J. DIFFERENTIAL GEOMETRY
35 (1992) 415-427

DONALDSON'S POLYNOMIALS FOR $K3$ SURFACES

KIERAN G. O'GRADY

Let M be a smooth compact simply connected four-manifold with $b_2^+ = 2p+1$, $p \geq 1$. Donaldson [5], [7] has defined polynomials $\gamma_c \in \mathrm{Sym}^d H^2(S, \mathbf{Z})$ for all $c > \frac{3}{2}(p+1)$, where $d = 4c-3(p+1)$. The polynomials are invariant under diffeomorphisms and actually provide new C^∞ invariants [5], [7]. To define these invariants choose a generic metric, g, on M and consider X_c, the Uhlenbeck compactification of the moduli space $\mathscr{M}_c$ of g-anti-self-dual connections on the SU(2) bundle on M with $c_2 = c$ [7]. There is a map $\bar{\mu}\colon H_2(M) \to H^2(X_c)$ which extends the map $\mu\colon H_2(M) \to H^2(\mathscr{M}_c)$ obtained by slant product with $-\frac{1}{4}p_2(P)$, where P is the universal SO(3) bundle over $M \times \mathscr{M}_c$. One defines

$$\gamma_c(\Gamma) = \int_{[X_c]} \underbrace{\mu(\Gamma) \cup \mu(\Gamma) \cup \cdots \cup \mu(\Gamma)}_{d \text{ times}} .$$

If M is the smooth manifold underlying a projective complex surface S, and g is the Kähler metric associated to an ample divisor H, then, by a theorem of Donaldson [4], $\mathscr{M}_c \cong M_S(H, 0, c)$, where $M_S(H, 0, c)$ is the moduli space of rank-two vector bundles E on S with $c_1(E) = 0$ and $c_2(E) = c$, μ-stable with respect to H. By passing to the algebraic-geometric situation Donaldson has proved that, for a projective surface, $\gamma_c \neq 0$, at least for big c [5]. Not much is known about Donaldson's polynomials: R. Friedman and J. Morgan have partially computed γ_c for simply connected elliptic surfaces. In particular, let S be a $K3$ surface with $c \geq 4$, $d = 4c - 6 = 2n$, and q the quadratic form of S. They show that

$$\gamma_c = \frac{(2n)!}{2^n n!} q^n .$$

The aim of this paper is to give a different proof of this formula in the case where c is odd. We do this by defining a polynomial $\delta_c \in \mathrm{Sym}^d H^2(S, \mathbf{Z})$ analogous to γ_c, the difference being that instead of X_c we use the compactification of $M_S(H, 0, c)$ provided by the moduli space of semistable

Received April 27, 1990 and, in revised form, February 11, 1991.

sheaves. We prove that although γ_c and δ_c are not a priori equal, in fact they are the same polynomial (we prove this only for certain polarized $K3$ surfaces and a corresponding value of c, but our arguments can be generalized to any $K3$ surface); this should be generalizable to many other surfaces. Then we compute $\delta_c(\Gamma+\overline{\Gamma})$, where Γ is the Poincaré dual of a nonzero holomorphic two-form on S; it is plausible that the method we employ can be applied to any surface. The result follows because γ_c is a multiple of a power of the quadratic form for a $K3$ surface.

Notation. Let E be a coherent torsion-free sheaf on a projective surface S, and let H be the hyperplane class on S. Then we say E is μ-stable (respectively semistable) if $\mu(F) < \mu(E)$ (respectively $\leq$) for every subsheaf $F \hookrightarrow E$, where $\mu(G) = (c_1(G)\cdot H)/\operatorname{rank}(G)$. We say E is stable (respectively semistable) if $p_F(n) < p_E(n)$ (respectively $\leq$) for all subsheaves $F \hookrightarrow E$ and all $n \gg 0$, letting $p_G(n) = \chi(G(n))/\operatorname{rank}(G)$, i.e., if E is stable (semistable) according to Gieseker and Maruyama. Both notions of stability depend on the polarization chosen, so to be precise one should always specify H. We denote by $M_S(H, c_1, c_2)$ the moduli space of rank-two locally free sheaves, E, on S, μ-stable with respect to H, with $c_1(E) = c_1$ and $c_2(E) = c_2$. We let $\overline{M}_S(H, c_1, c_2)$ be the moduli space of rank-two torsion-free sheaves, E, on S, Gieseker-Maruyama semistable with respect to H, with $c_1(E) = c_1$ and $c_2(E) = c_2$; it is a projective scheme [8], [10]. There is a natural embedding $\iota: M_S(H, c_1, c_2) \hookrightarrow \overline{M}_S(H, c_1, c_2)$, and $\iota(M_S(H, c_1, c_2))$ is clearly open in its closure, but a priori it need not be that $\overline{M}_S(H, c_1, c_2)$ is the closure of $\iota(M_S(H, c_1, c_2))$: there could possibly exist components all of whose points parametrize sheaves which are not locally free. When $c_1 = 0$ and $c_2 = c$, and there is no confusion about S and H, we will abbreviate $M_S(H, c_1, c_2)$ and $\overline{M}_S(H, c_1, c_2)$ to M_c and $\overline{M}_c$ respectively. Let E^{**} be the double dual of E. By the canonical sequence of E we will mean the exact sequence

$$0 \to E \to E^{**} \to Q \to 0,$$

where Q is a sheaf which naturally lives on Y, the zero-dimensional subscheme of S defined by the ideal sheaf $\operatorname{Ann} Q$. For such Q and Y we set $l(Q) = h^0(Q)$ and $l(Y) = h^0(\mathcal{O}_Y)$. In general we will denote by $[X]$ the equivalence class of an object X for an appropriate equivalence relation. So, for example, if E is an H-semistable sheaf, then $[E]$ will be a point in an appropriate moduli space, if $Z \subset S$ is a zero-dimensional subscheme, then $[Z]$ will be the corresponding point in the appropriate Hilbert scheme, etc.

