196. The Theory of Nuclear Spaces Treated by the Method of Ranked Space. IV

By Yasujirô NAGAKURA
Science University of Tokyo

(Comm. by Kinjirô KUNUGI, M. J. A., June 12, 1971)

§ 5. The completion of the linear ranked space Φ, (2).

Lemma 20. *Let $\hat{\Phi}_0$ be the subset of $\hat{\Phi}$ consisting of those equivalence classes which contain an R-Cauchy sequence $\{g_n\}$ for which $g_1=g_2=g_3=\cdots$.*

The mapping T of Φ onto $\hat{\Phi}_0$, which maps $g\in\Phi$ to the class $\hat{g}$ containing the sequence consisting of a single element g, is bijective and we have $g\in V_i\ (0,r,m)$ if and only if $\hat{g}\in\hat{V}_i\ (0,r,m)$.

Proof. Let g and f be two different elements in Φ. Then there exists no class containing two sequences $\{g_n\}$ and $\{f_n\}$ with $g_n=g$, $f_n=f$ for every n.

Because if it is not true, $\{g_n\}$ and $\{f_n\}$ are equivalent. And then there exists a fundamental sequence of neighbourhoods $\{V_{\gamma_i}(0,r_i,m_i)\}$ such that $g_i-f_i\in V_{\gamma_i}(0,r_i,m_i)$ for every i, that is, $g-f\in V_{\gamma_i}(0,r_i,m_i)$ for every i. This implies $g=f$ by Lemma 8 in [4].

Next, we shall prove that $g\in V_i\ (0,r,m)$ implies $\hat{g}\in\hat{V}_i\ (0,r,m)$. Since we have $V_i\ (0,1,m)=U_i\ (0,\varepsilon_i,m)$ by the paper [4], we obtain $V_i\ (0,r,m)=U_i\ (0,r\varepsilon_i,m)$. Hence we have

$$\left\|\sum_{k=1}^{m}\lambda_{k,n_{i-1},n_i}(g,\varphi_{k,n_i})\varphi_{k,n_{i-1}}\right\|<r\varepsilon_i.$$

Then there exists some number r', $0<r'<r$ such that

$$\left\|\sum_{k=1}^{m}\lambda_{k,n_{i-1},n_i}(g,\varphi_{k,n_i})\varphi_{k,n_{i-1}}\right\|<r'\varepsilon_i<r\varepsilon_i.$$

Consequently we obtain $g\in V_i\ (0,r',m)$. By Definition 5, this shows

$$\hat{g}\in\hat{V}_i\ (0,r,m).$$

Conversely, if we have $\hat{g}\in\hat{V}_i\ (0,r,m)$, there exist some number r', $0<r'<r$ and some integer N such that

$$g_n=g\in V_i\ (0,r',m)\qquad \text{if } n\geqq N.$$

And then we obtain $g\in V_i\ (0,r,m)$.

Theorem 2. *The set $\hat{\Phi}_0$ is dense in $\hat{\Phi}$.*

Proof. Let $\hat{g}$ be any element in $\hat{\Phi}$. And let an R-Cauchy sequence $\{g_n\}$ belong to $\hat{g}$. Then there exists a fundamental sequence of neighbourhoods $\{V_{\gamma_i}(0,r_i,m_i)\}$, such that the relations $n\geqq i$ and $m\geqq i$ imply

$$g_n-g_m\in V_{\gamma_i}(0,r_i,m_i).$$

Let $\hat{g}_n$ be the class containing the reptitive sequence $g_n, g_n, \cdots$.

Then we have $\hat{g}_n \in \hat{\Phi}_0$. Since by Lemma 20 we have that the relations $n \geqq i$ and $m \geqq i$ imply $\hat{g}_n - \hat{g}_m \in \hat{V}_{r_i}(0, r_i, m_i)$, the sequence $\{\hat{g}_n\}$ in $\hat{\Phi}_0$ is an R-Cauchy sequence.

