Rational curves on a general heptic fourfold

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Abstract

We show that there are no rational curves of degree d, $2 \leq d \leq 15$, on a general heptic hypersurface in $\mathbb{P}^5$.

1 Introduction

In this paper we prove the following result, which is an extension of the main result in [25]. Let the ground field be the field $\mathbb{C}$ of complex numbers.

Theorem 1.1. *A general heptic fourfold $F_7 \subset \mathbb{P}^5$ contains no rational curve of degree d, for $2 \leq d \leq 15$.*

Shin [25, Theorem 1.1.] has shown this for smooth curves for $2 \leq d \leq 11$. His methods are similar to those used in [16] for rational curves of degree at most 9 on a general quintic threefold. Our methods are those used by Cotterill [7] to show that there are only finitely many rational curves of degree 10 on a general quintic threefold in $\mathbb{P}^4$. Given a rational curve C in $\mathbb{P}^5$, we consider a general hyperplane section Γ of it and its generic initial ideal $gin(I_\Gamma)$. We can classify the possible generic initial ideals (see Lemma 2.3). Knowing the generic initial ideal $gin(I_\Gamma)$ we are able to bound the arithmetic genus $g(C)$ and $h^1(\mathcal{I}_C(7))$ (see Lemma 2.7). These bounds are sufficient to prove certain codimensional bounds that are used to prove Theorem 1.1. The number of cases we have to consider is considerably larger than in [7], however we are able to deal with each separate case more swiftly.

The following conjecture seems reasonable. It is also the expected result from naive dimension counts. For the number of lines see more below.

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Conjecture 1.2. *The only rational curves on a general heptic fourfold $F_7 \subset \mathbb{P}^5$ are 698005 lines.*

Theorem 1.1 is intended as a step on the way towards establishing this conjecture. We have not, however, through our work with the cases $d \leq 15$, been able to observe a pattern general enough to give a proof excluding curves of any degree d from lying on a fourfold as described.

We can coarsely count the number of rational curves of degree d on a general hypersurface of degree e in $\mathbb{P}^n$ as being zero, finite (and non-zero), or infinite. The following results sum up the present knowledge concerning this issue, as far as we know, and puts Theorem 1.1 in context.

Let F_e be a general hypersurface of degree e in $\mathbb{P}^n$.

1. If $e \geq 2n-2$, $n \geq 4$ or $e \geq 5$, $n = 3$, then F_e contains no rational curves of any degree.

2. If $e = 2n-3$, $n \geq 6$, then F_e contains finitely many lines and no rational curves of degree greater than one.

3. If $e = 3$ and $n = 3$, then F_3 contains twenty-seven lines and infinitely many rational curves of every degree greater than one.

4. If $e \leq 2n-4$, $n \geq 3$, then F_e contains infinitely many rational curves of every degree.

The case $e \geq 2n-1$, $n \geq 3$, was proved by Clemens [5]. The case $e \geq 2n-2$, $n \geq 4$, was proved by Voisin [26, 27]. The case $e = 2n-3$, $n \geq 6$, was proved by Pacienza [23]. For the cubic surface in $\mathbb{P}^3$ see [15, Section V.4.].

Moreover a formula for the number of lines in the $e = 2n-3$ case was found by Harris [14]. See more details below for the $n = 5$ case.

There are three cases not treated above: $e = 4$, $n = 3$; $e = 5$, $n = 4$; and $e = 7$, $n = 5$. The first of these cases is the quartic surface X_4 in $\mathbb{P}^3$. Segre [24] asserted that X_4 contains finitely many rational curves for every degree a multiple of 4 and 0 for all other degrees. It is easy to show that there are no rational curves of degrees not a multiple of 4, so the difficulty lies in the existence and finiteness of rational curves of degree a multiple of 4. In a modern setting the authors of [6] show this for degrees 4, 8 and 12. For other degrees they are able to show finiteness, but not existence. In recent years Gromov-Witten theory has been used to study rational curves on $K3$ surfaces (see for example [30]), though the problem of rational curves on X_4 remains open.

The second case, the quintic threefold $X_5 \in \mathbb{P}^4$, has been much studied (see for example [4, 10, 16, 17, 18, 22]). Clemens' conjecture states that in this case there should be a finite number of rational curves of all degrees. Wang [28, 29] has recently submitted an attempt to prove Clemens' conjecture.

The last case not treated in the above theorem is the heptic fourfold F_7 in $\mathbb{P}^5$, which is the topic of this paper. Some partial results are known. Voisin [26, 27] has shown that there are at most a finite number of rational curves of any degree, and the result by Shin ([25]) was referred to above. For $d = 1$, it is known that a general heptic fourfold $F_7 \subset \mathbb{P}^5$ contains 698005 disjoint lines all with normal bundle

$$\mathcal{N}_{L/F_7} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

This is true, since by [20, Exercise V.4.5] there exists a line on F_7, and all the lines on F_7 are disjoint with normal bundle as stated.

By [14, p. 708] the number of lines is

$$698005 = 7 \cdot 7! \cdot \sum_{0 \le k \le 3} \left(\frac{(2k)!}{k!(k+1)!} \cdot \sum_{\substack{I \subset \{1,2,3\} \\ \#I = 3-k}} \prod_{i \in I} \frac{(7-2i)^2}{i(7-i)} \right).$$

Moreover $\mathcal{N}_{L/F_7} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, so $H^0(\mathcal{N}_{L/F_7}) = 0$. Thus L has no first order deformations in F_7, implying that L occurs with multiplicity one in F_7.

Instead of using [20, Exercise V.4.5], the methods of [18, Appendix A] can be used to show that the lines have normal bundle as stated.

Here is some notation frequently used: Let C be a curve in $\mathbb{P}^n$ given by the ideal $\mathcal{I}_C = \mathcal{I}_{C,\mathbb{P}^n}$. The arithmetic genus of C is denoted by $g(C)$.

The space of morphisms $f : \mathbb{P}^1 \to \mathbb{P}^n$ of degree d is denoted $M_d(n)$. This space has dimension $(n+1)(d+1)-1$. We abuse notation and write $C \in M_d(n)$ where C is the image of f. We denote by $M_{d,i}(n)$ the locally closed subset of $M_d(n)$ corresponding to the curves C such that $h^1(\mathcal{I}_C(7)) = i$.

We use a * when we restrict ourselves to smooth unparametrized curves. For example, $M_d(n)^*$ is the open subspace of the Hilbert scheme of $\mathbb{P}^n$ parametrizing the smooth and irreducible rational curves of degree d. This space has dimension $(n+1)(d+1)-4$.

We are particularly interested in the case $n = 5$. Let $\mathbb{F} = \mathbb{P}H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(7))$ be the parameter space of hypersurfaces of degree 7 in $\mathbb{P}^5$. Then $\mathbb{F} \cong \mathbb{P}^N$, where $N = \binom{12}{7} - 1 = 791$.