1. Lemma 1. *Let S be a $K3$ surface, H a polarization on S, and E an H-semistable rank two torsion-free sheaf on S, and let $c_1(E) = 0$, $c_2(E) = c$ with c odd. Then E is stable.*

Proof. In Gieseker's notation

$$p_E(n) = \tfrac{1}{2}H^2n^2 - c/2 + 2.$$

Let $F \to E$ be a rank-one subsheaf of E. Then

$$p_F(n) = \tfrac{1}{2}H^2n^2 + (\det F \cdot H)n + \tfrac{1}{2}(\det F)^2 - c_2(F) + 2.$$

If E were semistable, there would exist F such that $p_F(n) = p_E(n)$. This is impossible because the constant coefficient of $p_F(n)$ is an integer (the intersection form is even), while the constant coefficient of $p_E(n)$ is not integer.

Corollary. *Let c be odd. If $\overline{M}_c$ is not empty, then it is smooth of dimension $4c - 6$, and there exists a universal sheaf over $S \times \overline{M}_c$.*

Proof. By the lemma, if $[E] \in \overline{M}_c$, then E is stable, hence simple. By a result of Mukai [13, Theorem 0.3], $\overline{M}_c$ is smooth at $[E]$ of dimension $4c - 6$. Again by a theorem of Mukai [13, Theorem A.6] a universal sheaf exists.

Proposition 1. *Let S be a $K3$ surface whose Picard group is generated by the ample divisor H, and let $H^2 = 2m$, and $c = 2m + 3$. Then $\overline{M}_c$ is irreducible and birational to the Hilbert scheme of zero-dimensional subschemes of S of length $4m + 3$.*

Proof. If $[E] \in \overline{M}_c$ let $F = E \otimes H$. Then $c_1(F) = 2H$ and $c_2(F) = 4m + 3$.

Claim 1. *The sheaf F fits into the exact sequence*

$$(*) \qquad 0 \to \mathcal{O}_S \to F \to I_Z(2H) \to 0,$$

where $Z \subset S$ is a zero-dimensional subscheme of length $4m + 3$.

Proof. By Riemann-Roch, $h^0(F) + h^2(F) \geq 1$; let us prove that $h^0(F) \geq 1$. By considering the canonical sequence of F we see that $h^2(F) = h^2(F^{**})$. By Serre duality, $h^2(F^{**}) = h^0(F^*)$; if $h^0(F^*) > 0$ there is an injection $\mathcal{O}_S(kH) \to F^*$, $k \geq 0$, hence an injection $\mathcal{O}_S((2+k)H) \to F^{**}$ and consequently a map $I_Z((2+k)H) \to F$ for some zero-dimensional $Z \subset S$. This clearly contradicts the stability of F, hence $h^2(F) = 0$ and $h^0(F) \geq 1$. From the stability of F it follows that any nonzero section has isolated zeros, hence it defines an injection $\mathcal{O}_S \to F$ with quotient a torsion-free rank-one sheaf $\mathcal{L}$ which is isomorphic to $I_Z(2H)$ for some zero-dimensional subscheme $Z \subset S$. Since $c_2(F) = 4m + 3$, the length of Z is $4m + 3$.

If F fits into the exact sequence $(*)$, then the following equalities hold:

(i) $h^0(F) = h^0(I_Z(2H)) + 1$.

(ii) $h^0(I_Z(2H)) + 1 = h^1(I_Z(2H))$.

(iii) $h^1(I_Z(2H)) = \dim \operatorname{Ext}^1(I_Z(2H), \mathscr{O}_S)$.

The first two equalities follow from the long exact cohomology sequences associated to $(*)$ and the exact sequence $0 \to I_Z(2H) \to \mathscr{O}_S(2H) \to \mathscr{O}_Z(2H) \to 0$, respectively. Equality (iii) follows from Serre duality.

Claim 2. *Let $Z \subset S$ be a zero-dimensional subscheme of length $4m+3$ such that, if $Z' \subset Z$ is a subscheme of length $4m+2$ with $h^0(I_{Z'}(2H)) = 0$, then there is a unique stable locally free sheaf F fitting into the exact sequence $(*)$.*

Proof. By our hypothesis $h^0(I_Z(2H)) = 0$, hence by (ii) and (iii) there is a unique nontrivial extension, F, of $I_Z(2H)$ by $\mathscr{O}_S$. Since Z satisfies the Cayley-Bacharach property relative to $|2H|$, the sheaf F is locally free. Let $0 \to \mathscr{O}_S(kH) \to F$ be a sublinebundle. Since, by (i), $h^0(F) = 1$, we must have $k \leq 0$, i.e., F is stable.

Definition 1. Let $\mathscr{H}_c$ be the Hilbert scheme of zero-dimensional subschemes of S of length $4m + 3$, and let $U_c \subset \mathscr{H}_c$ be the open subset defined by

$$U_c = \{Z | h^0(I_Z(2H)) = 0 \text{ and the corresponding extension } (*) \text{ is stable.}\}.$$

By Riemann-Roch, $h^0(2H) = 4m + 2$, hence if $Z \subset S$ is a generic zero-dimensional subscheme of length $4m + 3$, then $h^0(I_Z(2H)) = 0$ and, for any subscheme $Z' \subset Z$ of length $4m + 2$, $h^0(I_{Z'}(2H)) = 0$. By Claim 2 we conclude that U_c is not empty. Let V_c be the open subset of $\overline{M}_c$ defined by

$$V_c = \{[E] | h^0(E \otimes H) = 1\}.$$

The previous discussion defines an isomorphism $f: U_c \xrightarrow{\sim} V_c$ which extends to a rational map $\bar{f}: \mathscr{H}_c \to \overline{M}_c$.

