nu} &= (-1)^\alpha [\mathcal{B}_{\pm}]_{\mu',\nu'+\alpha\frac{\pi}{h};\mu,\nu+\alpha\frac{\pi}{h}}, \end{aligned} \tag{5.2}$$

where α is an arbitrary integer. This reflects the existence of another “dual” quantum group with parameter $\hat{h} = \pi^2/h$ which commutes with the present one up to a sign. We will not consider this group here, since we only include one screening charge, but the equations just written are important for the coming discussion where a shift of this type will be needed. To shorten the discussion we will only deal with positive half-integer spins J . The generalization is straightforward. We will first discuss the case of generic h . Then it is trivial to verify that

$$\begin{aligned} [\mathcal{A}_-]_{J,\nu;J+1,\nu} &= [\mathcal{A}_+]_{-J,\nu;-(J+1),\nu} = 0, \\ [\mathcal{B}_-]_{\mu,-J-1;\mu,-J} &= [\mathcal{B}_+]_{\mu,J+1;\mu,J} = 0. \end{aligned} \tag{5.3}$$

For generic h , these relations give us, up to the shift Eqs. (5.2), all the highest/lowest weight states. Consider first the \mathcal{A} algebra, ignoring the ν index since these generators do no act upon it. The states with $\mu = (J + 1)$, and $\mu = -(J + 1)$ are lowest weight and highest weight states, respectively. For $J > 0$, we get two disjoint semi-infinite representations with $\mu = J + 1 + n$ and $\mu = -(J + 1 + n)$, n non-negative integer, respectively. Next concerning the \mathcal{B} algebra, the states with $\nu = J$ and $\nu = -J$ are lowest weight and highest weight states, respectively. For positive J , we get a finite dimensional representation if $2J$ is integer. Let us now show that these simple facts allow us to recover the three cases which were discussed earlier [15]. There it was shown that the type of operator algebras is specified by the number – going from one to three – of triangular inequalities satisfied by each vertex. These cases were called TI1, TI2, TI3 in ref. [15]. To specify the spins associated with ϖ and ρ , one lets $\varpi = \varpi_0 + 2J_3, \rho = \varpi_0 + 2J_2$, where $\varpi_0 = 1 + \frac{\pi}{h}$ is such that the corresponding eigenvalue of L_0 vanishes. In the TI3 case, one has three triangular inequalities between J, J_2, J_3 , so that the latter are all half integers. This gives

$$\begin{aligned} J + J_3 - J_2 &\equiv J + (\varpi - \varpi_0)/2 - (\rho - \varpi_0)/2 = J + \nu \in \mathbb{Z}_+, \\ J - J_3 + J_2 &\equiv J - (\varpi - \varpi_0)/2 + (\rho - \varpi_0)/2 = J - \nu \in \mathbb{Z}_+, \end{aligned}$$

$$-J + J_3 + J_2 \equiv -J + (\varpi - \varpi_0)/2 + (\rho - \varpi_0)/2 = \mu - J - 1 - \frac{\pi}{h} \in \mathcal{Z}_+. \quad (5.4)$$

The first two inequalities give back the range of the finite dimensional \mathcal{B} representation. For the matrices \mathcal{A} , we see that the lowest value coincides with the lowest weight found above up to a shift of $\frac{\pi}{h}$, which may be incorporated easily by using Eq. (5.2). Thus the last inequality corresponds to the semi-infinite representation with lowest weight which we found earlier. The other semi-infinite \mathcal{A} representation is easily seen to correspond to negative spins using the formulae just recalled, and we leave it out for the present time. For the case TI2, one only imposes

$$J + J_3 - J_2 \equiv J + (\varpi - \varpi_0)/2 - (\rho - \varpi_0)/2 = J + \nu \in \mathcal{Z}_+,$$

$$J - J_3 + J_2 \equiv J - (\varpi - \varpi_0)/2 + (\rho - \varpi_0)/2 = J - \nu \in \mathcal{Z}_+.$$

Thus the \mathcal{B} representation is still finite dimensional, while the \mathcal{A} representation has no lower or upper bound. This is consistent with the above discussion because for the latter the lowest weight state is never reached since $J_2 + J_3$ is not half integer. For the TI1 case, that is for

$$J + J_3 - J_2 \equiv J + (\varpi - \varpi_0)/2 - (\rho - \varpi_0)/2 = J + \nu \in \mathcal{Z}_+,$$

the \mathcal{B} representation is semi-infinite. Again this is in agreement with the above.

Next, it is interesting to consider the particular case $J = 0$. There are two possibilities. First, if we use the formulae with $2J$ integer, and let $J = 0$, the \mathcal{B} representation has dimension one, and is restricted to $\nu = 0$. For the \mathcal{A} representation, we get the range $\mu \geq \varpi_0$. Thus the spin zero representation is non trivial. As we will see, in Sect. 7, this explains why our coproduct does not possess a counit in the usual sense. Another type of spin zero representation is obtained by considering the TI1 case, where J may take continuous values, and letting $J \rightarrow 0$. Then the \mathcal{B} representation is semi-infinite. This corresponds to the case of the powers of the screening operator which will be studied in the forthcoming subsection.

Finally let us briefly turn to the case where q is a root of unity. For a rational theory, with $C = 1 - 6(p - p')^2/pp'$, the spectrum of Virasoro weights is given by

$$\Delta_{r,t} = \frac{(p'r - pt)^2 - (p - p')^2}{4pp'}.$$

The correspondence with the present formalism is such that we have $h = -p'\pi/p$, and $r = 2J + 1$, $t = 2\hat{J} + 1$, where \hat{J} specifies the representation of the dual quantum group with parameter $\hat{h} = -p\pi/p'$. As shown by BPZ, the set of primary fields with $1 \leq r \leq p-1$, $1 \leq t \leq p'$, r, t integers form a closed OPA, if one identifies the operators with quantum numbers (r, t) and $(p - r, p' - t)$. Let us show that our representation theory correctly gives the corresponding truncation of the spectrum of zero modes, assuming that we consider our ψ fields with $0 \leq 2J \leq p - 2$ according to the limits just recalled. Since we do not include the second screening charge, we only discuss the case $t = 1$, i.e. $\hat{J} = 0$. Consider the \mathcal{A} representation, with lowest weight vector $\mu = \varpi_0 + J$. Making use of the explicit expression Eqs. (5.1), one sees that

