A NOTE ON DIVISIBILITY SEQUENCES*

MORGAN WARD

1. **Introduction.** A sequence of rational integers

$$(u):\quad u_0,\ u_1,\ u_2,\ \cdots,\ u_n,\ \cdots$$

is called a *divisibility sequence* if u_r divides u_s whenever r divides s, and any integer M dividing terms of (u) with positive suffix is called a divisor of (u). The suffix s is called a *rank of apparition of* M if $u_s \equiv 0 \pmod{M}$, but $u_r \not\equiv 0 \pmod{M}$ if r is a proper divisor of s. It follows from a previous note of mine in this Bulletin (Ward [1]) that a necessary and sufficient condition that every divisor of (u) shall have only *one* rank of apparition is that (u) have the following property:

A. *If $c=(a, b)$, then $u_c=(u_a, u_b)$ for every pair of terms u_a, u_b of (u).*

Assume that no $u_r=0$, $(r>0)$. Then we may introduce numbers

$$[n, r] = u_n \cdot u_{n-1} \cdot \cdots \cdot u_{n-r+1} / u_1 \cdot u_2 \cdot \cdots \cdot u_r,$$
$$r = 1, \cdots, n;\ n = 1, 2, \cdots,$$

which we call the *binomial coefficients belonging to* (u).†

In a previous paper (Ward [1]), I proved a result equivalent to the following theorem:

THEOREM 1. *If every divisor of (u) has only one rank of apparition, the binomial coefficients belonging to (u) are rational integers.*

I give here a simple sufficient condition for integral binomial coefficients applicable when the divisors of (u) have several ranks of apparition.

2. **Main theorem.** Let (v) be any sequence of rational integers subject to the single condition $v_r \neq 0$, $(r>0)$. The sequence (u) will be said to have the property C if

$$u_n = \prod_{d \mid n} v_d,$$

the product being extended over all divisors d of n.

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† If $u_n = n$, they reduce to ordinary binomial coefficients. For their properties for general (u), see Ward [2].

THEOREM 2. *Every sequence* (u) *with property* C *is a divisibility sequence, and all of its associated binomial coefficients are rational integers.*

The proof is immediate. The sequence (u) is obviously a divisibility sequence, and no $u_r=0$, $(r>0)$. Any one of the binomial coefficients of (u) may be put in the form

$$u_1 \cdot u_2 \cdot \cdots \cdot u_{n+m} / u_1 \cdot u_2 \cdot \cdots \cdot u_n \cdot u_1 \cdot u_2 \cdot \cdots \cdot u_m.$$

But if $[x/d]$ denotes as usual the greatest integer in x/d, v_d appears in the denominator of the expression above $[n/d]+[m/d]$ times, and in the numerator $[(n+m)/d]$ times. Since

$$\left[\frac{n+m}{d}\right] \geqq \left[\frac{n}{d}\right] + \left[\frac{m}{d}\right],$$

the expression is an integer. In like manner, all the multinomial coefficients belonging to (u) (Ward [2]) may be shown to be integral.

3. **An application.** Let α, β be distinct algebraic integers, and let $\mathfrak{F}$ be the smallest normal field containing both α and β. Define a sequence (u) by

$$u_n = \prod_S (\alpha^n - \beta^n),$$

where the product is extended over all automorphisms S of $\mathfrak{F}$, so that u_n is a rational integer.

If $Q_d(x, y)$ is the homogeneous cyclotomic polynomial of degree $\phi(d)$, then

$$u_n = \prod_{d|n} v_d,$$

where

$$v_d = \prod_S Q_d(\alpha, \beta).$$

Since the v_d are rational integers, it follows from Theorem 2 that all of the binomial coefficients belonging to (u) are rational integers provided that no $v_d=0$; that is, provided that α/β is not a root of unity.

This result applies to the Lucasian sequences studied in Ward [3] which appear to include all extant instances of divisibility sequences satisfying a linear recursion relation.

4. **Conclusion.** Sequences with property C have another interesting property which is stated in the following theorem:

THEOREM 3. *If* (u) *has property* C, *then the prime divisors of* (u) *and* (v) *are identical. Furthermore the ranks of apparition of any prime in* (u) *and in* (v) *are the same.*

The first part of this theorem is obvious. D. H. Lehmer has proved that every rank of apparition of a prime p in (u) is a rank of apparition of p in (v) (Lehmer [1], p. 462). The converse is immediate. Since (v) is not in general a divisibility sequence, a place of apparition of p in (u) need not be a place of apparition of p in (v).

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M. WARD

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CALIFORNIA INSTITUTE OF TECHNOLOGY