The use of norm attainment

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1 Introduction

The purpose of this short note is to gather a number of natural examples, belonging to various domains of analysis, where a result from fundamental functional analysis provides concrete inequalities in a unified manner. A familiar technique in this respect is to apply Baire category theorem via the uniform boundedness principle. This note displays arguments of a different kind, where the leading role is played by James' fundamental characterization of weak compactness [J], and Simons' inequality [S] which proves it in the separable case (see [Pf1], [Pf2] for recent and deep progress in the non-separable frame).

Let X be a Banach space, and let S be a norm-closed subspace of the dual space X^*. We denote B_S its closed unit ball. The space S is called separating if : $x^*(x) = 0$ for all $x^* \in S$ implies that $x = 0$. It is called norming if the functional $N(x) = \sup\{|x^*(x)|;\ x^* \in B_S\}$ is an equivalent norm on X and it is called 1-norming if N is equal to the original norm on X.

Straightforward applications of the Hahn-Banach theorem show that S is separating if and only if it is weak* dense in X^*, and that it is norming if and only if the weak* closure of B_S contains λB_{X^*} for some $\lambda > 0$, and finally that it is 1-norming if and only if $\lambda = 1$, in other words if and only if this weak* closure is equal to B_{X^*}. Easy examples show that these three notions are distinct : for instance, any hyperplane $H = Ker(x^{**})$ with $x^{**} \in X^{**}\backslash X$ is norming, but it is 1-norming if and only if $\|x + x^{**}\| \geqslant \|x\|$ for all $x \in X$. Also, take $X = c_0(\mathbf{N})$ equipped with its natural norm, and write $\mathbf{N}$ as a disjoint union of infinite sets $\bigcup_{j\geqslant 0} I_j$ with $0 \notin I_0$. Let S be the subspace of $l_1(\mathbf{N})$ consisting of all $y = (y_n)$ such that for all $k \in I_0$

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$$y_k = (k^{-1}) \sum_{n \in I_k} y_n.$$

It is easy to check that S is a separating subspace of $l_1 = c_0^*$ which is not norming (hint : consider the unit vector basis (e_k) of c_0 and pick $k \in I_0$). A more elaborate version of this example will be considered below (see the proof of Proposition 4).

We denote by $NA(X)$ the subset of X^* consisting of all linear forms which attain their norm, in other words which attain their supremum on B_X. It is natural to connect attainment of bounds with compactness, and the connection materializes in James's fundamental result [J]. But it is somewhat surprising that the distinction between separating and norming space S, which amounts to investigate the existence of certain constants, is related with the inclusion of S in $NA(X)$. However, for separable spaces X it is so : separating subspaces of $NA(X)$ are actually isometric preduals of X and thus in particular they are 1-norming. Separability is not a matter of convenience in this context : we will investigate what goes on in the non-separable case, where in general this "norming for free" property fails. We will also display several concrete examples where our unifying approach explains the (sometimes unexpected) availability of a metric control : special subspaces of L^1, little Lipschitz spaces, weighted spaces of holomorphic functions, spaces of compact operators.

These examples illustrate the versatility of the following principle, which seems worth keeping in mind : *norm-attainment provides isometric norming for free.*

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2 Results

The following remarkable result goes back to Y. I. Petunin and A. N. Plichko [PP]. Communication from and to Ukraine was somewhat uneasy at the time (1974) and thus their theorem was rediscovered later on (see e.g. Theorem IV. 2 in [G]). We will outline the proof for completeness.

Theorem 1 :*Let X be a separable Banach space, and let S be a separating norm-closed subspace of X^*, contained in $NA(X)$. Then S is an isometric predual of X.*

Proof : Let $Q_S : X^{**} \to S^*$ be the canonical quotient map of restriction to S. Since S is separating, the restriction of Q_S to X is one-to-one. Since S is contained in $NA(X)$, the set $Q_S(B_X) = B$ is a boundary of B_{S^*}. That is, every $s \in S$ attains its norm at some $s^* \in B$. Since X is separable, the boundary B is norm-separable as well.

It follows from Simons inequality [S] that if a bounded closed convex set C has a separable boundary B, then $C = \overline{conv}(B)$ (see [G], Theorem III.3 where a more general result is shown, or [G2]). With the above notation, it follows that $B_{S^*} = \overline{conv}(Q_S(B_X))$ and thus $B_{S^*} = \overline{Q_S(B_X)}$. But this implies immediately (see the proof of the open mapping theorem) that X is isometric to S^*.

Remark 2 : The above proof actually shows the following : if X is separable and S is any closed subspace of X^*, then S is contained in $NA(X)$ if and ONLY IF the unit ball of X is compact for the topology $t_p(S)$ of pointwise convergence on S. The space S is separating if and only if the topology $t_p(S)$ is moreover Hausdorff.

Theorem 1 implies immediately the following "norming for free" statement. It amounts to weaken $t_p(S)$-compactness to $t_p(S)$-closedness.