$$[(\mathcal{A}_+)^{p-1-2J}]_{\varpi_0+p-1-J, \nu; \varpi_0+J, \nu}^{(J)} = 0.$$

Thus the range of μ is $\varpi_0 + J \leq \mu \leq \varpi_0 + p - 1 - J$. Concerning the \mathcal{B} representation, it is easily seen that the range is still $-J \leq \nu \leq J$. Returning to ρ and ϖ , one verifies

that one has $\varpi_0 \leq \varpi \leq \varpi_0 + p - 2$, $\varpi_0 \leq \rho \leq \varpi_0 + p - 2$. Letting again $\varpi = \varpi_0 + 2J_3$, $\rho = \varpi_0 + 2J_2$, one sees that J_2 and J_3 vary over the same range as J , which is what we wanted to prove. There are of course many more points to discuss, such as the interpretation of the other representations, but we leave them for further study.

5.2 The internal symmetries of the Coulomb gas operators As another application we derive the transformation laws of the Coulomb gas operators: the Bäcklund free field and the screening operators. This will provide a concrete realization of the spin zero representation mentioned above. Since we have to change the operator normalizations we return to the earlier formulation using the notation $\psi_m^{(J)}$. It is easy to derive the explicit form of the action of $\mathcal{O}[T_-]_{\sigma_+}$ from Eqs. (4.1), (4.9). Next, we need to go from the ψ fields to the U fields introduced in ref. [10] whose normalization is well suited for discussing Coulomb gas operators. The correspondence is given by

$$\psi_m^{(J)}(\sigma) = \frac{1}{\beta_m^{(J)} \gamma_m^{(J)}} U_m^{(J)}(\sigma), \tag{5.5}$$

where

$$\beta_m^{(J)} = e^{ih(J+m)(\varpi - J+m)},$$

$$\gamma_m^{(J)} = \mu^{J+m} \frac{\rho(\varpi)}{\rho(\varpi + 2m)} \prod_1^{J+m} [\varpi + r]_q [\varpi + 2m - r]_{q^{-1}},$$

$$\rho = \sqrt{\Gamma(\varpi h/\pi) \Gamma(\varpi + 1) \Gamma_q(\varpi + 1)}, \quad \mu = -\frac{\pi^2}{h \sin h}.$$

Γ_q is the q deformed gamma function. This form is valid for arbitrary J provided $J + m$ is integer[18]. We get rid of ρ by transforming all the fields including the $\mathcal{O}[T_{\pm}]_{\sigma_{\pm}}$ generators, and forget about it. The action of the latter is straightforwardly deduced from the formulae just given:

$$\begin{aligned} \mathcal{O}[T_-]_{\sigma_+} U_m^{(J)}(\sigma) &= U_m^{(J)}(\sigma) e^{-ih(J+m)} \frac{[\varpi - 2m]_q [\varpi - J - m - 1]_q}{[\varpi]_q [\varpi - 1]_q} \mathcal{O}[T_-]_{\sigma_+} \\ &\quad + \mu U_{m-1}^{(J)}(\sigma) e^{ih(\varpi - J - m - \frac{1}{2})} [\varpi - 2m + 2]_q [J + m]_q \mathcal{O}[T_+]_{\sigma_+}, \\ \mathcal{O}[T_+]_{\sigma_+} U_m^{(J)}(\sigma) &= U_m^{(J)}(\sigma) e^{ih(J+m)} \frac{[\varpi + J - m + 1]_q}{[\varpi - 2m + 1]_q} \mathcal{O}[T_+]_{\sigma_+} \\ &\quad - \mu^{-1} U_{m+1}^{(J)}(\sigma) e^{-ih(\varpi - J - m - \frac{3}{2})} \frac{[J - m]_q}{[\varpi]_q [\varpi - 1]_q [\varpi - 2m - 1]_q} \mathcal{O}[T_-]_{\sigma_+}. \end{aligned} \tag{5.6}$$

Next, letting $J = 0$ gives the transformation properties of the screening operators,

$$\begin{aligned} \mathcal{O}[T_-]_{\sigma_+} S^m(\sigma) &= S^m(\sigma) e^{-ihm} \frac{[\varpi - 2m]_q [\varpi - m - 1]_q}{[\varpi]_q [\varpi - 1]_q} \mathcal{O}[T_-]_{\sigma_+} \\ &\quad + \mu S^{m-1}(\sigma) e^{ih(\varpi - m - \frac{1}{2})} [\varpi - 2m + 2]_q [m]_q \mathcal{O}[T_+]_{\sigma_+}, \\ \mathcal{O}[T_+]_{\sigma_+} S^m(\sigma) &= S^m(\sigma) e^{ihm} \frac{[\varpi - m + 1]_q}{[\varpi - 2m + 1]_q} \mathcal{O}[T_+]_{\sigma_+} \\ &\quad + \mu^{-1} S^{m+1}(\sigma) e^{-ih(\varpi - m - \frac{3}{2})} \frac{[m]_q}{[\varpi]_q [\varpi - 1]_q [\varpi - 2m - 1]_q} \mathcal{O}[T_-]_{\sigma_+}. \end{aligned} \tag{5.7}$$

As a check, one may verify that these formulae are consistent with the simple fusion algebra $S^m S^p \sim S^{m+p}$, which shows that these operators are indeed powers of the screening operator S . They provide a concrete realization of the spin zero representation of our internal symmetry group, where the \mathcal{B} representation is semi-infinite.