Since V_c is open (or by a dimension count), $\bar{f}$ is a birational map between $\mathscr{H}_c$ and one component of $\overline{M}_c$. We will be done if we can prove that there are no other components of $\overline{M}_c$. By the Corollary to Lemma 1 any component has dimension $4c - 6$, hence the following claim finishes the proof of the proposition.

Claim 3. *The codimension of $\overline{M}_c \backslash V_c$ in $\overline{M}_c$ is at least two (in fact equal to two).*

Proof. Let $[E] \in \overline{M}_c$. Then $F = E \otimes H$ fits into the exact sequence $(*)$, so we have to bound the number of moduli of stable nontrivial extensions which arise from $[Z] \in \mathscr{H}_c \backslash U_c$. Let $\varphi \colon S \to \mathbf{P}^{4m+1}$ be the map associated to the complete linear system $|2H|$. Let $[Z] \in \mathscr{H}_c$ vary in a family $\mathscr{F}$ for which $\dim \operatorname{Ext}^1(I_Z(2H), \mathscr{O}_S)$ is constant. Then the number of moduli of F's obtained as extensions $(*)$ is at most

$$\dim \mathscr{F} + \dim \operatorname{Ext}^1(I_Z(2H), \mathscr{O}_S) - 1 - (h^0(F) - 1) = \dim \mathscr{F},$$

where we have used the equalities (i), (ii), (iii) (this is the essential point). We stratify $\mathscr{H}_c \backslash U_c$ according to the dimension of $\operatorname{span} \varphi(Z)$ and its intersection with $\varphi(S)$; since $[Z] \notin U_c$, $\dim \operatorname{span} \varphi(Z) \leq 4m$. First, assume $\operatorname{span} \varphi(Z) \cap \varphi(S)$ is zero-dimensional. Then $d = \dim(\operatorname{span} \varphi(Z)) \leq 4m-1$. Since locally on $\mathscr{F}$ there is a subscheme $Z' \subset Z$ such that $\varphi(Z')$ spans $\varphi(Z)$ and $l(Z') = d+1$, there is an injection $\imath \colon \mathscr{F} \hookrightarrow \operatorname{Hilb}^{d+1}(S)$, and hence

$$\text{number of moduli of } F\text{'s} \leq 2(d+1) \leq 8m.$$

If $\operatorname{span} \varphi(Z) \cap \varphi(S)$ is a divisor D, then either $D \in |H|$ or $D \in |2H|$. In the first case the number of moduli is $\dim |H| + 4m + 3 = 5m + 5$, and in the second it is $\dim |2H| + 4m + 3 = 8m + 4$. Since $\dim \overline{M}_c = 8m + 6$ we conclude that $\operatorname{codim}(\overline{M}_c \backslash V_c, \overline{M}_c) \geq 2$.

2. Definition 2. Let c be odd, S be a $K3$ surface, H be a polarization on S, and $\mathscr{E}$ be a universal sheaf on $S \times \overline{M}_c$. Then we set

$$\nu \colon H_2(S, \mathbf{Z}) \to H^2(\overline{M}_c, \mathbf{Z})$$

to be the map given by $\nu(\Gamma) = c_2(\mathscr{E})/\Gamma$.

Notice that a universal sheaf is not unique, but ν does not depend on the choice of $\mathscr{E}$. Let X_c be Uhlenbeck's compactification [7] of the moduli space of connections on the SU(2)-bundle with $c_2 = c$, anti-self-dual with respect to the Kähler metric associated to H. Then one has the extended μ-map $\bar{\mu} \colon H_2(S) \to H^2(X_c)$. By a theorem of Donaldson [4] X_c and $\overline{M}_c$ are two (different) compactifications of M_c. If we restrict to M_c, then $\bar{\mu}$ and ν agree. Let $C \subset S$ be a curve and restrict the universal sheaf $\mathscr{E}$ to $C \times \overline{M}_c$. Choose $L \in \operatorname{Pic}^{g-1}(C)$, where g is the genus of C, and let $p \colon C \times \overline{M}_c \to C$ and $q \colon C \times \overline{M}_c \to \overline{M}_c$ be the projections. Then applying Grothendieck-Riemann-Roch to $\mathscr{F} = \mathscr{E} \otimes p^*(L)$ and q one gets

$$\nu(C) = -c_1(q_! \mathscr{F}).$$

This has an analogue in X_c—one chooses a spin structure on C, and $q_! \mathscr{F}$ is replaced by the determinant of the twisted Dirac operator.

One can choose a representative of $\nu(C)$ as follows: let

$$\Delta(C, L)_{\text{red}} = \{[E] | h^0(\mathscr{O}_C(E \otimes L)) > 0\}.$$

Then the Poincaré dual of $\nu(C)$ is represented by a cycle $\Delta(C, L)$ supported on $\Delta(C, L)_{\text{red}}$ (with positive coefficients). On the other hand, as is shown by Friedman and Morgan [7], $\Delta(C, L)$ restricted to M_c also represents $\mu(C)$. For this to make sense one has to choose L so that $\Delta(C, L)$ is a divisor (maybe empty), i.e., every component of $\overline{M}_c$ must contain a point $[E]$ such that $h^0(\mathcal{O}_C(E \otimes L)) = 0$. By a theorem of Raynaud [14] this is equivalent to $\mathcal{O}_C(E)$ being semistable. If C is an ample divisor and E is μ-stable with respect to C, then Mehta and Ramanathan [11] have shown that there exist $n > 0$ and $C' \in |nC|$ such that $\mathcal{O}_C(E)$ is stable. We will need the following stronger version due to Bogomolov [2, 11.8, Corollary 1].