Corollary 3 :*Let X be a separable Banach space. If a norm closed separating subspace S of X^* is contained in $NA(X)$, then S is 1-norming.*

It is natural to wonder whether assuming the separability of X is a simple convenience or whether it really matters in Corollary 3. It turns out that Simons inequality [S] is a substitute to Lebesgue's dominated convergence theorem when compactness and more generally topological regularity fails to hold, and it provides information on the behaviour of sequences as opposed to filters. We may therefore expect that separability is needed. It is indeed so, as shown by the following new example.

Proposition 4 : *Let $X = l_1([0,1])$ be the space of all atomic measures supported by $[0,1]$. There exists a norm-closed separating subspace of X^* which is contained in $NA(X)$ but which is not norming.*

Proof : Pick $\alpha > 1$ and let $M_\alpha = \{f \in C([0,1]);\ \alpha f(0) = \int_0^1 f(t)dt\}$. We consider M_α as a subspace of $X^* = l_\infty([0,1])$.

It is clear that M_α separates X : if not, it would be contained in a space $ker(\mu)$ with μ a measure with countable support on $[0,1]$, but then this measure would be proportional to $(\alpha\delta_0 - m)$ with m the Lebesgue measure, contradiction. Moreover M_α is contained in $NA(X)$ since any continuous function on $[0,1]$ attains its bounds. Finally, M_α is not λ-norming with $\lambda > (\alpha)^{-1}$. Indeed, if $\|f\|_\infty \leqslant 1$ then $|\int_0^1 f(t)dt| \leqslant 1$ and thus $|f(0)| \leqslant (\alpha)^{-1}$.

To conclude the proof, it suffices to consider X as the l_1-sum of countably many copies of itself, and the subspace S of its dual consisting of the c_0- direct sum of the spaces M_n, with $n \in \mathbf{N}$. It is easy to check that S is separating, contained in $NA(X)$ and not norming.

We now provide various examples where Theorem 1 and Corollary 3 can be applied. Let us note in passing that the separating spaces to which they apply are actually ideals (M-ideals and / or algebraic ideals) in their biduals.

Example 5 : Let X be a subspace of L^1 whose unit ball is closed for the topology of convergence in measure. Such subspaces are called "nicely placed" (see [G3], [HWW]). Following the notation of [GL], we denote by $X^\sharp$ the subspace of X^* consisting of the linear forms whose restriction to B_X is continuous in measure. By Komlos' theorem, any bounded sequence in L^1 has a subsequence whose Cesaro averages converge in measure. It follows that $X^\sharp$ is contained in $NA(X)$ and thus it is a predual of X as soon as it separates X. This result was shown in [GL] by different methods, using the Hewitt-Yoshida projection from $(L^1)^{**}$ onto L^1.

Example 6 : Let M be a compact metric space, equipped with a distance d. We choose a distinguished point 0 in M and we denote $Lip_0(M)$ the space of all Lipschitz functions on M which vanish at 0, equipped with the Lipschitz norm $\| \,.\, \|_L$ subordinated to d. This Banach space is isometric to a dual space. Its canonical predual is the norm closed linear span of the Dirac measures $\delta(x)$ when x runs into M. It is usually denoted $\mathcal{F}(M)$ (see [GK]) and called the free space over M. The subspace $lip_0(M)$ of $Lip_O(M)$ is defined as follows (see [W]) : if we let $\Delta = \{(x,x);\ x \in M\}$, then $f \in lip_0(M)$ if for all $\varepsilon > 0$, the set $\{(x,y) \in M^2 \backslash \Delta;\ |f(x) - f(y)| \geqslant \varepsilon d(x,y)\}$ is compact.

It is easily seen that if $f \in lip_0(M)$, there is $x \neq y$ in M such that $|f(x) - f(y)| = d(x,y)\|f\|_L$. In particular, f attains its norm on the linear form $(d(x,y))^{-1}(\delta(x) - \delta(y)) \in \mathcal{F}(M)$. Therefore we have $lip_0(M) \subset NA(\mathcal{F}(M))$. Now Theorem 1 implies that $lip_0(M)$ is an isometric predual of $\mathcal{F}(M)$ as soon as it separates it.

A case (which turns out to be the general case) when it happens is when $lip_0(M)$ uniformly separates M : that is, when there is some $a > 1$ such that for all $(p,q) \in M^2$, there is $f \in lip_0(M)$ such that $|f(p) - f(q)| = d(p,q)$ and $\|f\|_L \leqslant a$. It follows that for any finite subset $F \subset M$ and any $f \in Lip_0(M)$, there is $g \in lip_0(M)$ which coincide with f on F and such that $\|g\| \leqslant a\|f\|$. Indeed, we may assume $0 \in F$. For any $(p,q) \in F^2$, there exists by uniform separation some $f_{pq} \in lip_0(M)$ such that $f_{pq}(p) = f(p)$, $f_{pq}(q) = f(q)$ and $\|f_{pq}\|_L \leqslant a\|f\|_L$. It is clear that the function $g = \sup_{p \in F} \inf_{q \neq p} f_{pq}$ works. It easily follows that $lip_0(M)$ separates (and actually norms with $\lambda \geqslant a^{-1}$) the space $\mathcal{F}(M)$.