On the other hand, since we have $g_i - g_m \in V_{r_i}(0, r_i, m_i)$ for $m \geqq i$, we obtain $\hat{g}_i - \hat{g} \in \hat{V}_{r_i}(0, 2r_i, m_i)$ for every i. Since $\{\hat{V}_{r_i}(0, 2r_i, m_i)\}$ is a fundamental sequence of neighbourhoods, we obtain $\hat{g}_n \xrightarrow{R} \hat{g}$.

Definition 7. Let $P_{i,m}$ be a real valued function defined on the linear ranked space Φ such that

$$P_{i,m}(g) = \inf r, \qquad \text{where } g \in V_i(0, r, m).$$

Lemma 21. *We have* $P_{i,m}(g) = \left\| \sum_{k=1}^{m} (\lambda_{k, n_{i-1}, n_i}/\varepsilon_i)(g, \varphi_{k, n_i}) \varphi_{k, n_{i-1}} \right\|_{n_{i-1}}$.

Proof. It is easily verified.

Lemma 22. *The function $P_{i,m}$ is a semi-norm on Φ.*

Lemma 23. *We have $P_{i,m}(g) \geqq P_{j,m}(g)$ if $j \leqq i$.*

Lemma 24. *We have $P_{i,m}(g) \geqq P_{i,m'}(g)$ if $m' \leqq m$.*

Definition 8. Let $\hat{P}_{i,m}$ be a real valued function defined on the linear ranked space $\hat{\Phi}$ such that $\hat{P}_{i,m}(\hat{g}) = \inf r$, where $\hat{g} \in \hat{V}_i(0, r, m)$.

Lemma 25. *If an R-Cauchy sequence $\{g_n\}$ belongs to $\hat{g}$, we have*

$$\hat{P}_{i,m}(\hat{g}) = \lim_{n \to \infty} P_{i,m}(g_n).$$

Proof. Set briefly $\hat{P}_{i,m}(\hat{g}) = \alpha$. Then if $\alpha < r$, we have $\hat{g} \in \hat{V}_i(0, r, m)$. By Definition 5, there exist some number r', $0 < r' < r$ and some integer N such that $g_n \in V_i(0, r', m)$ for every $n \geqq N$. And then we have $P_{i,m}(g_n) \leqq r' < r$. Since the sequence $\{g_n\}$ is an R-Cauchy sequence, there exists a fundamental sequence of neighbourhoods, $\{V_{r_i}(0, r_i, m_i)\}$, such that the relations $k \geqq i$ and $h \geqq i$ imply $V_{r_i}(0, r_i, m_i) \ni g_k - g_h$.

On the other hand, there exists some integer j such that $V_i(0, r_j, m_i) \supseteqq V_{r_j}(0, r_j, m_j)$. And then we have that the relations $k \geqq j$ and $h \geqq j$ imply $|P_{i,m}(g_k) - P_{i,m}(g_h)| \leqq P_{i,m}(g_k - g_h) < r_j$. Hence $\{P_{i,m}(g_n)\}_n$ is a Cauchy sequence of real numbers. Then $\lim_{n \to \infty} P_{i,m}(g_n)$ exists and we have $\lim_{n \to \infty} P_{i,m}(g_n) \leqq \alpha$.

Conversely, if $\alpha > r > 0$, we have $\hat{g} \notin \hat{V}_i(0, r, m)$. And hence for every l with $0 < l < r$ and every integer N, there exists some integer $k > N$ such that $g_k \notin V_i(0, l, m)$, that is, $P_{i,m}(g_k) \geqq l$.

Since $\{P_{i,m}(g_n)\}$ is a Cauchy sequence, it follows $\lim_{n \to \infty} P_{i,m}(g_n) \geqq l$. And hence we have $\lim_{n \to \infty} P_{i,m}(g_n) \geqq r$. Consequently we obtain $\lim_{n \to \infty} P_{i,m}(g_n) \geqq \alpha$. This finishes the proof of the lemma.