Finally, the transformation laws of the Bäcklund free field are obtained by expanding in J for fixed screening number $n = J + m$, according to $U_m^{(J)} \sim (1 - \alpha_- J \vartheta) S^n$. Before expanding, it is useful to remark that most of the explicit dependence upon J for fixed $J + m$ may be removed by moving some factors to the left, and rewriting Eq. (5.6) as according to

$$\begin{aligned} \mathcal{O}[T_-]_{\sigma_*} U_m^{(J)}(\sigma) &= [\varpi]_q U_m^{(J)}(\sigma) e^{-ih(J+m)} \frac{1}{[\varpi]_q} \frac{[\varpi - J - m - 1]_q}{[\varpi - 1]_q} \mathcal{O}[T_-]_{\sigma_*} \\ &\quad + \mu [\varpi]_q U_{m-1}^{(J)}(\sigma) e^{ih(\varpi - J - m - \frac{1}{2})} [J + m]_q \mathcal{O}[T_+]_{\sigma_*}, \\ \mathcal{O}[T_+]_{\sigma_*} U_m^{(J)}(\sigma) &= \frac{[\varpi + J + m + 1]_q}{[\varpi + 1]_q} U_m^{(J)}(\sigma) e^{ih(J+m)} \mathcal{O}[T_+]_{\sigma_*} \\ &\quad - \mu^{-1} \frac{1}{[\varpi + 1]_q} U_{m+1}^{(J)}(\sigma) e^{-ih(\varpi - J - m - \frac{3}{2})} \frac{[J - m]_q}{[\varpi]_q [\varpi - 1]_q} \mathcal{O}[T_-]_{\sigma_*}. \end{aligned} \quad (5.8)$$

Expanding to the first order in J one finds¹⁵

$$\begin{aligned} \mathcal{O}[T_-]_{\sigma_*} \vartheta S^n(\sigma) &= [\varpi]_q \vartheta S^n(\sigma) e^{-ihn} \frac{1}{[\varpi]_q} \frac{[\varpi - n - 1]_q}{[\varpi - 1]_q} \mathcal{O}[T_-]_{\sigma_*} \\ &\quad + \mu [\varpi]_q \vartheta S^{n-1}(\sigma) e^{ih(\varpi - n - \frac{1}{2})} [n]_q \mathcal{O}[T_+]_{\sigma_*}, \\ \mathcal{O}[T_+]_{\sigma_*} \vartheta S^n(\sigma) &= \frac{[\varpi + n + 1]_q}{[\varpi + 1]_q} \vartheta S^n(\sigma) e^{ihn} \mathcal{O}[T_+]_{\sigma_*} \\ &\quad + \mu^{-1} \frac{1}{[\varpi + 1]_q} \vartheta S^{n+1}(\sigma) e^{-ih(\varpi - n - \frac{3}{2})} \frac{[n]_q}{[\varpi]_q [\varpi - 1]_q} \mathcal{O}[T_-]_{\sigma_*} \\ &\quad + \mu^{-1} \pi \alpha_- \frac{\cos(hn)}{\sin h} S^{n+1}(\sigma) e^{-ih(\varpi - n - \frac{3}{2})} \frac{1}{[\varpi]_q [\varpi - 1]_q} \mathcal{O}[T_-]_{\sigma_*}. \end{aligned} \quad (5.9)$$

In particular, for $n = 0$ one obtains the transformation laws of the Bäcklund field

$$\begin{aligned} \mathcal{O}[T_-]_{\sigma_*} \vartheta(\sigma) &= [\varpi]_q \vartheta(\sigma) \frac{1}{[\varpi]_q} \mathcal{O}[T_-]_{\sigma_*} \\ \mathcal{O}[T_+]_{\sigma_*} \vartheta(\sigma) &= \vartheta(\sigma) \mathcal{O}[T_+]_{\sigma_*} + \mu^{-1} \pi \alpha_- \frac{1}{\sin h} S(\sigma) e^{-ih(\varpi - \frac{3}{2})} \frac{1}{[\varpi]_q [\varpi - 1]_q} \mathcal{O}[T_-]_{\sigma_*}. \end{aligned} \quad (5.10)$$

6. Operator Realization of $U_q(\mathfrak{sl}(2)) \otimes U_q(\mathfrak{sl}(2))$

In this section, we depart from Liouville theory and consider a conformal theory where the extended symmetry we just unravelled would be operatorially realized. The primary

¹⁵ The notation $\vartheta S^n(\sigma)$ means the regularized product of ϑ with S^n at the point σ .

fields will be assumed to be of the form $\Psi_{\rho\varpi}^{(J)}(\sigma)$. They are of the same type as the Liouville ones, but act on a larger Hilbert space. Their transformation laws are defined by the usual co-product action, using the co-product displayed on Eqs. (4.21), (4.22). We are looking for a linear map \mathcal{O} from the matrices $[A^\lambda], [B^\lambda], [q^{\Omega_L}], [q^{\Omega_R}]$ to operators on the extended Hilbert space, which generates this co-product action. Again, \mathcal{O} will not be demanded to be an algebra homomorphism. We thus require for $\mathcal{O}(\mathcal{A}_{2\lambda})$ and $\mathcal{O}(\mathcal{B}_{2\lambda})$,

$$\begin{aligned} \mathcal{O}(\mathcal{A}_{2\lambda})\Psi_{\rho,\varpi} &= \sum_{\rho',\varpi'} \Psi_{\rho',\varpi'} [[A^{-\lambda}]_{\rho',\varpi';\rho,\varpi} \mathcal{O}(\mathcal{A}_{2\lambda}) + [B^\lambda]_{\rho',\varpi';\rho,\varpi} \mathcal{O}(\mathcal{B}_{2\lambda})], \\ \mathcal{O}(\mathcal{B}_{2\lambda})\Psi_{\rho,\varpi} &= \sum_{\rho',\varpi'} \Psi_{\rho',\varpi'} [[A^\lambda]_{\rho',\varpi';\rho,\varpi} \mathcal{O}(\mathcal{B}_{2\lambda}) + [B^{-\lambda}]_{\rho',\varpi';\rho,\varpi} \mathcal{O}(\mathcal{A}_{2\lambda})]. \end{aligned} \tag{6.1}$$

Likewise, we would like to construct an operatorial realization of the matrices Ω_L, Ω_R . The form of the co-product Eq. (4.22) dictates that

$$\begin{aligned} \mathcal{O}(q^{\Omega_L})\Psi_{\rho,\varpi} &= \Psi_{\rho,\varpi} q^\rho \mathcal{O}(1), \\ \mathcal{O}(q^{\Omega_R})\Psi_{\rho,\varpi} &= \Psi_{\rho,\varpi} \mathcal{O}(q^{\Omega_R}). \end{aligned} \tag{6.2}$$