The incidence scheme I_d is

$$I_d := \{(C, F_7) \in M_d(5) \times \mathbb{F} | C \subset F_7\}$$

with projections $p_M : I_d \to M_d(5)$ and $p_{\mathbb{F}} : I_d \to \mathbb{F}$. We write $I_{d,i}(5) := p_M^{-1}(M_{d,i}(5))$.

The plan for the paper is as follows: In Section 2 we describe the method for finding the possible hyperplane generic initial ideals and for determining bounds on $g(C) + h^1(\mathcal{I}_C(7))$. Section 3 contains a proof of the main result. In Section 4 we find the dimension of some subsets of I_d^*, the incidence scheme of smooth rational curves in a heptic. This section is independent of the rest of the paper, and gives an indication of some of the difficulty of using the same method as for $d \le 15$ to prove Conjecture 1.2 for higher d. Section 5 lists some of the hyperplane generic initial ideals, which are used in the proof of Theorem 1.1 in Section 3.

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a rewritten version of parts of the first author's Ph.D.-thesis at the University of Bergen.

2 Hyperplane generic initial ideals

In this section we describe hyperplane generic initial ideals. The material is mostly a shortened and generalized version of Section 1 of [7].

2.1 Monomial ideal trees

Let T be a labelled tree with a root vertex $v_\emptyset$, labelled $\emptyset$, the rest of the vertices labelled with the alphabet $\{x_0, x_1, \dots, x_n\}$, and such that if a vertex v labelled x_i is closer to the root vertex $v_\emptyset$ than a vertex w labelled x_j, then $i \leq j$. We call such a tree T a *monomial ideal tree*.

Let v be a terminating vertex, or *leaf*, of a monomial ideal tree. Then there exists a unique path from the root vertex $v_\emptyset$ to v. This path determines a sequence of labels $\emptyset, x_{i_1}, x_{i_2}, \dots, x_{i_l}$ which gives the monomial $\mathbf{x}_v = x_{i_1} x_{i_2} \cdots x_{i_l}$ of degree l. The monomial ideal tree T thus determines a monomial ideal $I(T)$ generated by the monomials $\mathbf{x}_v$ for v a leaf.

Given a monomial ideal I, the unique minimal generating set of I determines a monomial ideal tree $T(I)$ whose leaves correspond to the minimal generators of I. We choose $T(I)$ such that if $\mathbf{x}_a = x_{i_1} x_{i_2} \cdots x_{i_l}$ ($i_1 \leq \cdots \leq i_l$) and $\mathbf{x}_b = x_{j_1} x_{j_2} \cdots x_{j_l}$ ($j_1 \leq \cdots \leq j_l$) are two minimal generators with $i_1 = j_1, \dots, i_r = j_r$, then the paths from the root vertex $v_\emptyset$ that determine $\mathbf{x}_a$ and $\mathbf{x}_b$ coincide for the first r steps. $T(I)$ is then uniquely determined and the map $I \mapsto T(I)$ is injective.

Definition 2.1. Let T be a tree labelled with the alphabet $\{\emptyset, x_0, x_1, \dots, x_m\}$ and v a leaf. If v is labelled x_r, then a *Λ_n-rule applied to* v is the gluing at v, for $r \leq i \leq n$, of edges e_i terminating in vertices v_i labelled x_i. If v is labelled $\emptyset$, then a *Λ_n-rule applied to* v is the gluing at v, for $0 \leq i \leq n$, of edges e_i terminating in vertices v_i labelled x_i.

Example 2.2. Let $T(I)$ be the monomial ideal tree given by the monomial ideal I. Applying a Λ_n-rule, $n \geq l$, to the leaf v corresponding to the monomial $x_0^{j_0} x_1^{j_1} \cdots x_l^{j_l}$ gives a new monomial ideal tree $T(I)'$. The monomial ideal $I(T(I)')$ is then the monomial ideal given by the same minimal generating set as I except that the monomial $x_0^{j_0} x_1^{j_1} \cdots x_l^{j_l}$ is replaced by the monomials $x_0^{j_0} x_1^{j_1} \cdots x_l^{j_l+1}$, $x_0^{j_0} x_1^{j_1} \cdots x_l^{j_l} x_{l+1}, \dots,$ and $x_0^{j_0} x_1^{j_1} \cdots x_l^{j_l} x_n$. We can thus talk interchangeably of Λ_n-rules as operations on either trees or ideals.

2.2 Generic initial ideal

Throughout we use the reverse lexicographic order, or reflex order, for monomials.

Let $I \subset \mathbb{C}[x_0, \dots, x_n]$ be any ideal. The initial ideal $in(I)$ is the ideal generated by the leading terms of elements in I. I is *Borel fixed* if it is fixed under the action of upper triangular matrices in $PGL(n+1)$. I is Borel fixed if and only if I is generated by monomials and for every monomial $P \in I$,

$$P^* := x_i/x_j \cdot P \tag{1}$$

also belongs to I for all $x_j|P$ and $i < j$ ([9, Theorem 15.23]). A Borel fixed ideal is saturated if and only if none of its minimal generators are divisible by x_n ([11, Corollary 2.10]). For a subset $S \subset \mathbb{C}[x_0, \ldots, x_n]$ we write $B(S)$ for the smallest Borel fixed ideal containing S.

There exists a Zariski open subset $U \subset PGL(n+1)$ such that $in(g(I))$ is constant and Borel fixed for $g \in U$ ([11, Theorem 1.27]). The *generic initial ideal* (or just *gin*) $gin(I)$ is this constant monomial ideal $in(g(I))$.

The C-M regularity of I is equal to the maximal degree of a minimal generator of $gin(I)$ ([11, Theorem 2.27]).In particular I and $gin(I)$ have the same regularity.

See [11] and [9, Section 15.9] for more on generic initial ideals.

2.3 Gins of non-degenerate irreducible curves and their hyperplane sections

Let C be a non-degenerate curve in $\mathbb{P}^n$ and $\Gamma := C \cap H$ be a general hyperplane section. We can always assume that $\mathbb{P}^n$ is given by homogeneous coordinates $x_0, x_1, \ldots, x_n$ such that H is given by $x_n = 0$.

The following result characterizes possible gins of hyperplane sections of irreducible curves.

Lemma 2.3. *Let C be a non-degenerate curve in $\mathbb{P}^n$ and Γ be a general hyperplane section of this curve. The minimal generating set of $gin(\mathcal{I}_\Gamma)$ is given by applying a finite number of Λ_{n-2}-rules to the tree consisting of the lone vertex $\emptyset$. The tree corresponding to $gin(\mathcal{I}_\Gamma)$ satisfies the following properties:*

1. *The regularity of the hyperplane gin is equal to the maximal degree of a leaf.*
2. *The number of non-leaf vertices equals the degree of the curve.*

Proof. This is just [7, Lemma 1.2.1. and 1.2.2.] extended to any projective space. Note that what Cotterill calls a Λ-rule is here called a Λ_{n-2}-rule. ∎

Remark 2.4. $gin(\mathcal{I}_\Gamma)$ must contain a minimal generator of the form x_{n-2}^λ for some $\lambda > 0$. Otherwise, the vanishing locus $V(gin(\mathcal{I}_\Gamma))$ would contain the line $x_1 = \cdots = x_{n-3} = 0$, so $\dim V(gin(\mathcal{I}_\Gamma)) \geq 1 > 0 = \dim V(\mathcal{I}_\Gamma)$ which contradicts the dimension theorem [8, Theorem 9.3.11].