Lemma 26. *The function $\hat{P}_{i,m}$ is a semi-norm on $\hat{\Phi}$.*

Proof. It is evident.

Lemma 27. *The quotient space $\hat{\Phi}/M_{i,m}$, where $M_{i,m} = \{\hat{g} \in \hat{\Phi}; \hat{P}_{i,m}(\hat{g}) = 0\}$, is a finite dimensional space.*

Proof. Let $\hat{g}$ be an element in $\hat{\Phi}$, and let an R-Cauchy sequence $\{g_n\}$ belong to $\hat{g}$. Then there exists a fundamental sequence of neighbourhoods $\{V_{r_j}(0, r_j, m_j)\}$ such that the relations $h \geqq j$ and $l \geqq j$ imply $V_{r_j}(0, r_j, m_j) \ni g_h - g_l$. And then the relations $\gamma_j \geqq i$ and $m_j \geqq m$ imply $V_i(0, r_j, m) \ni g_h - g_l$ for $h, l \geqq j$. Hence we have

$$(1/\varepsilon_i)\left\| \sum_{k=1}^{m} \lambda_{k,n_{i-1},n_i}(g_h - g_l, \varphi_{k,n_i})\varphi_{k,n_{i-1}} \right\| < r_j,$$

and it follows

$$|(g_h, \varphi_{k,n_i}) - (g_l, \varphi_{k,n_i})| < \varepsilon_i r_j / \lambda_{k,n_{i-1},n_i}, \qquad \text{for } 1 \leqq k \leqq m.$$

Then $\{(g_h, \varphi_{k,n_i})\}_h$ is a Cauchy sequence of real numbers, thus there exists some number α_k such that $(g_h, \varphi_{k,n_i}) \to \alpha_k$ $h \to \infty$.

Now, we set $(1/\varepsilon_i)\left(\sum_{k=1}^{m} \lambda_{k,n_{i-1},n_i}\alpha_k\varphi_{k,n_{i-1}}\right) = \hat{g}_{(m)}$. Thus the mapping $F_{i,m}$ of $\hat{g}$ to $\hat{g}_{(m)}$ is a homomorphism of $\hat{\Phi}$ onto $\hat{\Phi}_{(i,m)}$, where

$$\hat{\Phi}_{(i,m)} = \left\{\sum_{k=1}^{m} \lambda_{k,n_{i-1},n_i}(g, \varphi_{k,n_i})\varphi_{k,n_{i-1}}\,;\, g \in \Phi\right\}.$$

Moreover we can easily verify $\hat{P}_{i,m}(\hat{g}) = P_{i,m}(\hat{g}_{(m)})$. And hence $F_{i,m}$ induces an isomorphism of $\hat{\Phi}/\ker(F_{i,m})$ onto $\hat{\Phi}_{(i,m)}$. Next, we shall prove $\ker(F_{i,m}) = M_{i,m}$. Because if $\hat{g}$ belongs to $\ker(F_{i,m})$, we have $\hat{g}_{(m)} = 0$. Then there exists an R-Cauchy sequence $\{g_n\}$ belonging to $\hat{g}$ such that $(g_n, \varphi_{k,n_i}) \to 0$ as $n \to \infty$ for every $k = 1, \cdots, m$. And hence we have $P_{i,m}(g_n) = (1/\varepsilon_i)\left\| \sum_{k=1}^{m} \lambda_{k,n_{i-1},n_i}(g_n, \varphi_{k,n_i})\varphi_{k,n_{i-1}} \right\| \to 0$. It follows $\hat{P}_{i,m}(\hat{g}) = 0$, that is, $\hat{g} \in M_{i,m}$.