To simplify the formulae we dropped the superscript (J) and the σ dependences. The latter are similar to what we encountered previously, and will be re-established at the end. Concerning Ω_R , the above co-product structure is rather particular in that the matrix Ω_R does not appear at all in the linear actions on the Ψ fields. Let us note from the start that this co-action only implies that $\mathcal{O}(q^{\Omega_R})$ commutes with all the Ψ 's and thus at this level only tells us that $\mathcal{O}(q^{\Omega_R})$ is a central charge. To establish a link with the matrix Ω_R , we will impose later on that the (FP) commutation relations of $\mathcal{O}(q^{\Omega_R})$ with the other generators should reproduce the corresponding matrix algebra. Concerning operator products, we will assume that the matching condition Eq. (4.16) holds, and only consider products of the type $\Psi_{\rho_1,\varpi_1}\Psi_{\varpi_2,\rho_2}$. Then it is easy to verify that, on such products, the action just defined has the same form as above, with the matrices replaced by their co-products Eqs. (4.21), (4.22). This makes use of the special tensor product which obeys Eq. (4.23), and of the consistency relation

$$\Psi_{\rho,\varpi} q^\varpi \mathcal{O}(1) = \Psi_{\rho,\varpi} \mathcal{O}(q^{\Omega_L}). \tag{6.3}$$

In the above equation, contrary to the co-product actions to the right, the matrix Ω_R does appear. It is easy to check that we may consistently assume that the Ψ fields satisfy the same fusion and braiding relations as the Liouville ψ fields. Indeed, Eqs. (4.55) and (4.62) which were consequences of the covariance of the ψ field operator algebra under $\mathcal{O}(T_\pm)$, imply the covariance of the fusion and braiding of the Ψ fields under the action of $\mathcal{O}(\mathcal{A}_{2\lambda})$, and $\mathcal{O}(\mathcal{B}_{2\lambda})$.

The next step is to study what is the algebra satisfied by the operators $\mathcal{O}(\mathcal{A}_{2\lambda})$, $\mathcal{O}(\mathcal{B}_{2\lambda})$, $\mathcal{O}(q^{\Omega_L})$ and $\mathcal{O}(q^{\Omega_R})$. Of course, we are dealing with FP relations similar to Eqs. (3.30), (3.31), although this is hidden since we do not write the σ variables. As expected we will find a suitable extension of the matrix algebra Eqs. (4.11)–(4.14). First using these matrix relations, we derive from Eqs. (6.1), (6.2), (6.3) the following operatorial relations

$$\mathcal{O}(q^{\Omega_L}) \mathcal{O}(\mathcal{A}_{2\lambda}) = q^{2\lambda} \mathcal{O}(\mathcal{A}_{2\lambda}) \mathcal{O}(q^{\Omega_L}), \tag{6.4}$$

$$\mathcal{O}(q^{\Omega_L}) \mathcal{O}(\mathcal{B}_{2\lambda}) = q^{-2\lambda} \mathcal{O}(\mathcal{B}_{2\lambda}) \mathcal{O}(q^{\Omega_L}), \tag{6.5}$$

$$[\mathcal{O}(\mathcal{A}_{2\lambda}), \mathcal{O}(\mathcal{B}_{2\lambda})] = C_1^\lambda \mathcal{O}([\Omega_L]_q), \quad (6.6)$$

$$\mathcal{O}(\mathcal{A}_{2\lambda}) \mathcal{O}(\mathcal{A}_{-2\lambda}) - \mathcal{O}(\mathcal{B}_{2\lambda}) \mathcal{O}(\mathcal{B}_{-2\lambda}) = C_2^\lambda \mathcal{O}([\Omega_L]_q), \quad (6.7)$$

where the C_i^λ are central terms which commute with the Ψ fields. Let us note that any operator that commutes with the Ψ 's automatically commutes with $\mathcal{O}(q^{\Omega_L})$ as well, as a trivial consequence of the first of Eqs. (6.2). Thus

$$[C_i^\lambda, \mathcal{O}(q^{\Omega_L})] = 0 \quad (6.8)$$

We do not assume, however, that the C_i^λ commute with the other generators, and it will turn out that in fact they don't. Moreover we find that the quadratic operator $[\mathcal{O}(\mathcal{A}_{2\lambda}), \mathcal{O}(\mathcal{B}_{-2\lambda})]$ and another operator noted $X^\lambda(\mathcal{A}, \mathcal{B})$ transform among themselves under the co-product action. The operator X^λ is given by

$$X^\lambda(\mathcal{A}, \mathcal{B}) = (\mathcal{O}([\Omega_L - 2\lambda]_q))^{-1} Z^\lambda(\mathcal{A}, \mathcal{B}) - (\mathcal{O}([\Omega_L + 2\lambda]_q))^{-1} Z^{-\lambda}(\mathcal{A}, \mathcal{B}),$$

where

$$Z^\lambda(\mathcal{A}, \mathcal{B}) = \mathcal{O}(\mathcal{A}_{2\lambda}) \mathcal{O}(\mathcal{A}_{-2\lambda}) - \mathcal{O}(\mathcal{B}_{-2\lambda}) \mathcal{O}(\mathcal{B}_{2\lambda}).$$

In order to proceed further, we have to make ansätze. First, it is easy to see from the definitions that $\mathcal{O}(\mathcal{A}_{2\lambda})$ and $\mathcal{O}(\mathcal{B}_{-2\lambda})$ act the same way, the matrices involved being the same. It is therefore natural to postulate that

$$[\mathcal{O}(\mathcal{A}_{2\lambda}), \mathcal{O}(\mathcal{B}_{-2\lambda})] = 0. \quad (6.9)$$