Theorem (Bogomolov). *Let S be a projective surface, H an ample line bundle on S, and E an H μ-stable rank-two vector bundle over S with Chern classes c_1, c_2. Then there exists a number $k(c_1, c_2)$, depending on c_1 and c_2 but not on E, such that if $k \geq k_0$ and C is any smooth curve in $|kH|$, then $E_{|C}$ is stable.*

Definition 3. Let S, H, c be as in Definition 2, and let $d = 4c - 6 = \dim \overline{M}_c$. We define $\delta_c \in \operatorname{Sym}^d(H^2(S, \mathbf{Z})) \cong \operatorname{Sym}^d(H_2(S, \mathbf{Z})^*)$ by setting

$$\delta_c(\Gamma) = \nu(\Gamma)^d \quad \text{for } \Gamma \in H_2(S, \mathbf{Z}).$$

The polynomial δ_c depends a priori on the polarization chosen to define $\overline{M}_c$ and on the polarized $K3$ S, so whenever we want to stress this dependence we denote it by $\delta_c(S, H)$. It is clearly analogous to Donaldson's polynomial γ_c, but it is not a priori obvious that they are equal.

Lemma 2. *Let (S, H) be a polarized $K3$ surface, let c be odd, and assume $\overline{M}_c$ is not empty. Then $\gamma_c(H) = \delta_c(S, H)(H)$.*

Proof. The proof follows Donaldson's method for proving that $\gamma_c(H) \neq 0$ [5]. Let $d = \dim \overline{M}_c = 4c - 6$. We will show that for k large enough one can choose smooth curves $C_i \in |kH|$, $i = 1, \cdots, d$, and line bundles $L_i \in \operatorname{Pic}^{g-1}(C_i)$, where g is the genus of C_i, such that the representatives $\Delta(C_i, L_i)$ of $\nu(kH)$ intersect only in M_c and the intersection is a finite set of points (a priori it could be empty, but in fact our main theorem shows it is not). Let g_H be the Kähler metric associated to the polarization H. Then, as we will see, g_H and the $\Delta(C_i, L_i)$'s define an admissible system in the terminology of Donaldson [5], hence the intersection of their restrictions to M_c computes $\gamma_c(H)$, but then, since there is no point of intersection on $\overline{M}_c \backslash M_c$, $\gamma_c(H) = \delta_c(H)$.

We introduce the following notation: $\Delta_l(C, L) = \Delta(C, L)_{|M_l}$. We also need to observe that the set $\mathcal{S} = \{F \in \operatorname{Pic}(S) | -c \leq F^2 \leq 0,\ F \cdot H = 0\}$

is finite: this follows from the Hodge index theorem and the fact that S is regular. By Bogomolov's Theorem there exists k such that if $C \in |kH|$ and $[E] \in M_l$ for $l \leq c$, then $E_{|C}$ is stable; clearly we can also assume that $|kH|$ is very ample.

Claim. *We can choose smooth curves $C_i \in |kH|$ and line bundles $L_i \in \mathrm{Pic}^{g-1}(C_i)$ for $i = 1, \cdots, d$ such that*

(1) *no three of the C_i's intersect,*

(2) *for all $i \leq d$, if $F \in \mathcal{S}$ then $h^0(L_i \otimes F_{|C_i}) = 0$,*

(3) *$\Delta_l(C_1, L_1)_{\mathrm{red}} \cap \cdots \cap \Delta_l(C_n, L_n)_{\mathrm{red}}$ is empty or has codimension n for any $n \leq d$.*

Proof of claim. By induction on n. If $n = 1$ let $\{[E_1], \cdots, [E_r]\}$ be a finite set of μ-stable rank-two vector bundles on S with $c_1 = 0$ and $c_2 \leq c$ such that any irreducible component of M_l for $l \leq c$ contains at least one $[E_s]$. Let $C_1 \in |kH|$ be any smooth curve. Since $E_{s|C_1}$ is stable for all s, there exists $L_1 \in \mathrm{Pic}^{g-1}(C_1)$ such that $h^0(E_{s|C_1} \otimes L_1) = 0$ for all s; since $\mathcal{S}$ is finite we can further insure that $h^0(L_1 \otimes F_{|C_1}) = 0$. With this choice of (C_1, L_1), $\Delta_l(C_1, L_1)_{\mathrm{red}}$ is a divisor for all $l \leq c$. Now assume $(C_1, L_1), \cdots, (C_m, L_m)$ satisfy (1), (2), (3) with d replaced by m. Then let $\{[E_1], \cdots, [E_r]\}$ be a finite set as above such that for all $l \leq c$ each irreducible component of $\Delta_l(C_1, L_1)_{\mathrm{red}} \cap \cdots \cap \Delta_l(C_m, L_m)_{\mathrm{red}}$ contains at least one $[E_s]$. Furthermore, let $C_{m+1} \in |kH|$ be any smooth curve such that $C_1, \cdots, C_{m+1}$ satisfy (1). Then we choose $L_{m+1} \in \mathrm{Pic}^g(C_{m+1})$ such that $h^0(E_{s|C_{m+1}} \otimes L_{m+1}) = 0$ for all s and $h^0(L_{m+1} \otimes F_{|C_{m+1}}) = 0$ for all $F \in \mathcal{S}$. Clearly with these choices $(C_1, L_1), \cdots, (C_{m+1}, L_{m+1})$ satisfy (1), (2), (3), hence the proof is complete.