Remark 2.5. The regularity is of the hyperplane gin is bounded above by $\lceil (\deg(C) - 1)/(n-1) \rceil + 1$ ([2]).

Lemma 2.6. *Let C be a non-degenerate curve in $\mathbb{P}^n$ and Γ be a general hyperplane section of this curve. Then $gin(\mathcal{I}_C)$ is given by applying a finite number of Λ_{n-1}-rules to $gin(\mathcal{I}_\Gamma)$.*

Furthermore, $h^1(\mathcal{I}_C(7))$ equals the number of Λ_{n-1}-rules applied to vertices of degree eight or greater.

Proof. For the first statement see [7, p. 1840]. Note that what Cotterill calls a C-rule is here called a Λ_{n-1}-rule. The second statement is (almost) [7, Lemma 1.4.3.]. ■

Let C be a non-degenerate curve in $\mathbb{P}^n$ of genus g and Γ be a general hyperplane section of this curve. $I = gin(\mathcal{I}_\Gamma)$ is an ideal in $\mathbb{C}[x_0, \cdots, x_{n-1}]$. Let I' be the extension of I to $\mathbb{C}[x_0, \cdots, x_n]$. Then $I' = I_{C_\Gamma}$, where C_Γ is the cone with vertex $(0, \cdots, 0, 1) \in \mathbb{P}^n$ over the zero-dimensional scheme defined by the vanishing of I in the hyperplane H. Write $g_\Gamma := g(C_\Gamma)$ and $i := h^1(\mathcal{I}_C(7))$.

Lemma 2.7. *[7, Lemmas 1.5.1. and 1.5.2.] With notation as above, $g_\Gamma - g$ is the number of Λ_{n-1}-rules applied to I to give $gin(\mathcal{I}_C)$. Furthermore*

$$g + h^1(\mathcal{I}_C(7)) \leq g_\Gamma,$$

and if C is m-regular, then

$$g_\Gamma = dm + 1 - \binom{m+n}{n} + h^0(\mathcal{I}_{C_\Gamma}(m)).$$

Remark 2.8. In fact, $g + h^1(\mathcal{I}_C(7))$ is g_Γ minus the number of Λ_{n-1}-rules applied to vertices of degree less than eight when obtaining $gin(\mathcal{I}_C)$ from $gin(\mathcal{I}_\Gamma)$ (see [7, proof of Lemma 1.5.1.]).

Remark 2.9. For a curve C spanning a $\mathbb{P}^r$ inside $\mathbb{P}^5$ we have $h^1(\mathbb{P}^r, \mathcal{I}_{C,\mathbb{P}^r}(7)) = h^1(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7))$. This follows as (2-2) of [16]. Hence we may apply Lemma 2.7 with $h^1(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7))$ in the place of $h^1(\mathbb{P}^r, \mathcal{I}_{C,\mathbb{P}^r}(7))$.

3 Towards the proof of the main theorem

To a great extent the strategy consists of bounding the dimensions of the various pieces of the incidence I_d, by bounding the dimensions of pieces of $M_d(5)$, and the dimension of $p_M^{-1}(C)$, for the C in each of the pieces. Recall the definitions of these objects from the last part of Section 1.

Lemma 3.1. *Rational curves of degree d in reduced, irreducible hyperquadrics determine a locus of codimension at least $2d - 19$ in $M_d(5)$ ($d \geq 10$), $2d - 13$ in $M_d(4)$ ($d \geq 7$), and $2d - 8$ in $M_d(3)$ ($d \geq 5$).*

Proof. We argue along the lines of [7, Lemma 2.2.2.]. First of all we note that the space of rational curves of fixed degree d in a projective homogeneous space is irreducible of the expected dimension (see [19]). Every smooth hyperquadric is a homogeneous space and they are all isomorphic, so the codimension of curves lying on smooth hyperquadrics is the expected one, i.e.,

$$2d + 1 - \binom{n+2}{2} + 1$$

in $M_d(n)$.

We move on to the case where Q is a singular hyperquadric. So Q must be a cone over a quadric $\tilde{Q}$ of rank $2, \ldots, n-1$.

Let $n = 5$, $\text{rank}(Q) = 4$, and $Hom_d(\mathbb{P}^1, Y)$ be the affine parameter space of morphisms $\mathbb{P}^1 \to Y$. Projecting from the vertex of Q defines a rational map $\pi : Hom_d(\mathbb{P}^1, \mathbb{P}^5) \to Hom_d(\mathbb{P}^1, \mathbb{P}^4)$. Assume that the vertex has coordinates $(0, 0, 0, 0, 0, 1)$ and let $f : \mathbb{P}^1 \to \mathbb{P}^5$ be a morphism given by the sections $f_i \in H^0(\mathcal{O}_{\mathbb{P}^1}(\deg(f)))$ for $i = 0, \ldots, 5$. Then π is the rational map given by $(f_0, \ldots, f_5) \mapsto (f_0 \ldots, f_4)$. The affine fibre dimension of π is $d+1$ over any degree d map f whose image avoids the vertex. Such a map $f \in Hom_d(\mathbb{P}^1, Q)$ is sent by π to a map $\tilde{f} \in Hom_d(\mathbb{P}^1, \tilde{Q})$, where $\tilde{Q}$ is a smooth quadric threefold in $\mathbb{P}^4$. Using [19], the dimension of $Hom_d(\mathbb{P}^1, \tilde{Q})$ is the expected one, i.e., $(5d+5) - (2d+1) = 3d + 4$. Quadric cones determine a codimension 1 subset in the set of hyperquadrics in $\mathbb{P}^5$, i.e., quadric cones determine a set of dimension 19. The maps f whose image contains the vertex give a proper closed subset of $Hom_d(\mathbb{P}^1, Q)$. Thus we conclude that the dimension of the union of the spaces $Hom_d(\mathbb{P}^1, Q)$ for all Q is at most $(d+1) + (3d+4) + 19 = 4d + 24$. Thus this case has codimension at least $(6d+6) - (4d+24) = 2d - 18$ in $M_d(5)$.