Second, by analogy with the matrix algebra, we assume that

$$\mathcal{O}(q^{\Omega_R}) \mathcal{O}(\mathcal{A}_{2\lambda}) = \mathcal{O}(\mathcal{A}_{2\lambda}) F^\lambda(\mathcal{O}(q^{\Omega_R})), \quad (6.10)$$

$$\mathcal{O}(q^{\Omega_R}) \mathcal{O}(\mathcal{B}_{2\lambda}) = \mathcal{O}(\mathcal{B}_{2\lambda}) G^\lambda(\mathcal{O}(q^{\Omega_R})), \quad (6.11)$$

where F and G are unknown functions of one variable. The co-product action on Ψ (Eqs. (6.1), (6.2)) implies that $F^\lambda = G^\lambda$. Note that the operator $\mathcal{O}(q^{\Omega_R})$ may still be replaced by an arbitrary function, say K , of itself since we only know that it commutes with all the Ψ 's. This replacement is equivalent to changing

$$F^\lambda(\mathcal{O}(q^{\Omega_R})) \rightarrow K(F^\lambda(K^{-1}(\mathcal{O}(q^{\Omega_R}))), \quad (6.12)$$

where K^{-1} is the inverse function. We will consider only the class of functions F for which there exists a function K such that Eq. (6.12) gives $F^+(x) \rightarrow xq$. This choice is precisely what is needed to reproduce the matrix algebra on the operatorial level (up to central terms). Passing $\mathcal{O}(q^{\Omega_R})$ from left to right on both sides of Eq. (6.7), and making use of Eqs. (6.10), (6.11), one sees that F^λ obey the relation $F^\lambda(F^{-\lambda}(x)) = x$. Thus there exists a redefinition of $\mathcal{O}(q^{\Omega_R})$ such that $F^\lambda(x) \rightarrow xq^{2\lambda}$. We adopt this choice hereafter, so that Eqs. (6.10), (6.11) become

$$\mathcal{O}(q^{\Omega_R}) \mathcal{O}(\mathcal{A}_{2\lambda}) = q^{2\lambda} \mathcal{O}(\mathcal{A}_{2\lambda}) \mathcal{O}(q^{\Omega_R}), \quad (6.13)$$

$$\mathcal{O}(q^{\Omega_R}) \mathcal{O}(\mathcal{B}_{2\lambda}) = q^{2\lambda} \mathcal{O}(\mathcal{B}_{2\lambda}) \mathcal{O}(q^{\Omega_R}). \quad (6.14)$$

Let us now prove that the relations just written allow us to derive the operator algebra, and that it coincides with the one we obtained for matrices (Eqs. (4.11–4.14)) up to central terms. First Eq. (6.9) implies that $X^\lambda(\mathcal{A}, \mathcal{B}) = 0$ and making use of Eq. (6.7) one finds

$$[\mathcal{O}(\mathcal{A}_{2\lambda}), \mathcal{O}(\mathcal{A}_{-2\lambda})] = \frac{1}{q + q^{-1}} (C_2^\lambda \mathcal{O}([\Omega_L - 2\lambda]_q) - C_2^{-\lambda} \mathcal{O}([\Omega_L + 2\lambda]_q)), \tag{6.15}$$

$$[\mathcal{O}(\mathcal{B}_{2\lambda}), \mathcal{O}(\mathcal{B}_{-2\lambda})] = \frac{1}{q + q^{-1}} (C_2^{-\lambda} \mathcal{O}([\Omega_L - 2\lambda]_q) - C_2^\lambda \mathcal{O}([\Omega_L + 2\lambda]_q)). \tag{6.16}$$

In general, we must verify that our assumptions are consistent with higher commutators (typically Jacobi identities). By the same argument as for Eq. (6.8), we have

$$[\mathcal{O}(q^{\Omega_R}), \mathcal{O}(q^{\Omega_L})] = 0. \tag{6.17}$$

Next, passing $\mathcal{O}(q^{\Omega_R})$ from left to right on both sides of Eq. (6.6), and making use of Eqs. (6.17), (6.13), (6.14) one sees that

$$C_1^\lambda = 0.$$

Furthermore, commuting $\mathcal{O}(q^{\Omega_R})$ with both sides of Eq. (6.7), and taking Eqs. (6.17) and (6.13) into account, one obtains

$$[C_2^\lambda, \mathcal{O}(q^{\Omega_R})] = 0. \tag{6.18}$$

Now let us consider the Jacobi identity between $\mathcal{O}(\mathcal{A}_{2\lambda}), \mathcal{O}(\mathcal{A}_{-2\lambda}), \mathcal{O}(\mathcal{B}_{2\lambda})$. Since we now know that the last operator commutes with the first and with the second, this gives

$$[[\mathcal{O}(\mathcal{A}_{2\lambda}), \mathcal{O}(\mathcal{A}_{-2\lambda})], \mathcal{O}(\mathcal{B}_{2\lambda})] = 0,$$

and, according to Eq. (6.7),

$$[(C_2^\lambda \mathcal{O}([\Omega_L - 2\lambda]_q) - C_2^{-\lambda} \mathcal{O}([\Omega_L + 2\lambda]_q)), \mathcal{O}(\mathcal{B}_{2\lambda})] = 0.$$

Exchanging the role of A and B , one also deduces that

$$[(C_2^{-\lambda} \mathcal{O}([\Omega_L - 2\lambda]_q) - C_2^\lambda \mathcal{O}([\Omega_L + 2\lambda]_q)), \mathcal{O}(\mathcal{A}_{2\lambda})] = 0.$$

In order to solve these equations, we look for operators that commute with $\mathcal{O}(\mathcal{B}_{2\lambda})$ or $\mathcal{O}(\mathcal{A}_{2\lambda})$. Combining Eqs. (6.13), (6.14) with Eqs. (6.4), and (6.5), respectively, one deduces that

$$[\mathcal{O}(q^{\Omega_R - \Omega_L}), \mathcal{O}(\mathcal{A}_{2\lambda})] = 0, \quad [\mathcal{O}(q^{\Omega_R + \Omega_L}), \mathcal{O}(\mathcal{B}_{2\lambda})] = 0.$$

Thus we are lead to make the ansatz

$$\frac{1}{q + q^{-1}} (C_2^+ \mathcal{O}([\Omega_L - 1]_q) - C_2^- \mathcal{O}([\Omega_L + 1]_q)) = c^+ \mathcal{O}(q^{\Omega_R + \Omega_L}) - c^- \mathcal{O}(q^{-\Omega_R - \Omega_L}),$$

$$\frac{1}{q + q^{-1}} (C_2^- \mathcal{O}([\Omega_L - 1]_q) - C_2^+ \mathcal{O}([\Omega_L + 1]_q)) = d^+ \mathcal{O}(q^{\Omega_R - \Omega_L}) - d^- \mathcal{O}(q^{-\Omega_R + \Omega_L}),$$

where c^\pm , and d^\pm commute with all operators, and C_2^λ is taken to be independent of $\mathcal{O}(q^{\Omega_L})$. Note that these last relations could not involve higher powers of $\mathcal{O}(q^{\pm(\Omega_R + \Omega_L)})$, since by assumption the left hand sides are linear in $\mathcal{O}(q^{\pm\Omega_L})$. Solving the two above equations, one finds that $d^\pm = c^\pm$, and

$$\begin{aligned} C_2^+ &= c^+ q \mathcal{O}(q^{\Omega_R}) - c^- q^{-1} \mathcal{O}(q^{-\Omega_R}), \\ C_2^- &= c^+ q^{-1} \mathcal{O}(q^{\Omega_R}) - c^- q \mathcal{O}(q^{-\Omega_R}). \end{aligned} \quad (6.19)$$