Now let us show that $\Delta(C_1, L_1)_{\mathrm{red}} \cap \cdots \cap \Delta(C_d, L_d)_{\mathrm{red}} \subset M_c$. Assume there exists

$$(*) \qquad [E] \in \Delta(C_1, L_1)_{\mathrm{red}} \cap \cdots \cap \Delta(C_d, L_d)_{\mathrm{red}}$$

with $[E] \in \overline{M}_c \backslash M_c$. Consider the canonical sequence of E,

$$0 \to E \to E^{**} \to \mathcal{Q} \to 0.$$

Let $Z \subset S$ be the zero-dimensional subscheme whose ideal sheaf is $\mathrm{Ann}\,\mathcal{Q}$, let Z_{red} be the reduced Z, and let $c_2(E^{**}) = l$. Then $c_2(E^{**}) + l(\mathcal{Q}) = c$. If $[E] \in \Delta(C_i, L_i)$, then $h^0(E^{**}_{|C_i} \otimes L_i) > 0$ or $Z_{\mathrm{red}} \cap C_i \neq \emptyset$. Since E is Gieseker-Maruyama stable, the double dual E^{**} is μ-semistable. We distinguish two cases.

First case: *E^{**} is μ-stable.* Since $[E] \notin M_c$, we have $E^{**} \not\cong E$, hence $l < c$. Let $a = \#\{i | [E^{**}] \in \Delta_i(C_i.L_i)\}$ and $b = \#\{i | Z_{\mathrm{red}} \cap C_i \neq \varnothing\}$; then by $(*)$ $a + b \geq d$. From our choice of the (C_i, L_i)'s it follows that $a \leq \dim M_l = 4l - 6$. On the other hand clearly $b \leq 2(\#Z_{\mathrm{red}}) \leq 2l(\mathscr{Q}) = 2(c - l)$, hence $d \leq a + b \leq 2c + 2l - 6 < 4c - 6 = d$, which is absurd.

Second case: *E^{**} is μ-semistable but not stable.* Let F be the semistabilizing line bundle of E^{**}, i.e., $F \cdot H = 0$ and E^{**} fits into

$$(**) \qquad 0 \to F \to E^{**} \to I_W \otimes F^* \to 0,$$

where $W \subset S$ is a zero-dimensional subscheme. From $(**)$ we get that $c_2(E^{**}) = l(W) - F^2$, by the Hodge index theorem $F^2 \leq 0$, hence $-c \leq F^2 \leq 0$, i.e., $F \in \mathscr{S}$. If Z is, as above, the subscheme on which $\mathscr{Q}$ lives, then $[E] \in \Delta(C_i, L_i)$ implies that one of the following holds:

(1) $h^0(E^{**}_{|C_i} \otimes L_i) > 0$.
(2) $W_{\mathrm{red}} \cap C_i \neq \varnothing$.
(3) $Z_{\mathrm{red}} \cap C_i \neq \varnothing$.

Since $F \in \mathscr{S}$, we know that (1) cannot hold. Let α, β be the number of i's such that (2), (3) hold, respectively. Clearly $\alpha \leq 2(\#W_{\mathrm{red}}) \leq l(W) \leq 2l$ and $\beta \leq 2(c - l)$, hence $d \leq \alpha + \beta \leq 2c < 4c - 6 = d$, which is absurd.

Next we claim that the Kähler metric g_H and the $\Delta(C_i, L_i)$'s define an admissible system, as defined by Donaldson [5]. In fact we only have to notice that, by a theorem of Mukai [13, Theorem 0.3], M_l is smooth and of the expected dimension (if not empty) whatever l is; but then our choice of the (C_i, L_i)'s ensures that the $\Delta(C_i, L_i)$'s define an admissible system. By Donaldson's Proposition 3.6 [5] the intersection number $\Delta_c(C_1, L_1) \cdot \cdots \cdot \Delta_c(C_d, L_d)$ is equal to $\gamma_c(kH)$. On the other hand, since the $\Delta(C_i, L_i)$'s do not intersect in $\overline{M}_c \backslash M_c$, $\Delta_c(C_1, L_1) \cdot \cdots \cdot \Delta_c(C_d, L_d) = \delta_c(S, kH)(kH)$, hence we conclude that $\gamma_c(kH) = \delta_c(kH)$.

The following lemma is well known in the case of locally free sheaves.

Lemma 3. *Let S be a $K3$ surface, let $A \subset \mathrm{Pic}(S)$ be the subset of ample line bundles, and let $R_c = \{F \in \mathrm{Pic}(S) | -c \leq F^2 \leq 0\}$. The set of walls $W_c = \{F^{\perp} \subset \mathrm{Pic}(S) | F \in R\}$ determined by R_c partitions the ample cone $A \otimes \mathbf{R}$ into chambers. Let H_1, H_2 be polarizations on S and assume that they belong to the same open chamber of $A \otimes \mathbf{R}$. Then $\overline{M}_S(H_1, 0, c) \cong \overline{M}_S(H_2, 0, c)$.*

Proof. We must show that a sheaf E cannot be H_2-semistable and H_1 nonsemistable (then we exchange the roles of H_1 and H_2). Let

$$(*) \qquad 0 \to I_{\Gamma}(F) \to E \to I_{\Gamma'}(-F) \to 0$$

be an H_1 desemistabilizing sequence. Let $\gamma = h^0(\mathscr{O}_{\Gamma})$ and $\gamma' = h^0(\mathscr{O}_{\Gamma'})$.