The other cases with a singular hyperquadric are treated similarly. The rank 3 case with $n = 5$ gives codimension at least $(6d+6) - ((d+1) + (2d+3) + 17) = 3d - 16$. The rank 3 case with $n = 4$ gives codimension at least $(5d+5) - ((d+1) + (2d+3) + 13) = 2d - 12$. The rank 2 case with $n = 5$ gives codimension at least $(6d+6) - ((d+1) + (d+2) + 14) = 4d - 13$. The rank 2 case with $n = 4$ gives codimension at least $(5d+5) - ((d+1) + (d+2) + 11) = 3d - 9$. The rank 2 case with $n = 3$ gives codimension at least $(4d+4) - ((d+1) + (d+2) + 9) = 2d - 8$. (We use that the quadric hyperquadrics of rank m in $\mathbb{P}^n$ determine a codimension $\binom{n-m+1}{2}$ subset in the set of hyperquadrics in $\mathbb{P}^n$, see [1, Exercise II.A-2].) ■

Lemma 3.2. *For* $2 \leq d \leq 9$,

$$\dim I_d < N + 3 = 794.$$

Proof. Let $M_d^r(n)$ be the space of morphism $f : \mathbb{P}^1 \to \mathbb{P}^n$ of degree d such that the image C spans an r-plane. Then

$$\dim M_d^5(5) = \dim M_d(5) = 6d + 5, \tag{2}$$

$$\dim M_d^4(5) = \dim M_d(4) + \dim Gr(5, 6) = 5d + 9, \tag{3}$$

$$\dim M_d^3(5) = \dim M_d(3) + \dim Gr(4, 6) = 4d + 11, \tag{4}$$

and

$$\dim M_d^2(5) = \dim M_d(2) + \dim Gr(3, 6) = 3d + 11. \tag{5}$$

Let $C \in M_{d,i}^r(5) := M_d^r(5) \cap M_{d,i}(5)$ with arithmetic genus $g(C) = g$. Now $p_M^{-1}(C)$ is the projectivization of the kernel of $r : H^0(\mathbb{P}^5, \mathcal{O}_{C,\mathbb{P}^5}(7)) \to H^0(C, \mathcal{O}_C(7))$. We have the exact sequence

$$0 \to H^0(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7)) \to H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(7)) \\ \to H^0(C, \mathcal{O}_C(7)) \to H^1(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7)) \to 0.$$

which gives

$$\dim p_M^{-1}(C)+1 = h^0(\mathbb{P}^5,\mathcal{I}_{C,\mathbb{P}^5}(7)) \le N+1-(7d+1-g)+h^1(\mathbb{P}^5,\mathcal{I}_{C,\mathbb{P}^5}(7)). \tag{6}$$

Let

$$M_{d,I}(5)=\{C\in M_d(5)|gin(I_\Gamma)=I\}$$

and

$$I_{d,I}=\{(C,F_7)\in I_d|gin(I_\Gamma)=I\}=p_M^{-1}(M_{d,I}(5)).$$

Then, with $r(C):=\dim Span(C)$, we have

$$\dim I_{d,I}\le \max_{C\in M_{d,I}(5)}(\dim M_d^{r(C)}(5)+\dim p_M^{-1}(C)). \tag{7}$$

Here $I_{d,I}$ is empty for all but a finite number of monomial ideals I. Thus I_d is a (disjoint) union of finite number of subsets of the type $I_{d,I}$, and it is enough to show that $\dim I_{d,I}(5)<N+3$ for all I.

Notation 3.3. In the rest of this section we denote the integer $h^1(\mathbb{P}^5,\mathcal{I}_{C,\mathbb{P}^5}(7))$ by i.

By equations (2)–(7), it is thus enough to show

$$g+i<d-1 \text{ for } C\in M_d^5(5), \tag{8}$$
$$g+i<2d-5 \text{ for } C\in M_d^4(5), \tag{9}$$
$$g+i<3d-7 \text{ for } C\in M_d^3(5), \tag{10}$$

and

$$g+i<4d-7 \text{ for } C\in M_d^2(5). \tag{11}$$

If $C\in M_d^2(5)$, then $gin(\mathcal{I}_\Gamma)=(x_0^d)$. By Lemma 2.7

$$g+i\le g_\Gamma=\frac{d^2-3d}{2}+1.$$

In particular $g+i<4d-7$ for $d\le 9$.

If $C\notin M_d^2(5)$, then C has a hyperplane gin of one of the types given in Section 5 below. The tables of Section 5 give the necessary bounds for $g+i$ using Lemma 2.7. ∎

Remark 3.4. Lemma 3.2 does not hold for $d\ge 10$. Choose $C\in M_d^2(5)$ and $C\subset F_7$ with $P:=Span(C)$. We assume that P is given by $x_3=x_4=x_5=0$. Then $C\subset P\cap F_7$, so $P\subset F_7$ for $d\ge 8$. Thus F_7 is given by

$$x_5f_6(x_0,\dots,x_5)+x_4g_6(x_0,\dots,x_4)+x_3h_6(x_0,\dots,x_3)=0$$

for some degree 6 polynomials f, g and h. These can be chosen in

$$\binom{11}{5}+\binom{10}{4}+\binom{9}{3}=756$$

ways. Hence $\dim p_M^{-1}(C) = 755$ and

$$\dim I_d(5) \geq \dim M_d^2(5) + \dim p_M^{-1}(C) = 766 + 3d.$$

In particular, $\dim I_d(5) > N + 3$ for $d \geq 10$.

Proof of Theorem 1.1. By the above lemma, $\dim I_d < N + 3 = 794$ for $2 \leq d \leq 9$. Since we are considering parametrized curves all the non-empty fibres of the projection $p_{\mathbb{F}}$ have dimension at least three. In particular, $\dim p_{\mathbb{F}}(I_d) < N = \dim \mathbb{F}$ and $p_{\mathbb{F}}^{-1}(F_7) = \emptyset$ for general F_7 and $2 \leq d \leq 9$.

We can write I_d as a (disjoint) union of a finite number of subsets of the type $I_{d,I}$. If $\dim I_{d,I} < N + 3$, then $p_{\mathbb{F}}(I_{d,I})$ is not dense in $\mathbb{F}$. Thus it is enough to show that the I with $\dim I_{d,I} \geq N + 3$ does not occur for general F_7.

If $C \in M_d^2(5)$ and $C \subset F_7$ with $P := Span(C)$ and $d \geq 8$, then $P \subset F_7$ as in the above remark. Since F_7 contains a plane, F_7 cannot be a general hypersurface. In fact, by the above remark, the heptic hypersurfaces containing a plane determine a subset of codimension at least $(792 - 756 - \dim Gr(3,6)) = 27$ in $\mathbb{F}$.

For the rest of the proof we assume $C \notin M_d^2(5)$. Using Lemma 3.1 and arguing as in Lemma 3.2, we see that if C lies on a quadric, the following inequalities are enough to show $\dim I_{d,I} < N + 3$.

$$g + i < 3d - 20 \text{ for } C \in M_d^5(5) \text{ and } d \geq 10, \tag{12}$$

$$g + i < 4d - 18 \text{ for } C \in M_d^4(5) \text{ and } d \geq 7, \tag{13}$$

and

$$g + i < 5d - 15 \text{ for } C \in M_d^3(5) \text{ and } d \geq 5. \tag{14}$$

C has a hyperplane gin of one of the types given in Section 5 below. Together with Lemma 2.7, the tables of Section 5 give bounds for $g + i$. For now we exclude the case $I = (x_0^2, x_0x_1^7, x_1^8)$, for $d = 15$, and $I = (x_0^2, x_0x_1^6, x_1^8)$, for $d = 14$, which we will come back to below. Using equations (8)–(14) we see that the inequalities are satisfied in all cases for which C is contained in a quadric or C_Γ is not contained in a quadric. If C_Γ is contained in a quadric but C is not, then use Remark 2.8 to get better bounds for $g + i$.