Finally, substituting these last two relations in Eqs. (6.7), one finally derives the operator algebra. It is given by

$$\begin{aligned} [\mathcal{O}(A_{2\lambda}), \mathcal{O}(B_{2\mu})] &= [\mathcal{O}(q^{2A_3}), \mathcal{O}(q^{2B_3})] = 0, \\ [\mathcal{O}(A_{2\lambda}), \mathcal{O}(q^{2B_3})] &= [\mathcal{O}(q^{2A_3}), \mathcal{O}(B_{2\mu})] = 0, \\ \mathcal{O}(q^{2A_3})\mathcal{O}(A_{2\lambda}) &= q^{4\lambda} \mathcal{O}(A_{2\lambda})\mathcal{O}(q^{2A_3}), \\ [\mathcal{O}(A_+), \mathcal{O}(A_-)] &= c^+ \mathcal{O}(q^{2A_3}) - c^- \mathcal{O}(q^{-2A_3}), \\ \mathcal{O}(q^{2B_3})\mathcal{O}(B_{2\lambda}) &= q^{4\lambda} \mathcal{O}(B_{2\lambda})\mathcal{O}(q^{2B_3}), \\ [\mathcal{O}(B_+), \mathcal{O}(B_-)] &= c^+ \mathcal{O}(q^{2B_3}) - c^- \mathcal{O}(q^{-2B_3}), \\ \mathcal{O}(A_+)\mathcal{O}(A_-) - \frac{c^+ q \mathcal{O}(q^{2A_3}) + c^- q^{-1} \mathcal{O}(q^{-2A_3})}{q - q^{-1}} &= \\ \mathcal{O}(B_+)\mathcal{O}(B_-) - \frac{c^+ q \mathcal{O}(q^{2B_3}) + c^- q^{-1} \mathcal{O}(q^{-2B_3})}{q - q^{-1}} &= \end{aligned} \quad (6.20)$$

We have defined, as for matrices,

$$\begin{aligned} \mathcal{O}(q^{2A_3}) &= \mathcal{O}(q^{\Omega_R})\mathcal{O}(q^{\Omega_L}), \\ \mathcal{O}(q^{2B_3}) &= \mathcal{O}(q^{\Omega_R})\mathcal{O}(q^{-\Omega_L}). \end{aligned} \quad (6.21)$$

The next step is to discuss the σ dependence. In analogy with the case of Liouville (for $U_{\sqrt{q}}(sl(2))$), it is natural to assume that $\mathcal{O}(A_{\pm})$, $\mathcal{O}(B_{\pm})$ depend upon one point (have gradation one), and that $\mathcal{O}(q^{\Omega_L})$, $\mathcal{O}(q^{\Omega_R})$ have gradation zero. Then the central terms c^{\pm} have gradation two, that is, would be written explicitly as $c^{\pm}(\sigma_1, \sigma_2)$. With this, it is straightforward to re-establish the σ dependences. We will not do it explicitly. The commutators written above are actually not true ones, but instead are similar to the left-hand side of Eq. (3.31). One may verify that the FP version of Jacobi identity holds, so that our discussion indeed makes sense.

In order to make a closer contact with our Liouville discussion, let us show finally that we may construct generators of the $U_{\sqrt{q}}(sl(2))$ algebra also in the present framework. Taking $c^+ = c^- = c$ for simplicity one defines

$$\mathcal{O}(T_{\pm}) = (\mathcal{O}(\lfloor \Omega_R / 2 \rfloor_q))^{-1} (\mathcal{O}(A_{\pm}) + \mathcal{O}(B_{\mp})). \quad (6.22)$$

Making use of the commutation relations derived above one deduces

$$\begin{aligned} [\mathcal{O}(T_+), \mathcal{O}(T_-)] &= c' \mathcal{O}(\lfloor \Omega_L \rfloor_q), \\ \mathcal{O}(q^{\Omega_L}) \mathcal{O}(T_{\pm}) &= q^{\pm 1} \mathcal{O}(T_{\pm}) \mathcal{O}(q^{\Omega_L}). \end{aligned} \quad (6.23)$$

It is easily seen that the co-product action of the operators just defined is the same as the action of the $U_{\sqrt{q}}(sl(2))$ generators in Liouville theory (Eq. (4.5)). This is a consistent co-product action since it follows from the co-product definition Eq. (4.21) that, for matrices

$$\Lambda(T_{2\lambda}) = A^{\lambda} \underbrace{\otimes}_{\wedge} T_{2\lambda} + B^{-\lambda} \underbrace{\otimes}_{\wedge} T_{-2\lambda}. \quad (6.24)$$

7. Novel Hopf-like Algebraic Structure

As we already mentioned, our internal symmetry algebra $U_q(sl(2)) \otimes U_q(sl(2))$ does not obey the usual axioms of a Hopf algebra, since in particular it does not admit a counit in the usual sense. However, in this last section we show that it does possess an algebraic structure which is a natural generalization of the Hopf algebra axioms. For orientation let us recall them. Let \mathcal{G} be a Hopf algebra. It is equipped with a multiplication m , a comultiplication Λ , an antipode $s : \mathcal{G} \rightarrow \mathcal{G}$ and a counit $\epsilon : \mathcal{G} \rightarrow \mathcal{C}$ (\mathcal{C} the set of complex numbers), with the following properties

$$m(a \otimes 1) = m(1 \otimes a) = a, \quad m(m \otimes \text{id}) = m(\text{id} \otimes m) = m, \tag{7.1}$$

$$(\Lambda \otimes \text{id})\Lambda = (\text{id} \otimes \Lambda)\Lambda, \quad \Lambda(a)\Lambda(b) = \Lambda(ab), \tag{7.2}$$