Then $c = -F^2 + \gamma + \gamma'$, hence

$$(\dagger) \qquad F^2 \geq -c.$$

Assume $F \cdot H_1 > 0$ and $F \cdot H_2 < 0$. Then by the Hodge index theorem $F^2 < 0$, and by $(\dagger)$ H_1 and H_2 cannot belong to the same chamber, impossible. If $F \cdot H_1 > 0$ and $F \cdot H_2 = 0$, again by Hodge index $F^2 < 0$, and by $(\dagger)$ and our hypothesis it is impossible. If $F \cdot H_1 = 0$ either $F = 0$ or $F^2 < 0$. By $(\dagger)$ and our hypothesis $F^2 < 0$ is impossible. If $F = 0$, since $I_\Gamma(F)$ is H_1 desemistabilizing, $-\gamma > -c/2$, but $-\gamma \leq -c/2$ since E is H_2 semistable, impossible.

Corollary. *Let S be a $K3$ surface, H a polarization on S, and c an odd number. Assume $\overline{M}_c$ is not empty, and H does not lie on a wall of W_c. Then*

$$\gamma_{c|\operatorname{Pic}(S)} = \delta_c(S, H)_{|\operatorname{Pic}(S)}.$$

Proof. Let C_H be the intersection of the open chamber containing H and $\operatorname{Pic}(S)$, and let $H_i \in C_H$. By Lemma 3 we know that $\delta_c(S, H)(H_i) = \delta_c(S, H_i)(H_i)$, and, by Lemma 2, $\delta_c(S, H_i)(H_i) = \gamma_c(H_i)$, hence $\delta_c(S, H) \times (H_i) = \gamma_c(H_i)$. The set of lines $\{[H_i] | H_i \in C_h\}$ is a Zariski dense subset of $\mathbf{P}(\operatorname{Pic}(S) \otimes \mathbf{R})$, hence the two homogeneous polynomials $\gamma_{c|\operatorname{Pic}(S)}$ and $\delta_c(S, H)_{|\operatorname{Pic}(S)}$ must be equal.

Lemma 4. *Let S be a $K3$ surface, H be a primitive polarization on S, $H^2 = 2m$, $c = 2m+3$, and $d = 4c-6$. Let $q \in \operatorname{Sym}^2(H^2(S, \mathbf{Z}))$, $h \in H^2(S, \mathbf{Z})$ be the intersection form and $c_1(H)$ respectively. Then $\delta_c(S, H)$ is a polynomial in q and h, i.e.,*

$$\delta_c(S, H) = a_0 q^{d/2} + a_1 q^{d/2-1} h^2 + \cdots + a_{d/2} h^d$$

for some rational numbers $a_0, a_1, \cdots, a_{d/2}$.

Proof. The surface S belongs to the family $\mathscr{B}$ of all $K3$ surfaces with a primitive polarization of degree H^2, which will be surfaces in a fixed $\mathbf{P}^r$, $r = h^0(S, nH) - 1$ $(n \geq 3)$. By Gieseker and Maruyama's theorem ([8], [10]), there is a relative moduli space $\mathscr{M}$ of H-semistable sheaves over $\mathscr{B}$. Let $\pi: \mathscr{M} \to \mathscr{B}$ be the projection. By Proposition 1, $\pi(\mathscr{M})$ contains the dense subset $\mathscr{B}_0 \subset \mathscr{B}$ parametrizing surfaces whose Picard group has rank one. Since π is proper, we conclude that $\pi(\mathscr{M}) = \mathscr{B}$. We would like to have a relative universal sheaf on $\mathscr{S} \times_{\mathscr{B}} \mathscr{M}$, where $\mathscr{S}$ is the universal $K3$ with a primitive polarization of degree $2m$, in order to compare the polynomials $\delta_c(S_0, H_0)$ and $\delta_c(S_1, H_1)$ for two surfaces. A relative universal sheaf might not exist, although there is one of each fiber $S \times \overline{M}_c$. But, by using a criterion of Maruyama [10, Proposition 6.10], as modified by Mukai [13, Theorem A.6], we conclude that there exists a finite covering map $\phi\colon \widetilde{\mathscr{B}} \to \mathscr{B}$ such that there is a "universal sheaf" on $\widetilde{\mathscr{S}} \times_{\widetilde{\mathscr{B}}} \widetilde{\mathscr{M}}$ where $\widetilde{\mathscr{S}} = \mathscr{S} \times_{\mathscr{B}} \widetilde{\mathscr{B}}$. In fact let $H_1, H_2, \cdots, H_{d-3}$ be fixed generic hyperplanes and let $\widetilde{\mathscr{B}} \subset S \times \mathscr{B}$ be defined by $\widetilde{\mathscr{B}} = \{(P, b) | P \in H_1 \cap \cdots \cap H_{d-3} \cap S\}$. By definition on $\widetilde{\mathscr{S}}$ there is a section Δ of the map to $\widetilde{\mathscr{B}}$; hence the sheaf $\mathscr{O}_\Delta$. When restricted to $S \subset \mathscr{S}$, $\mathscr{O}_\Delta$ is $\mathscr{O}_P$ and $\chi(\mathscr{O}_P(E)) = 2$; hence Mukai's criterion [13, Theorem A.6] applies in this relative case and we conclude that there exists a "universal sheaf". Let $\alpha\colon [0, 1] \to \widetilde{\mathscr{B}}$ be a path with end points corresponding to surfaces S_0 and S_1, and let $\alpha_*\colon H_2(S_0) \to H_2(S_1)$ be the natural map. Hence we conclude that $\delta_c(S_0, H_0)(v) = \delta_c(S_1, H_1)(\alpha_*(v))$. Now fix one polarized $K3$, S; then $\delta_c(S, H)$ is invariant under the action of the fundamental group of $\widetilde{\mathscr{B}}$. Since the image of $\pi_1(\widetilde{\mathscr{B}})$ in the group of isometries of $H_2(S)$ is of finite index in the subgroup fixing h, we conclude, as in [6], that $\delta_c(S, H)$ is of the given form.