As an example take $I = B(x_1x_3) + B(x_2^3x_3) + B(x_3^5)$, the last case listed in Table 2. If C is a curve such that $I = gin(\mathcal{I}_\Gamma)$, then $\deg C = 15$ (the number of non-leaf vertices of $T(I)$) and C spans all of $\mathbb{P}^5$. From Lemma 2.7 we get $g + i \leq g_\Gamma = 20$. Here I contains a quadratic generator, so C_Γ is contained in a quadric. If C is also contained in a quadric, equation (12) gives the necessary bound $g + i \leq 3d - 20 = 25$. If C is not contained in a quadric, then $gin(\mathcal{I}_C)$ cannot contain a quadratic generator. Since I contains seven quadratic generators this means that $gin(\mathcal{I}_C)$ is obtained from I by applying at least seven Λ_4-rules (Lemma 2.6). Thus, by Remark 2.8, $g + i \leq g_\Gamma - 7 = 13$. This clearly satisfies equation (8), $g + i < d - 1 = 14$.

For the two remaining cases $I = (x_0^2, x_0x_1^7, x_1^8)$ and $I = (x_0^2, x_0x_1^6, x_1^8)$ the above method does not give the necessary inequalities. We must therefore exclude these two cases by different methods.

We now consider the case $I = (x_0^2, x_0x_1^7, x_1^8)$ with $d = 15$. We first show that Γ lies on a rational normal curve. Let $R := k[x_0, x_1, x_2]/I_\Gamma$ and

$$a(R) := \max\{i \geq 0 | \dim(R_i) - \dim(R_i - 1) \neq 0\} - 1.$$

By [9, Theorem 15.26], with $R' := k[x_0, x_1, x_2]/gin(I_\Gamma) = k[x_0, x_1, x_2]/I$, we have $a(R) = a(R')$. Since $\dim(R'_6) = 13$ and $\dim(R'_j) = 15$, $j \geq 7$, this gives $a(R) = 6$. Thus Γ lies on a rational normal curve by [21, Lemma 2.5]. Since $\Gamma \subset \mathbb{P}^2$, we have that Γ is contained in a conic S. Here $\Gamma \subset S \cap F_7$, so S is contained in F_7 by Bezout's theorem. The $d = 2$ case of the Theorem gives that F_7 cannot be a general heptic fourfold.

For the case $I = (x_0^2, x_0x_1^6, x_1^8)$ with $d = 14$, we have $g_\Gamma = 36$. If C is contained in a quadric, then Equation (14) gives the necessary inequality. If C is not contained in a quadric, then Lemma 2.7 and Remark 2.8 gives $g + i \leq 35$. If $g + i \leq 34$, then Equation (10) holds. If $g + i = 35$, then either i is non-zero or $g = 35$. But i non-zero occurs only for C in a positive-codimensional set inside $M^3_{14}(5)$ by [3], and g positive also occurs only in such a positive-codimensional set, so, $g + i \leq 35$ is good enough. ∎

Remark 3.5. For smooth curves of degree less than ten, the Theorem can easily be proved using regularity instead. By [12, Theorem 1.1.] C is 8-regular, i.e., i. Thus $g = i = 0$ and the necessary bounds from the proof of Lemma 3.2 are obviously satisfied. It should also be possible to give an alternative proof for singular curves of degree less than ten arguing along the lines of [16].

Remark 3.6. We have seen that the method using hyperplane gins to get sufficiently low bounds on g and i to conclude that there are no rational curves of degree d on a general heptic in $\mathbb{P}^5$ does not work for high d in $M^2_d(5)$ ($d \geq 10$) and $M^3_d(5)$ ($d \geq 14$). We give an example showing that this is also the case for $d \geq 20$ for curves in $M^5_d(5)$.

Let $d = 4a$ and consider the ideal

$$I = B(x_1x_3) + (x_2^4, x_2^3x_3^{a-2}, x_2^2x_3^{a-1}, x_2x_3^a, x_3^{a+1}).$$

When $a > 3$, this ideal satisfies the properties we know a hyperplane gin of a curve of degree d have to satisfy, i.e., Lemma 2.3, Remark 2.5 and Borel fixedness (see Equation (1)). Then

$$h^0(\mathcal{I}_{C_\Gamma}) = \binom{a+4}{5} + 2\binom{a+3}{4} + 2\binom{a+2}{3} + 2\binom{a+1}{2} + \binom{a}{3} + 4$$

and

$$g_\Gamma = d(a+1) + 1 - \binom{a+6}{5} + h^0(\mathcal{I}_{C_\Gamma}) = 2a^2 - 2a - 1.$$

This gives $g + i \leq 2a^2 - 2a - 1$. In particular, since $a \geq 4$, we get $d - 2 < 2a^2 - 2a - 1$. If $a \geq 5$, even the improved bound $g + i \leq 2a^2 - 2a - 8$, which we get by assuming that C is not contained in a hyperquadric, is not good enough to give the codimension needed.

4 Subsets of I_d^*

This section is independent of the rest of our paper, and we see that the strategy of limiting the dimension of the components of the incidence I_d is difficult to apply for large d, since we can produce components with bigger dimension than $\mathbb{F}$. The explicit subsets of I_d that we study, do not dominate $\mathbb{F}$.

The following proposition is a heptic reworking of [17, Lemma 2.1]. The proof follows the proof of that result closely. Recall the notation introduced in the introduction.

Proposition 4.1. *If $d \le 112$, then $I^*_{d,0}$ is smooth, irreducible and of dimension $792 - d$. Moreover, for $d \ge 2$, its image $p_{\mathbb{F}}(I^*_{d,0})$ is not dense in $\mathbb{F}$ and its closure $\bar{I}^0_{d,0}$ is a component of I^*_d. If $d \ge 113$, then $I^0_{d,0}$ is empty.*

Proof. Fix $C \in M_d(5)$. $C \in M_{d,0}(5)$ if and only if the map

$$H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(7)) \to H^0(C, \mathcal{O}_C(7)) \tag{15}$$

is surjective. The source and target have dimensions 792 and 7d+1 respectively, for smooth C. By the maximal-rank theorem [3, Theorem 1] surjectivity holds for general $C \in M_d(5)$ when $d \le 112$. For $d = 113$ injectivity (and surjectivity) holds for general C. For such smooth C, in $M_{113,0}(5)$, the kernel $H^0(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7))$ is then empty, and so is $I_{113,0}$ then. If $d \ge 114$, then surjectivity cannot hold for smooth C, so $M^*_{d,0}(5)$ is empty, and so is $I^*_{d,0}$.