$$(\epsilon \otimes \text{id})\Lambda = (\text{id} \otimes \epsilon)\Lambda = \text{id}, \tag{7.3}$$

$$\epsilon(ab) = \epsilon(a)\epsilon(b), \tag{7.4}$$

$$s(ab) = s(b)s(a). \tag{7.5}$$

$$\Lambda(s) = (s \otimes s)P\Lambda, \tag{7.6}$$

$$m(s \otimes \text{id})\Lambda(a) = m(\text{id} \otimes s)\Lambda(a) = \epsilon(a).1, \tag{7.7}$$

where P is the permutation operator. Let us recall that our algebra may be expressed in two equivalent ways, that is, either in terms of the generators $A_{\pm}, B_{\pm}, q^{\varpi_R}, q^{\varpi_L}$, or in terms of the generators $\mathcal{A}_{\pm}, \mathcal{B}_{\pm}, q^{A_3}, q^{B_3}$. Each set has its virtues and drawbacks, so that they should be used according to the question addressed. A general generator will be denoted K^a or \mathcal{K}^a depending upon the description chosen. For the coproduct, which was defined in Sect. 4, we have a more standard form in terms of the K generators. The corresponding structure constant Λ_{bc}^a will be defined such that

$$\Lambda(K^a) = \Lambda_{bc}^a K^b \otimes K^c. \tag{7.8}$$

Due to the matching condition, this does not uniquely specify Λ_{bc}^a . However, some of the equations we will check later on remove this ambiguity. The appropriate choice is that Eq. (4.21) and (4.22) hold directly. However, for the unit element of \mathcal{G} , we have to define the structure constant such that

$$\Lambda(\mathbf{1}) = q^{-\varpi_R} \otimes q^{\varpi_L}. \tag{7.9}$$

This is consistent since the right hand side is equal to $\mathbf{1} \otimes \mathbf{1}$, but Λ_{bc}^a should be defined without making this replacement¹⁶. Next Eqs. (7.2) mean that the coproduct is coassociative and preserves the algebra. These properties were verified in Sect. 4 using the \mathcal{K} form, where however one cannot define structure constants similar to Λ_{bc}^a . Let us next discuss the counit. From Eq. (7.5), and the fact that ϵ is a number, we see that it defines a one dimensional representation of \mathcal{G} . Equation (7.3) means that that its coproduct with any other representation gives back the same representation only – hence for the usual $U_q(sl(2))$, the counit is simply the one dimensional spin zero representation. However, as we already observed, in our case the relevant spin zero representation is infinite dimensional. From the viewpoint of conformal theory, this is natural, since the corresponding $\psi_{\rho,\omega}^{(0)}$ is proportional to the identity operator in the Hilbert space, while

¹⁶ Note that this is also consistent with the coproduct action due to Eq. (6.3).

a one dimensional representation for \mathcal{A} would correspond to a projector onto a single Verma module. Since the fusion of the identity operator with any other ψ gives back the same operator, one sees that the present infinite dimensional representation with $J = 0$ should play the role of counit, and we next show that this is true. Since the identity operator does not shift the zero modes, it is consistent that this counit of a novel type be restricted to $\rho = \varpi$. Thus, making use of Eq. (5.1) we define the counit as given by the spin zero representation, with $B = 0$. This may be written compactly, for a general element K^a , as

$$\epsilon(K^a)_{\rho, \varpi, \rho', \varpi'} = \epsilon(K^a)_{\varpi} \delta(K^a)_{\varpi, \varpi'} \delta_{\rho, \varpi} \delta_{\rho', \varpi'}, \tag{7.10}$$

where

$$\epsilon(A_{\pm})_{\varpi} = \pm 1, \quad \epsilon(B_{\pm})_{\varpi} = 0, \quad \epsilon(q^{\varpi R})_{\varpi} = q^{\varpi}, \quad \epsilon(q^{\varpi L})_{\varpi} = q^{\varpi}, \tag{7.11}$$

$$\delta(\mathcal{A}_{\pm})_{\varpi, \varpi'} = \delta_{\varpi', \varpi \pm 1} = \delta(B_{\pm})_{\varpi, \varpi'}, \quad \delta(q^{\mathcal{A}_3})_{\varpi, \varpi'} = \delta(q^{\mathcal{B}_3})_{\varpi, \varpi'} = \delta_{\varpi', \varpi}. \tag{7.12}$$

One may verify that the coproduct of this representation with any spin J representation gives a single representation with the same spin J . Indeed one has (the upper indices specify the representations)

$$\Lambda(K^a)^{(J_1, 0)}_{\rho'_1, \varpi'_1, \rho'_2, \varpi'_2; \rho_1, \varpi_1, \rho_2, \varpi_2} = [\mathcal{K}^a]^{(J_1)}_{\rho'_1, \varpi'_1; \rho_1, \varpi_1} \delta_{\rho'_2, \varpi'_2} \delta(K^a)_{\varpi_2, \varpi'_2} \delta_{\rho_2, \varpi_2} \delta_{\rho_2, \varpi_1}, \tag{7.13}$$

with similar equations after reversing the orders. It is easy to see that the right hand side of the equation just written satisfies the same spin J_1 relations as the matrices we started from, although it acts non trivially in the second space. Note that a trivial action in the second space would have been inconsistent since our $\underbrace{\otimes}$ definition which appears in the coproduct respects the matching condition Eq. (4.16), so that, if indices are shifted in the first space, there must be also some shift in the second. The equations just written are the analogues of Eqs. (7.3). It is easy to see that the left and right hand sides contain the same factors, so that we may rewrite in general, making use of the form Eq. (7.10),

$$\Lambda_{bc}^a(K^b)_{\rho, \varpi, \rho', \varpi'} \epsilon(K^c)_{\varpi'} = \Lambda_{bc}^a \epsilon(K^b)_{\rho'} (K^c)_{\rho, \varpi, \rho', \varpi'} = (K^a)_{\rho, \varpi, \rho', \varpi'}, \tag{7.14}$$

which is very similar to Eq. (7.3). The only difference is that, here, ϵ has an index.