Conversely if $\hat{g}$ belongs to $M_{i,m}$, we have $\hat{P}_{i,m}(\hat{g}) = 0$. Then there exists an R-Cauchy sequence $\{g_n\}$ belonging to $\hat{g}$ such that $\lim_{n\to\infty} P_{i,m}(g_n) = 0$. Hence we have $(g_n, \varphi_{k,n_i}) \to 0$ as $n \to \infty$ for $k = 1, \cdots, m$. It follows $\hat{g}_{(m)} = 0$.

Now, we shall define a norm on $\hat{\Phi}/M_{i,m}$ such that $\|\hat{g} + M_{i,m}\| = \inf \hat{P}_{i,m}(\hat{f})$, where $\hat{f} \in \hat{g} + M_{i,m}$ and $\hat{g} \in \hat{\Phi}$. Thus we shall prove $\|\hat{g} + M_{i,m}\| = P_{i,m}(\hat{g}_{(m)})$. Because let $\hat{f}$ belong to $\hat{g} + M_{i,m}$ such that $\hat{f} = \hat{g} + \hat{g}'$, $\hat{g}' \in M_{i,m}$.

Hence we have

$$|P_{i,m}(\hat{g}_{(m)}) - P_{i,m}(\hat{f}_{(m)})| = |\hat{P}_{i,m}(\hat{g}) - \hat{P}_{i,m}(\hat{f})| \leqq \hat{P}_{i,m}(\hat{g} - \hat{f}) = 0.$$

Consequently we obtain $\|\hat{g} + M_{i,m}\| = \hat{P}_{i,m}(\hat{g}) = P_{i,m}(\hat{g}_{(m)})$.

Theorem 3. *A bounded infinite set in $\hat{\Phi}$ is sequential compact.*

Proof. Let B be a bounded infinite set in $\hat{\Phi}$. By the definition of the boundness in linear ranked space (in [6]), there exist a fundamental sequence of neighbourhoods $\{\hat{V}_{r_i}(0, r_i, m_i)\}$ and numbers $C_i (i = 1, 2, \cdots)$ such that $C_i\hat{V}_{r_i}(0, r_i, m_i) \supset B$ for every integer i.

And then the relation $\hat{g} \in B$ implies $\hat{P}_{r_{i,m}}(\hat{g}) \leqq C_i r_i$ for every integer i.

Case I. Suppose B/M_1, where $M_1 = \{\hat{g} \in \hat{\Phi}\,;\, \hat{P}_{r_{1,m_1}}(\hat{g}) = 0\}$.

If B/M_1 is consisted of some finite element of the coset; $B_{1,1}$, $B_{1,2}$, $\cdots$, $B_{1,n(1)}$, some coset $B_{1,i(1)}$ has to contain infinite elements in $\hat{\Phi}$. Next, suppose $B_{1,i(1)}/M_2$, where $M_2=\{\hat{g}\in\hat{\Phi}\,;\,\hat{P}_{r_2,m_2}(\hat{g})=0\}$. If $B_{1,i(1)}/M_2$ is consisted of some finite element of the coset; $B_{2,1}, B_{2,2}, \cdots, B_{2,n(2)}$, some coset $B_{2,i(2)}$ has to contain infinite elements in $\hat{\Phi}$. In general, suppose that $B_{k,i(k)}/M_{k+1}$ $(k=1,2,\cdots)$ has finite element of the coset, thus $B_{k,i(k)}$ $(k=1,2,\cdots)$ has infinite elements in $\hat{\Phi}$.

And then we have $B_{1,i(1)}\supseteqq B_{2,i(2)}\supseteqq\cdots\supseteqq B_{k,i(k)}\supseteqq\cdots$. If we take a sequence $\{\hat{g}_k\}\subset\hat{\Phi}$ such that $\hat{g}_k\in B_{k,i(k)}$ and $\hat{g}_k\neq\hat{g}_h$ if $k\neq h$, then the sequence $\{\hat{g}_k\}$ is a Cauchy sequence in $\hat{\Phi}$ with respect to every semi-norm $\hat{P}_{r_i,m_i}$ $(i=1,2,\cdots)$. Because for any $\varepsilon>0$ and any $\hat{P}_{r_i,m_i}$, the relations $h\geqq j$ and $k\geqq j$ imply $\hat{P}_{r_j,m_j}(\hat{g}_h-\hat{g}_k)=0<\varepsilon$, that is $\hat{g}_h-\hat{g}_k\in\hat{V}_{r_j}(0,\varepsilon,m_j)$.