Assume $d \le 112$. Then $H^0(\mathbb{P}^5, \mathcal{I}_{C,\mathbb{P}^5}(7))$ is non-empty of dimension $791 - 7d$ for $C \in I^*_{d,0}(5)$, and $\dim I^*_{d,0}(5) = 792 - d$. Moreover $M_d(5)^*$ is smooth, irreducible and of dimension $6d + 2$ [25, Lemma 2.2]. Let $\mathbf{C}$ be the universal curve in $\mathbb{P}^5 \times M_d(5)^*$, with ideal $\mathcal{I}_{\mathbf{C}}$. Then $\mathcal{I}_{\mathbf{C}}$ is flat over $M_d(5)^*$, and $h^1(\mathcal{I}_{\mathbf{C}}(7))$ is upper semi-continuous. So $M_{d,0}(5)^*$ is open in $M_d(5)^*$. Let $C \in M_{d,0}(5)^*$, i.e., $H^1(\mathcal{I}_C(7)) = 0$. Then the direct image $\mathcal{Q}$ is locally free on $M_{d,0}(5)^*$, and its formation commutes with base change to the fibres. Hence $I^*_{d,0} = \mathbb{P}(\mathcal{Q}^* | M_{d,0}(5)^*)$. The map in equation (15) is surjective, so $h^0(\mathcal{I}_C(7)) = 791 - 7d$. Thus, for $d \le 112$, $I^*_{d,0}$ is smooth, irreducible and of dimension $(790 - 7d) + (6d + 2) = 792 - d$. Since $M_{d,0}(5)^*$ is open in $M_d(5)^*$, we see that $I^*_{d,0}$ is open in I^*_d. Hence the closure of $I^*_{d,0}$ is a component, since $I^*_{d,0}$ is non-empty and irreducible. Since $\dim I^*_{d,0} = 792 - d < 791 = \dim \mathbb{F}$, for $d \ge 2$, its image in $\mathbb{F}$ cannot be dense. ∎

The following proposition is based on [17, Lemma 3.1]. Our proof follows the proof of that result closely.

Let J^e_d be the set of pairs $(C, F_7) \in I^*_d$ such that C spans a $\mathbb{P}^3$ and lies on a smooth surface of degree e in $\mathbb{P}^3$.

Proposition 4.2. *The dimension of the above sets are as follows:*

$$\begin{aligned} \dim J^2_d &= 2d + 743 \text{ for } d \ge 14, \\ \dim J^3_d &= d + 732 \text{ for } d \ge 21, \\ \dim J^4_d &\le 733 \text{ for } d \ge 28, \\ \dim J^5_d &\le 744 \text{ for } d \ge 35, \\ \dim J^6_d &\le 766 \text{ for } d \ge 42. \end{aligned}$$

Furthermore, $p_{\mathbb{F}}(J_d^e)$ *is not dense in* $\mathbb{F}$ *for* $e \leq 6$ *and* $d \geq 7e$.

Proof. Fix $(C, F_7) \in J_d^e$. Then C lies on some smooth surface S of degree e in $\mathbb{P}^3$. If $d \geq e^2$, then S is uniquely determined. Otherwise, C would lie on (and, in fact, be equal to) the intersection of two different smooth surfaces of degree e in $\mathbb{P}^3$, and would thus have non-zero genus.

If $d \geq 7e$, then S lies in F_7. Otherwise, the intersection of S and F_7 would be a curve containing C. So C would be equal to this intersection and have non-zero genus.

Varying $(C, F_7) \in J_d^e$, we form the space $\tilde{J}_d^e$ of corresponding triples (C, S, F_7). If $e \leq 7$ and $d \geq 7e$, then, by the above, the projection $\tilde{J}_d^e \to J_d^e$ is bijective. So J_d^e and $\tilde{J}_d^e$ have the same dimension and image in $\mathbb{F}$.

We compute this dimension and bound the dimension of the image for $2 \leq e \leq 6$. The fibre of $\tilde{J}_d^e$ over a pair (S, F_7) consists of all $C \in S$. So it has dimension $2d - 1$ if $e = 2$, $d - 1$ if $e = 3$, and at most 0 if $4 \leq e \leq 6$ (see [17, proof of Lemma 3.1]).

The F_7 containing a fixed S form a space of dimension $h^0(\mathcal{I}_S(7)) - 1$. Using the exact sequence

$$0 \to \mathcal{I}_{\mathbb{P}^3} \to \mathcal{I}_S \to \mathcal{I}_{S/\mathbb{P}^3} \to 0$$

and that the third term is equal to $\mathcal{O}_{\mathbb{P}^3}(-e)$, we see that

$$\begin{aligned} h^0(\mathcal{I}_S(7)) &= h^0(\mathcal{I}_{\mathbb{P}^3}(7)) + h^0(\mathcal{O}_{\mathbb{P}^3}(7-e)) \\ &= \binom{12}{5} - \binom{10}{3} + \binom{10-e}{3} = 672 + \binom{10-e}{3}. \end{aligned}$$

The various S in a fixed $\mathbb{P}^3$ forms a space of dimension $\binom{3+e}{3} - 1$, and the various H form a $Gr(4,6)$ of dimension 8. Hence the various pairs (S, F_7) form a space of dimension

$$\begin{aligned} (672 + 56 - 1) + (10 - 1 + 8) &= 744 \text{ if } e = 2, \\ (672 + 35 - 1) + (20 - 1 + 8) &= 733 \text{ if } e = 3, \\ (672 + 20 - 1) + (35 - 1 + 8) &= 733 \text{ if } e = 4, \\ (672 + 10 - 1) + (56 - 1 + 8) &= 744 \text{ if } e = 5, \\ (672 + 4 - 1) + (84 - 1 + 8) &= 766 \text{ if } e = 6. \end{aligned}$$

These numbers are less than 791, so the image $p_{\mathbb{F}}(J_d^e)$ is not dense in $\mathbb{F}$ for $e \leq 6$ and $d \geq 7d$.

Furthermore,

$$\begin{aligned} \dim J_d^2 &= 2d + 743 \text{ for } d \geq 14, \\ \dim J_d^3 &= d + 732 \text{ for } d \geq 21, \\ \dim J_d^4 &\leq 733 \text{ for } d \geq 28, \\ \dim J_d^5 &\leq 744 \text{ for } d \geq 35, \\ \dim J_d^6 &\leq 766 \text{ for } d \geq 42. \end{aligned}$$

∎

Corollary 4.3. *If* $d \geq 24$*, then* $\dim I_d^* > \dim \mathbb{F} = 791$*.*

Proof. If $d \geq 24$, then $\dim I_d^* \geq \dim J_d^2 > 791 = \dim \mathbb{F}$. ∎

Remark 4.4. In particular, Lemma 3.2 cannot be extended to $d \geq 24$ even if we restrict ourselves to smooth curves. This does not mean that Conjecture 1.2 does not hold since neither J_d^2 nor J_d^3 are dense in $\mathbb{F}$. In Remark 3.4 we show that $\dim I_d > 791 + 3$ for $d \geq 10$ (remember that I_d takes into account the parametrization of C, while I_d^* does not).

Remark 4.5. If $d \geq 16$, then $\dim I_d^* \geq \dim J_d^2 > 792 - d$. Thus [25, Proposition 2.1] does not hold for $d \geq 16$.