Let us now turn to the antipode. As discussed in our first paper along the same line[7], it appears when one goes from the right-action which we have been using so far here to the left-action. This is neatly done by making use of Eqs. (6.1), but one may also define the antipode by using the right-action of $\mathcal{O}(T_{\pm})$, since Eq. (6.24) shows that it acts by the same coproduct coefficient and matrices A^{λ} and B^{λ} as $\mathcal{O}(\mathcal{A}_{2\lambda})$ (or $\mathcal{O}(\mathcal{B}_{-2\lambda})$). Moreover, the left-action can be derived either directly by inverting relations Eq. (4.3), or Eq. (6.1), or else by hermitian conjugation of Eq. (4.3). In any case, one arrives at the following definition of the antipode, which it is simpler to handle in terms of $\mathcal{K}_{(S)}^a$ operators:

$$\mathcal{A}_{\pm}(S) = -([\Omega_R]_q)^{-1} \mathcal{A}_{\mp} [\Omega_R]_q, \quad \mathcal{B}_{\pm}(S) = -([\Omega_R]_q)^{-1} \mathcal{B}_{\pm} [\Omega_R]_q, \tag{7.15}$$

$$q_{(S)}^{\mathcal{A}_3} = q^{\mathcal{A}_3}, \quad q_{(S)}^{\mathcal{B}_3} = q^{-\mathcal{B}_3}. \tag{7.16}$$

Next, making use of Eqs. (4.21), (4.22) and the matching condition, one may verify that

$$\Lambda(A_{(S)}^{\lambda}) = A_{(S)}^{\lambda} \underbrace{\otimes}_{\mathcal{A}} A_{(S)}^{\lambda} + B_{(S)}^{\lambda} \underbrace{\otimes}_{\mathcal{B}} B_{(S)}^{-\lambda},$$

$$\Lambda(B_{(S)}^\lambda) = B_{(S)}^\lambda \underbrace{\otimes}_{\text{}} A_{(S)}^{-\lambda} + A_{(S)}^\lambda \underbrace{\otimes}_{\text{}} B_{(S)}^\lambda,$$

$$\Lambda(q_{(S)}^{\Omega_L}) = \mathbf{1} \underbrace{\otimes}_{\text{}} q_{(S)}^{\Omega_L} \quad \Lambda(q_{(S)}^{\Omega_R}) = q_{(S)}^{\Omega_R} \underbrace{\otimes}_{\text{}} \mathbf{1}. \tag{7.17}$$

These are the analogues of Eq. (7.6), since they take the same form as the coproduct of the K generators, with the two factors exchanged. On the other hand, the antipode just defined take the general form $\mathcal{K}_{(S)}^a = U^{-1} S_b^a \mathcal{K}^b U$ where S_b^a are very simple constants. This immediately allows us to verify the analogues of Eq. (7.5), namely that $\mathcal{K}_{(S)}^a$ satisfies the same algebra as \mathcal{K}^a if one reverses the order. Finally, one may check that

$$A_{bc}^a (K^c K_{(S)}^b)_{\rho, \varpi, \rho', \varpi'} = \epsilon(K^a)_{\varpi} \delta_{\rho \rho'} \delta_{\varpi \varpi'},$$

$$A_{bc}^a (K_{(S)}^c K^b)_{\rho, \varpi, \rho', \varpi'} = \epsilon(K^a)_{\rho} \delta_{\rho \rho'} \delta_{\varpi \varpi'}, \tag{7.18}$$

which are the analogues of Eq. (7.7).

8. Conclusion

The idea of using the primary fields in the smallest quantum group representation as generators has led to interesting developments. In our previous article, we recovered in this way the well-known quantum group symmetries of the covariant operator algebra; however, nontrivial central terms were found to appear in the corresponding generator algebra, and the free field zero mode was seen to play an unusual role. In this paper, we have analyzed the more familiar Bloch wave/Coulomb gas operator basis of conformal field theory using the same idea. The nonlinear character of the transformation relating the two bases has been seen to lead to a surprisingly different realization of the quantum group symmetry for the Bloch waves, with a novel $U_q(sl(2)) \otimes U_q(sl(2))$ structure with somewhat unusual properties emerging. It represents the underlying symmetry of the operator algebra exactly in the same way as $U_q(sl(2))$ did for the covariant basis. Our approach is constructive in the sense that we deduce the symmetry structure directly from the operator algebra under consideration, without a priori assumptions. We were thus lead to modify some of the standard Hopf algebra axioms in order to accommodate the properties of the above structure. In particular, the coproduct prescription differs from the standard one by the presence of two additional constraints (matching condition and equality of Casimir eigenvalues), which incorporate rather naturally the CFT features of the theory. Moreover, we were led to define the counit in terms of an infinite dimensional spin zero representation rather than a map to the complex numbers, since it corresponds to the identity operator in the operator algebra. In this way we arrived at a self-consistent symmetry structure, the representation theory of which reproduces the spectrum of operators of the theory.

Of course there remain many open questions, either of a mathematical or physical nature. Let us list some of them:

- A more systematic mathematical understanding of our $U_q(sl(2)) \otimes U_q(sl(2))$ and $U_{\sqrt{q}}(sl(2))$ structures is clearly desirable.
- We have discussed only briefly the case where q is a root of unity, the case of rational conformal field theory, where interesting subtleties are expected to appear. A systematic treatment of the Coulomb gas picture in this case is provided by Felder's formalism [32], which would have to be adapted to the slightly different formulation of vertex operators in the present framework.

- One would like to understand better the CFT framework of Sect. 6 for the operatorial realization of our extended symmetry group.
- Our discussion differs strongly from previous analyses of quantum group symmetries in rational CFT, and the connection is not obvious.
- So far we have not considered the question of how our generators act on the Hilbert space of states, rather than the covariant fields of the theory. There is a double enigma here: First, the connection between states and operators is nontrivial in view of the problem of the $SL(2)$ – invariant vacuum (in the CFT sense) [33]. Second, the existence of a vacuum state invariant under the quantum group as postulated in [19], assumes that the counit is of the usual type, i.e. a complex number. This problem must be reexamined from scratch with our counit of a novel type.
- In general one would like to put our results to some practical use in solving Liouville theory. One obvious application would be to use the quantum group generators for the classification of observables in the strong coupling theory [9], where no classical interpretation is available, and quantum group invariance becomes the sole defining property of observables.
- It would be interesting to see how our analysis extends to the higher Toda theories where the quantum group symmetry has a higher rank.

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