It follows now from Corollary 3 that if $lip_0(M)$ uniformly separates M for some $a > 1$, then it uniformy separates it for every $a > 1$. This intriguing fact (see [W], Cor. 3.3.5) is therefore a special case of the "norming for free" phenomenon expressed by Corollary 3. Theorem 1 asserts that under uniform separation one has $lip_0(M)^{**} = Lip_0(M)$.

Example 7 : Let U be a bounded open subset of $\mathbf{C}^n$, and v be a strictly positive continuous function on U. We define a weighted space of holomorphic functions on U as follows :

$$H_v(U) = \{f : U \to \mathbf{C} \text{ holomorphic} :\ \sup_U |vf| < \infty\}$$

The subspace $H_{v,0}(U)$ consists of all $f \in H_v(U)$ such that for any $\varepsilon > 0$, there is a compact subset $K \subset U$ such that $|f(z)v(z)| < \varepsilon$ if $z \in U \backslash K$. The space $H_v(U)$ equipped with its natural norm (the supremum norm of $|v.|$) is a dual space, since its unit ball is compact for the topology τ_K of compact convergence on U. Its predual $G_v(U)$ contains all Dirac measures and it follows that $H_{v,0}(U) \subset NA(G_v(U))$. It was shown in [BS] that if the unit ball of $H_{v,0}(U)$ is τ_K-dense in the unit ball of $H_v(U)$, then we have $H_{v,0}(U)^{**} = H_v(U)$. Theorem 1 (or Corollary 3) allows to weaken this condition : if the space $H_{v,0}(U)$ separates $G_v(U)$ then $H_{v,0}(U)^{**} = H_v(U)$ (and conversely of course). It is not known whether this separation property holds in full generality (see [BR]).

Example 8 : Let X be a separable reflexive space, and let $L(X)$ be the space of bounded linear operators from X to itself equipped with the operator norm. The space $L(X)$ is a dual space, its (unique) isometric predual is the projective tensor product $X \otimes_\pi X^*$ and the weak* topology coincide on the unit ball with the weak-operator topology. The subspace $K(X)$ of compact operators is contained in $NA(X \otimes_\pi X^*)$ since any compact operator on a reflexive space attains is norm. Hence if $K(X)$ separates $X \otimes_\pi X^*$, it follows by Theorem 1 that $K(X)^* = X \otimes_\pi X^*$ and $K(X)^{**} = L(X)$. This latter equation holds if and only if X has the compact approximation property (see [GS], Cor. 1.3). Since then the identity operator Id_X belongs to the weak* closure of the unit ball of $K(X)$, it follows that X has the metric compact approximation property.

A special case of the above is when X has the approximation property, since this property means that the canonical map $j : X \otimes_\pi X^* \to X \otimes_\varepsilon X^*$ is one-to-one (see [P], Theorem 0.3) and $X \otimes_\varepsilon X^*$ is separated by rank one operators. Hence Corollary 3 implies Grothendieck's classical result that approximation property implies metric approximation property for separable reflexive spaces.

Another application of Simons' inequality to the approximation property for non-reflexive spaces reads as follows : let X be a separable Banach space, and let (L_n) be an approximating sequence of finite rank operators, that is, $\lim_n L_n(x) = x$ for all $x \in X$. By the uniform boundedness principle, the space X has the bounded approximation property. Assume the following condition :

$(*)$ for any sequence $(\lambda_n)_{n \geqslant 1}$ of positive numbers such that $\sum_n \lambda_n = 1$, the operator $L = \sum_{n \geqslant 1} \lambda_n L_n$ attains its norm on X.

This condition $(*)$ implies via Simons inequality ([S]) applied to the boundary $B = S_X \otimes S_{X^*}$ that there is a sequence (C_k) of successive convex combinations of (L_n) such that $\lim_k \|C_k\| = 1$. In particular, the space X has the metric approximation property.

When $X = Y^*$ is isometric to a separable dual with the bounded approximation property, one can take a sequence $L_n = R_n^*$ of conjugate operators. Norm-attainment follows since then $L(B_X)$ is norm compact for any convex combination L of the L_n's, and we recover the well-known fact that a *separable* dual with the bounded approximation property has the metric approximation property ; in fact, Grothendieck showed that assuming the approximation property is already sufficient.

However nobody knows if *every* dual space with the approximation property has the metric approximation property (see [C], Pb. 3.8). An important special case of this problem is the open question : does the metric approximation property holds for any equivalent norm $\| \, . \, \|$ on $l_1(\mathbf{N})$? Condition $(*)$ could possibly help for this problem, although it seems difficult to check it without some compactness. Note that the question is open only for norms which are not dual norms, regardless of which isomorphic predual of $l_1(\mathbf{N})$ we pick.

We conclude this note with a much less ambitious problem, which amounts to ask whether the example of Proposition 4 is minimal.

Problem 9 : Let X be a Banach space which does not contain an isomorphic copy of $l_1([0,1])$. Let M be a separating separable subspace of X^* contained in $NA(X)$. Does it follow that M is an isometric predual of X ?

Note that ([BG], Lemma 2.8) provides a positive answer under a mild regularity assumption.

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