5 Possible hyperplane gins for $d \leq 15$

In this section we list the possible hyperplane gins $I = gin(\mathcal{I}_\Gamma)$ for $d = 14$ when C spans a $\mathbb{P}^3$, and all possible hyperplane gins for $d = 15$. The tables are made using Lemma 2.3, Remarks 2.5 and the fact that I is Borel fixed (see Equation (1)). For $d \leq 13$ there are no problematic cases. The same is true for $d = 14$ when C spans a $\mathbb{P}^4$ or a $\mathbb{P}^5$. Similar tables for all these (unproblematic) cases with $d \leq 14$ are listed in [13] and are available from the authors upon request.

For C that spans $\mathbb{P}^3$ we are able to say a bit more. By the above listed results we are able to say that

$$gin(\mathcal{I}_\Gamma) = (x_0^k, x_0^{k-1}x_1^{\lambda_{k-1}}, \ldots, x_0x_1^{\lambda_1}, x_1^{\lambda_0})$$

for some k and invariants $\lambda_0, \ldots, \lambda_{k-1}$. Gruson and Peskine showed that the invariants satisfy

$$\lambda_i - 1 \geq \lambda_{i+1} \geq \lambda_i - 2$$

for $i = 0, 1, \ldots, k-2$ (see [11, Corollary 4.8]). This enables us to exclude some additional cases.

We write h^0 for $h^0(\mathcal{I}_{C_\Gamma}(m))$ and m for the regularity of $gin(\mathcal{I}_\Gamma)$. From these we calculate g_Γ using Lemma 2.7.

We calculate $h^0(\mathcal{I}_{C_\Gamma}(m))$ by counting the monomials of degree m in $\mathcal{I}_{C_\Gamma}$. For example, take $I = (x_0^3, x_0^2x_1^3, x_0x_1^5, x_1^6)$ with $m = 6$ (see Table 1). Then $\mathcal{I}_{C_\Gamma}$ is the extension of I to $\mathbb{C}[x_0, \ldots, x_3]$. The monomials of degree 6 are

$$x_0^2x_1^4, x_0^2x_1^3x_2, x_0^2x_1^3x_3, x_0x_1^5, x_1^6,$$

and

$$x_0^3M(x_0, x_1, x_2, x_3),$$

where $M(x_0, x_1, x_2, x_3)$ is any of the 20 monomials of degree 3 in x_0, x_1, x_2, x_3.

This gives 25 monomials of degree 6, so $h^0(\mathcal{I}_{C_\Gamma}(m)) = 25$.

The tables are sorted by the number of quadratic relations in I (Recall the notation $B(S)$ from Subsection 2.2).

Table 1: $\deg C = 14$ and C spans $\mathbb{P}^3$

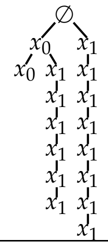
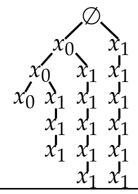
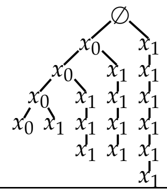
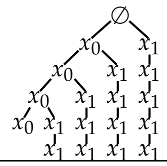
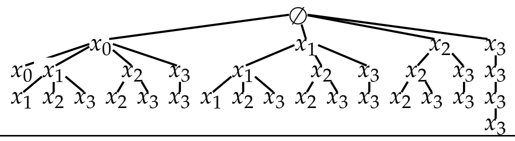
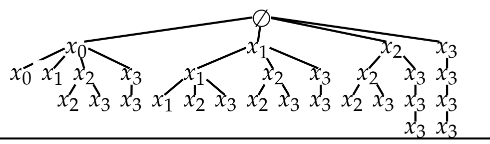
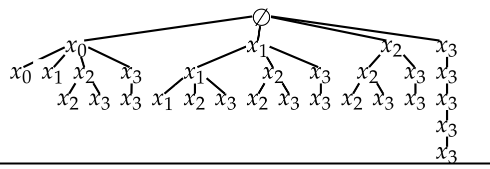
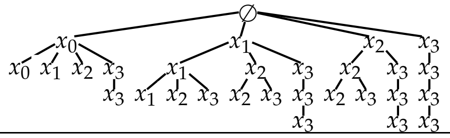
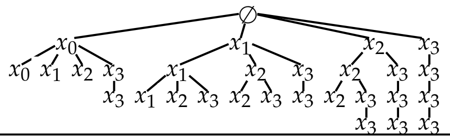
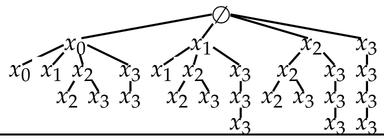
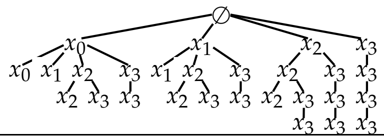
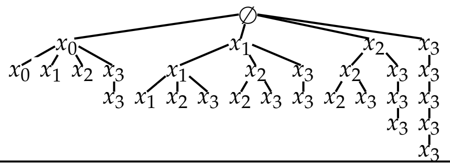
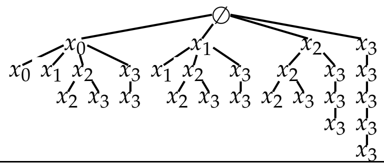
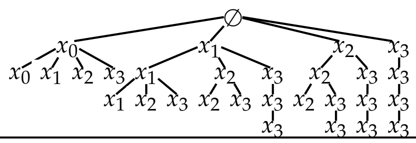
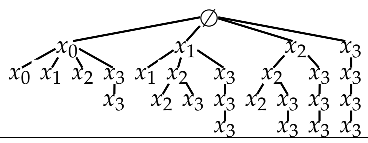
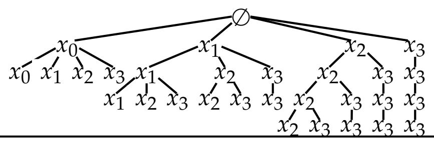
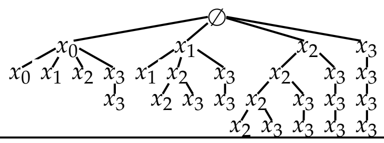
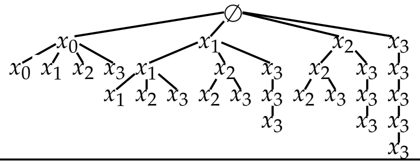
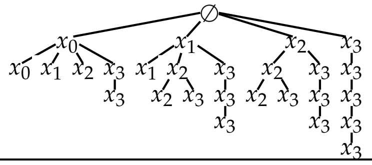
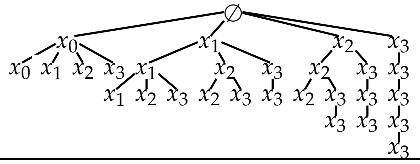
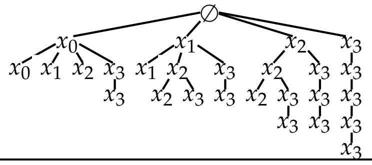
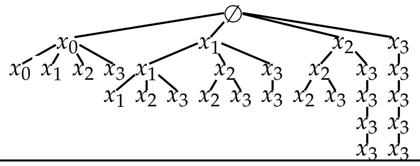
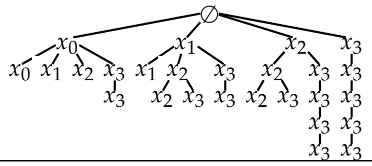
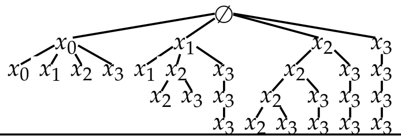
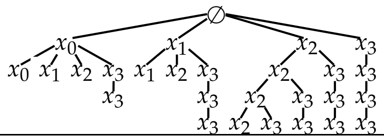
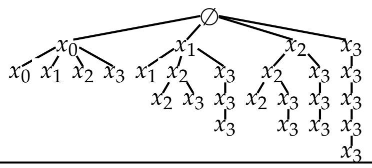
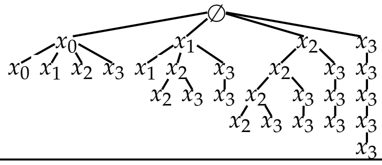
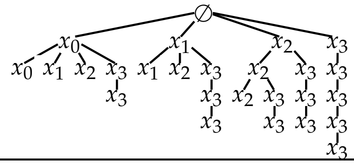
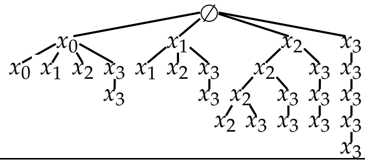
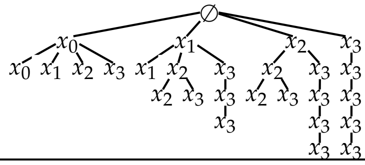
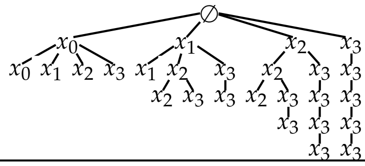
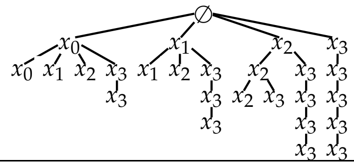
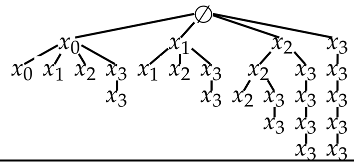
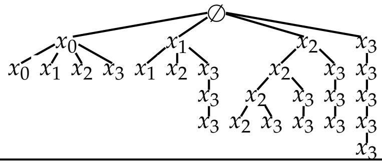
I	$T(I)$	m	h^0	g_Γ
$(x_0^2, x_0x_1^6, x_1^8)$		8	88	36
$(x_0^3, x_0^2x_1^3, x_0x_1^5, x_1^6)$		6	25	26
$(x_0^4, x_0^3x_1, x_0^2x_1^3, x_0x_1^4, x_1^6)$		6	23	24
$(x_0^4, x_0^3x_1^2, x_0^2x_1^3, x_0x_1^4, x_1^5)$		5	8	23