Proposition 2. *Let S be a $K3$ surface, H be a primitive polarization on S of degree $2m$, and $c = 2m + 3$. Then $\delta_c(S, H) = \gamma_c$.*

Proof. By Lemma 4, $\delta_c(S, H)$ is a polynomial in q and h; on the other hand, γ_c is a polynomial in q [7], hence we can write

$$(*) \qquad \delta_c(S, H) - \gamma_c = \sum_{i=0}^{d/2} a_i q^{d/2-i} h^{2i}.$$

Let (S, H) be a polarized $K3$ surface such that $\mathrm{Pic}(S) = \mathbf{Z}[H] \oplus \mathbf{Z}[L]$, where $H^2 = 2m$, $H \cdot L = a$, $L^2 = -2$ (i.e., L is a rational curve of degree a). Such an S exists if $a > 0$. As is easily checked, whatever a is, H will not lie on any wall of W_c (the notation is as in Lemma 3), hence by the Corollary to Lemma 3 we know that

$$(**) \qquad \gamma_{c|\,\mathrm{Pic}(S)} = \delta_c(S, H)_{|\,\mathrm{Pic}(S)}.$$

Let ϕ be the polynomial on the right side of $(*)$. We claim that $(**)$ implies $\phi = 0$. Assuming $\phi \neq 0$, we will arrive at a contradiction. Write $\phi = h^{2n}\psi$, where ψ is not divisible by h, so $\psi = \sum_{i=n}^{d/2} a_i q^{d/2-i} h^{2i-2n}$ and $a_n \neq 0$. Obviously $\psi_{|\,\mathrm{Pic}(S)} = 0$. Let $D \in \mathrm{Pic}(S)$ be a nonzero divisor class perpendicular to H. Then $\psi(D) = a_n q(D)^{d/2-n}$ and, since $D^2 \neq 0$, we get $a_n = 0$, which is a contradiction.

Corollary. *Let S be a $K3$ surface, H be a primitive polarization on S of degree $2m$, and $c = 2m + 3$. Then $\delta_c = aq^{d/2}$.*

3. Let S be a $K3$ surface, H be a primitive polarization on S, $H^2 = 2m$, and $c = 2m + 3$. Recall from §2 that there is an isomorphism

$f\colon U_c \xrightarrow{\sim} V_c$: if $[Z] \in U_c$, then $f([Z])$ is the isomorphism class of the unique nontrivial extension of $I_Z(2H)$ by $\mathscr{O}_S$. We will therefore identify U_c and V_c. Let $Y = S \times U_c$. By a standard construction [3] there exists a universal extension

$$0 \to \mathscr{O}_Y \to \mathscr{F} \to I_{\mathscr{Z}}(p_S^*(2H) \otimes p_{U_c}^*(L)) \to 0,$$

where $\mathscr{Z} \subset S \times U_c$ is the restriction of the universal subscheme on $S \times \mathscr{H}_c$ to $S \times U_c$, p_S and P_{U_c} are the projections, and L is a line bundle on U_c. If we tensor $\mathscr{F}$ by $p_S^*(-H)$, we get a universal sheaf $\mathscr{E}$ on $S \times U_c$ and consequently on $S \times V_c$:

$$0 \to \mathscr{O}_Y(p_S^*(-H)) \to \mathscr{E} \to I_{\mathscr{Z}}(p_S^*(H) \otimes p_{U_c}^*(L)) \to 0.$$

Now choose a nonzero holomorphic two-form, ω, on S. Let $\Gamma \in H_2(S)$ be the Poincaré dual to the class $[\omega] \in H^2(S)$ represented by ω, and let $\mathrm{P.D.}(\mathscr{Z})$ be the Poincaré dual of $\mathscr{Z}$. Then

$$c_2(\mathscr{E}) = p_S^*(-c_1(H)^2) - p_S^*(H)p_{U_c}^*(L) + \mathrm{P.D.}(\mathscr{Z}).$$

Since $[\omega] \cup c_1(H) = 0$, we see that

$$c_2(\mathscr{E})/\Gamma = \mathrm{P.D.}(\mathscr{Z}))/\Gamma,$$

so that $c_2(\mathscr{E})/\Gamma$ is represented by the form obtained by integrating $p_S^*(\omega)_{|\mathscr{Z}}$ along the fibers of p_{V_c}, i.e., the push-forward of $p_S^*(\omega)_{|\mathscr{Z}}$, which we will denote by $\omega^{(n)}$, $n = 4m+3$ (since V_c is identified with U_c, we can think of V_c as a subset of $\mathscr{H}_c$, and then $\omega^{(n)}$ is the restriction of a holomorphic form on $\mathscr{H}_c$ [1]). We have proved

Lemma 5. *Let $\pi\colon \mathscr{Z} \to V_c$ be the projection and let $\omega^{(n)} \in \Gamma(\Omega^{2,0}_{U-c})$ be the push-forward of $p_S^*(\omega)_{|\mathscr{Z}}$. Then $\nu(\Gamma)$ restricted to V_c is represented by $\omega^{(n)}$.*

Lemma 6. *There exists a unique holomorphic two-form on $\overline{M}_c$, $\tau_{\overline{M}_c}(\omega)$, extending $\omega^{(n)}$ and representing $\nu(\Gamma)$.*

Proof. The point is that, by the claim following Definition 1, $\mathrm{cod}(\overline{M}_c \backslash V_c, \overline{M}_c) = 2$, hence $\omega^{(n)}$ extends holomorphically to $\tau_{\overline{M}_c}(\omega)$. Since $[\tau_{\overline{M}_c}(\omega)]_{|V_c} = \nu(\Gamma)_{|V_c}$, we conclude that they are equal on the whole $\overline{M}_c$.