On the other hand, for any neighbourhood in $\hat{\Phi}$, $\hat{V}_i(0,\varepsilon,m)$, there exists some integer j such that $\hat{V}_i(0,\varepsilon,m)\supseteqq\hat{V}_{r_j}(0,\varepsilon,m_j)$. And then we have $\hat{V}_i(0,\varepsilon,m)\ni\hat{g}_h-\hat{g}_k$ for $h\geqq j$ and $k\geqq j$. Consequently we assert that the sequence $\{\hat{g}_k\}$ is an R-Cauchy sequence in $\hat{\Phi}$ and then there exists a limiting element in $\hat{\Phi}$.

Case II. Suppose that there exists some integer l such that for $k<l$, $B_{k,i(k)}/M_{k+1}$ has finite element of the coset and $B_{l,i(1)}/M_{l+1}$ has infinite elements of the coset. Let $\hat{f}$ belong to $\hat{g}+M_{l+1}$ (with $\hat{g}\in B_{l,i(l)}$) in $B_{l,i(l)}/M_{l+1}$. Since we can select $\hat{g}'$ in M_{l+1} such that $\hat{f}=\hat{g}+\hat{g}'$, we have $\hat{P}_{r_{l+1},m_{l+1}}(\hat{f})\leqq\hat{P}_{r_{l+1},m_{l+1}}(\hat{g})+\hat{P}_{r_{l+1},m_{l+1}}(\hat{g}')=\hat{P}_{r_{l+1},m_{l+1}}(\hat{g})\leqq C_{r_{l+1}}r_{r_{l+1}}$.

And hence we have $\|\hat{g}+M_{l+1}\|\leqq C_{r_{l+1}}r_{r_{l+1}}$. By Lemma 27, $\hat{\Phi}/M_{l+1}$ is finite dimensional and then $B_{l,i(l)}/M_{l+1}$ is bounded in the finite dimensional spase. Thus there exists a Cauchy sequence $\{B_{l+1,n}\}_n$ of cosets in $B_{l,i(l)}/M_{l+1}$ with respect to norm of $\hat{\Phi}/M_{l+1}$. If we take a sequence $\{\hat{g}_{1,n}\}$ such that $\hat{g}_{1,n}\in B_{l+1,n}$ for every integer n, the sequence $\{\hat{g}_{1,n}\}$ is a Cauchy sequence with respect to $\hat{P}_{r_{l+1},m_{l+1}}$, since we have $\hat{P}_{r_{l+1},m_{l+1}}(\hat{g})=\|g+M_{l+1}\|$ by the proof of Lemma 27.

On the other hand, since we have

$$\hat{V}_{r_{l+1}}(0,1,m_{l+1})\supseteqq\hat{V}_{r_{l+2}}(0,1,m_{l+2}),$$

it is clear that

$$\hat{P}_{r_{l+2},m_{l+2}}(\hat{g})\geqq\hat{P}_{r_{l+1},m_{l+1}}(\hat{g}).$$

Thus we obtain $M_{l+2}\subseteqq M_{l+1}$.

