181. A Criterion for the Existence of the Twosided Unity Element of Semigroups*)

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In this paper, which is a modified version of author's earlier article [4], written in Hungarian, will be given a necessary and sufficient condition, and its three corollaries, for the existence of the twosided unity element of a semigroup. Author's this criterion is left-right symmetric, and it can be transformed, almost trivially, also into a left-right nonsymmetric formally milder condition, such that it remains yet equivalent to the existence of the twosided unity element of the semigroup.

On the other hand, this criterion is an analogy of one among author's [5] criteria for the existence of the twosided unity element of an associative ring. Further criteria, for the existence of the unity element of a semigroup, were earlier discussed, in the joint paper [2] of S. Lajos and J. Szép, using the notion of so called magnifying elements.

The here used fundamental notions can be found e.g. in A. H. Clifford's and G. P. Preston's book [1] or in Ljapin's book [3]. Let S be an arbitrary semigroup, generally without zero element. An element r of a semigroup S with zero element z will be called right annihilator of S, if $sr=z$ for any $s \in S$ holds. Furthermore, the element r_1 of S is said to be right regular in S, if $xr_1=yr_1$ always implies $x=y$ $(x, y \in S)$. Left annihilator and left regularity are defined left-right dually. The center C of S is the set of all elements c, satisfying $cs=sc$ for any element $s(\in S)$, of S. Now let T be S itself, if S has zero z, and $T=S \cup z$ with $z^2=zs=sz=z$ for any $s \in S$, if $z \notin S$. T is called here the related semigroup of S. Then we have:

Theorem. *For an arbitrary semigroup S the following two conditions are equivalent:*

(I) *S has twosided unity element;*

(II) *S contains a right regular element r and a left regular element l, satisfying $rS \subseteq Sr$ and $Sl \subseteq lS$, furthermore the related semigroup T yet satisfies both of requirements:*

($*$) *T has no homomorphic image with nonzero right annihilators;*

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*) To Professor Jenö Szép on his 50th birthday.

(**) *T has no homomorphic image with nonzero left annihilators.*

Proof. (I) implies (II) trivially.

Conversely, let us assume, that condition (II) for the semigroup S holds, and we shall prove (I), as follows. Let R be an arbitrary right ideal of T. Then TR is a twosided ideal of T, consequently the Rees factor semigroup T/TR (cf. e.g. [1]) is a homomorphic image of T. Obviously $R \cup TR$ is contained in the right annihilator of T/TR, and therefore our requirement (*) implies the inclusion $R \subseteq TR$ for any right ideal R of T. Dually, by condition (**) also $L \subseteq L, T$ can be verified for any left ideal L of T.

Furthermore, by (II) there exist elements $r, l \in S$ with $rS \subseteq Sr$ and $Sl \subseteq lS$, such that $xr = yr$ and $lu = lv$ imply always $x = y$ and $u = v(x, y, u, v \in S)$. In particular, for the principal right ideal $R_s = s \cup z \cup sS$ of T and for the principal left ideal $L_s = s \cup z \cup Ss$ of $T(s \in S)$ we have, by the earlier statement, $R_s \subseteq TR_s$ and $L_s \subseteq L_s T$, consequently

$$s \in z \cup ((Ss \cup SsS) \cap (sS \cup SsS)) \qquad \text{for any } s \in S.$$

The substitutions $s = r$ and $s = l$, respectively, yield by $T^2 = T$, $S^2 = S$, $r \neq z \neq l$, $rS \subseteq Sr$ and $Sl \subseteq lS$, evidently

$$r \in Sr \quad \text{and} \quad l \in lS.$$

Therefore there exist elements s_1 and s_2 of S such that $r = s_1 r$ and $l = ls_2$ hold, which give $xr = xs_1 r$ and $ly = ls_2 y$ for any $x, y \in S$. Hence, the right regularity of r implies $xs_1 = x$ for any $x \in S$. Therefore s_1 is a right unity element of S. Dually can be shown, that s_2 is a left unity element of S, which yields at once, that $e = s_1 = s_2$ is the twosided unity element of S. Therefore, also (II) implies (I), indeed.

This concludes the proof of the theorem.

Corollary 1. *A semigroup S has twosided unity element if and only if it contains a central, regular nonzero element, and for the related semigroup T, defined before the theorem, of S,* (*) *and* (**) *are satisfied.*

Corollary 2. *A semigroup S, whose any nonzero element is twosided regular, has twosided unity element if and only if the center C of S differs from the (eventual) zero element of S, furthermore the related semigroup T or S satisfies* (*) *and* (**).

Corollary 3. *A commutative semigroup S, having only regular nonzero elements, contains twosided unity element if and only if the related semigroup T of S satisfies* (*) *and* (**).

References

[1] A. H. Clifford and G. B. Preston: The Algebraic Theory of Semigroups. I, II. Providence (1961, 1967).

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[5] ——: Einige Kriterien für die Existenz des Einselementes in einem Ring. Acta Sci. Math. Szeged, **18**, 261–272 (1967).