Remark. We have associated to $\omega \in H^0(K_S)$ a two-form on $\overline{M}_c$. One can show that $\tau_{\overline{M}_c}(\omega)$ is (up to a multiplicative constant) the symplectic form constructed by Mukai ([12], [15]).

Theorem. *Let S be a $K3$ surface, let $c = 2m+3$ be an odd number greater than 3, and let $n = 4m+3$. Then*

$$\gamma_c = \frac{(2n)!}{2^n n!} q^n .$$

Proof. Since all $K3$ surfaces are diffeomorphic, we can assume that S has a primitive polarization, H, of degree $2m$. By Proposition 3 we know that $\gamma_c = \delta_c(S, H)$. Let $\omega \in H^0(K_S)$ be a generator; we will compute $\delta_c(\Gamma + \overline{\Gamma})$. Let $d = 8m+6 = \dim \overline{M}_c$. By Lemma 6, $\nu_c(\Gamma + \overline{\Gamma})$ is represented by $\tau_{\overline{M}_c}(\omega) + \overline{\tau_{\overline{M}_c}(\omega)}$. Then

$$\delta_c(\Gamma + \overline{\Gamma}) = \int_{\overline{M}_c} \bigwedge^d (\tau_{\overline{M}_c}(\omega) + \overline{\tau_{\overline{M}_c}(\omega)}),$$

which is equal to

$$\int_{U_c} \bigwedge^d (\omega^{(n)} + \bar{\omega}^{(n)}).$$

Now let $S_0^{(n)} \subset U_c$ be the subvariety parametrizing the Z's such that $\operatorname{supp} Z$ consists of n distinct points, let S^n be the product of n copies of S, and S_0^n be the open subvariety mapping to $S_0^{(n)}$ by the obvious map. Denote this map by f, and the ith projection by $p_i \colon S^n \to S$. Then it is clear that $f^*(\tau_{\overline{M}_c}(\omega) + \overline{\tau_{\overline{M}_c}(\omega)}) = \sum_{i=1}^n p_i^*(\omega + \bar{\omega})$, so that

$$\begin{aligned} \int_{S^n} \bigwedge^d \left(\sum_{i=1}^n p_i^*(\omega + \bar{\omega}) \right) &= (2n)! \left(\int_S \omega \wedge \bar{\omega} \right)^n \\ &= \frac{(2n)!}{2^n} \left(\int_S (\omega + \bar{\omega}) \wedge (\omega + \bar{\omega}) \right)^n . \end{aligned}$$

The first equality holds because in the wedge product the only terms which give a nonzero integral are

$$\underbrace{p_1^*(\omega) \wedge p_1^*(\bar{\omega}) \wedge p_2^*(\omega) \wedge \cdots \wedge p_n^*(\bar{\omega})}_{2n}$$

and all its permutations. Since $\deg f = n!$, we have

$$\delta_c(\Gamma + \overline{\Gamma}) = \frac{(2n)!}{2^n n!} q(\Gamma + \overline{\Gamma})^n .$$

By Proposition 2 we conclude that

$$\gamma_c = \delta_c = \frac{(2n)!}{2^n n!} q^n .$$

Acknowledgments

It is a pleasure to thank Bob Friedman for various useful conversations. Thanks again to Friedman and to John Morgan for letting me read the preliminary version of their book on Gauge theory and the classification of smooth four-manifolds.

References

[1] A. Beauville, *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Differential Geometry **18** (1983) 755–782.

[2] F. A. Bogomolov, *Holomorphic tensors and vector bundles on projective varieties*, Math. USSR-Izv. **13** (1979) No. 3, 499–555.

[3] J. E. Brosius, *Rank-two vector bundles on a ruled surface*. I, Math. Ann. **265** (1983) 155–168.

[4] S. K. Donaldson, *Infinite determinants, stable bundles and curvature*, Duke Math. J. **54** (1987) 231–247.

[5] ——, *Polynomial invariants for smooth four-manifolds*, Topology **29** (1990) 257–315.

[6] R. Friedman, B. Moishezon & J. Morgan, *On the C^∞ invariance of the canonical class of certain algebraic surfaces*, Bull. Amer. Math. Soc. (N.S.) **17** (1987) 283–286.

[7] R. Friedman & J. Morgan, *Gauge theory and the classification of smooth four-manifolds*, Smooth four-manifolds and complex surfaces, Springer, Berlin, to appear.

[8] D. Gieseker, *On the moduli of vector bundles on an algebraic surface*, Ann. of Math. (2) **106** (1977) 45–60.

[9] A. Iarrobino, *Punctual Hilbert schemes*, Bull. Amer. Math. Soc. **78** (1972) 819–823.

[10] M. Maruyama, *Moduli of stable sheaves*. II, J. Math. Kyoto Univ. **18** (1978) 557–614.

[11] V. B. Mehta & A. Ramanathan, *Restriction of stable sheaves and representations of the fundamental group*, Invent. Math. **77** (1984) 163–172.

[12] S. Mukai, *Symplectic structure of the moduli space of sheaves on an abelian or $K3$ surface*, Invent. Math. **77** (1984) 101–116.

[13] ——, *On the moduli spaces of bundles on $K3$ surfaces*. I, *Vector bundles on algebraic varieties*, Oxford University Press, 1987.

[14] M. Raynaud, *Sections des fibrés vectoriels sur une courbe*, Bull. Soc. Math. France **110** (1982) 103–125.

[15] A. N. Tyurin, *Symplectic structures on the varieties of moduli of vector bundles on algebraic surfaces with $p_g > 0$*, Math. USSR-Izv. **33** (1989) 139–177.

COLUMBIA UNIVERSITY