Hence there cannot be two different elements $\hat{g}_{1,n}$, $\hat{g}_{1,m}$ with $n\neq m$ in the same coset with respect to M_{l+2}. And then $\{\hat{g}_{1,n}\}_n/M_{l+2}$ has infinite elements of the coset and it is bounded in $\hat{\Phi}/M_{l+2}$. Hence there exists a Cauchy sequence $\{\hat{g}_{2,n}\}_n$ in $\{\hat{g}_{1,n}\}_n$ with respect to $\hat{P}_{r_{l+2},m_{l+2}}$. We proceed by induction, obtaining sequences $\{\hat{g}_{k,n}\}_n$, each a subsequence of its predecessor. The diagonal sequence $\{\hat{g}_{n,n}\}_n$ is then a Cauchy sequence in $\hat{\Phi}$ with respect to every $\hat{P}_{r_i,m_i}$. And then it has a limiting element in $\hat{\Phi}$.

Theorem 4. *A bounded infinite set in the nuclear space Φ is sequential compact.*

Proof. Let B be a bounded infinite set in the nuclear space Φ. And then there exist numbers C_i $(i=1,2,\cdots)$ such that the relation $g\in B$ implies

$$\|g\|_{n_{i-1}}=\left\|\sum_{k=1}^{\infty}\lambda_{k,n_{i-1},n_i}(g,\varphi_{k,n_i})\varphi_{k,n_{i-1}}\right\|<C_i,$$

for every i.

And hence we have

$$\varepsilon_i P_{i,m}(g)=\left\|\sum_{k=1}^{m}\lambda_{k,n_{i-1},n_i}(g,\varphi_{k,n_i})\varphi_{k,n_{i-1}}\right\|<C_i.$$

Now, let $\hat{g}$ be an element in $\hat{\Phi}$, which contains an R-Cauchy sequence $\{g\}$ consisting of a single element g in Φ. Then we have

$$\hat{P}_{i,m}(\hat{g})=P_{i,m}(g)<C_i/\varepsilon_i,$$

and hence we obtain

$$\hat{g}\in\hat{V}_i(0,C_i/\varepsilon_i,m).$$

Consequently, $\{\hat{g}\}=\{\hat{g}\,;\,g\in B\}$ is a bounded infinite set in the linear ranked space $\hat{\Phi}$. Hence by Theorem 3, there exists an R-Cauchy subsequence $\{\hat{g}_n\}$ in $\{\hat{g}\}$.

By Lemma 13 in [4], for any $\hat{V}_i(0,\varepsilon,m)$, there exists some integer N such that the relations $n\geqq N$ and $m\geqq N$ imply $\hat{g}_n-\hat{g}_m\in\hat{V}_i(0,\varepsilon,m)$.

Since for every integer n, $\hat{g}_n$ is a equivalence class which contains an R-Cauchy sequence consisting of a single element g_n, then we have

$$g_n-g_m\in V_i(0,\varepsilon,m)\qquad\text{by Lemma 20.}$$

This shows that the sequence $\{g_n\}_n$ is an R-Cauchy sequence in Φ.

Hence by Lemma 4, the sequence $\{g_n\}_n$ is a Cauchy sequence in the nuclear space.

Since the nuclear space Φ is complete, the proof finishes.

References

[1] K. Kunugi: Sur la méthode des espaces rangés, I, II. Proc. Japan Acad., **42**, 318-322, 549–554 (1966).

[2] M. Washihara: On ranked spaces and linearity. II. Proc. Japan Acad., **45**, 238–242 (1969).

[3] Y. Nagakura: The theory of nuclear spaces treated by the method of ranked space. I. Proc. Japan Acad., **47**, 337–341 (1971).

[4] ——: The theory of nuclear spaces treated by the method of ranked space. II. Proc. Japan Acad., **47**, 342–345 (1971).

[5] ——: The theory of nuclear spaces treated by the method of ranked space. III. Proc. Japan Acad., **47**, 870–874 (1971).

[6] M. Washihara and Y. Yoshida: Remarks on bounded sets in linear ranked spaces. Proc. Japan Acad., **44**, 73–76 (1968).

[7] I. M. Gel'fand and N. Ya. Vilenkin: Generalized Functions, Vol. 4 (1964).