Table 2: $\deg C = 15$ and C spans $\mathbb{P}^5$

I	$T(I)$	m	h^0	g_Γ
$B(x_3^3)$		3	20	10
$(x_0^2, x_3^4) + B(x_2x_3^2)$		4	76	11
$B(x_0x_1) + B(x_1x_3^2, x_2^2x_3)$ $+B(x_3^4)$		4	77	12
$B(x_0x_1) + B(x_2x_3^2) + (x_3^5)$		5	189	13

I	$T(I)$	m	h^0	g_Γ
$B(x_0x_2) + B(x_2x_3^2) + B(x_3^4)$		4	78	13
$B(x_0x_2) + B(x_1x_3^2, x_2^3)$ $+B(x_3^4)$		4	78	13
$B(x_1^2) + B(x_0x_3^2, x_2^2x_3)$ $+B(x_3^4)$		4	78	13
$B(x_1^2) + B(x_1x_3^2, x_2^3)$ $+B(x_3^4)$		4	78	13
$B(x_0x_2) + B(x_1x_3^2, x_2^2x_3)$ $+(x_2x_3^3, x_3^5)$		5	190	14
$B(x_1^2) + B(x_1x_3^2, x_2^2x_3)$ $+(x_2x_3^3, x_3^5)$		5	190	14
$B(x_0x_3) + B(x_1x_2x_3, x_2^3)$ $+B(x_3^4)$		4	79	14
$B(x_0x_2, x_1^2)$ $+B(x_0x_3^2, x_1x_2x_3, x_2^3) + B(x_3^4)$		4	79	14
$B(x_0x_3) + B(x_1x_3^2) + B(x_3^4)$		4	79	14
$B(x_0x_2, x_1^2) + B(x_1x_3^2) + B(x_3^4)$		4	79	14

I	$T(I)$	m	h^0	g_Γ
$B(x_0x_3)+B(x_0x_3^2,x_2^2x_3)$ $+B(x_2x_3^3)+(x_3^5)$		5	191	15
$B(x_0x_2,x_1^2)+B(x_0x_3^2,x_2^2x_3)$ $+B(x_2x_3^3)+(x_3^5)$		5	191	15
$B(x_0x_3)+B(x_1x_3^2,x_2^3)$ $+B(x_2x_3^3)+(x_3^5)$		5	191	15
$B(x_0x_2,x_1^2)+B(x_1x_3^2,x_2^3)$ $+B(x_2x_3^3)+(x_3^5)$		5	191	15
$B(x_0x_3)+B(x_1x_3^2,x_2^2x_3)$ $+B(x_3^5)$		5	192	16
$B(x_0x_2,x_1^2)$ $+B(x_1x_3^2,x_2^2x_3)+B(x_3^5)$		5	192	16
$B(x_0x_3,x_1^2)$ $+B(x_1x_2x_3)+B(x_3^4)$		4	80	15
$B(x_1x_2)+(x_0x_3^2)+B(x_3^4)$		4	80	15
$B(x_0x_3,x_1^2)+B(x_1x_2x_3,x_2^3)$ $+B(x_2x_3^3)+(x_3^5)$		5	192	16

I	$T(I)$	m	h^0	g_Γ
$B(x_0x_3, x_1^2) + B(x_1x_3^2)$ $+B(x_2x_3^3) + (x_3^5)$		5	192	16
$B(x_1x_2) + (x_0x_3^2, x_2^3)$ $+B(x_2x_3^3) + (x_3^5)$		5	192	16
$B(x_1x_2) + B(x_1x_3^2)$ $+B(x_2x_3^3) + (x_3^5)$		5	192	16
$B(x_0x_3, x_1^2) + B(x_2^2x_3)$ $+(x_1x_3^3) + B(x_3^5)$		5	189	13
$B(x_0x_3, x_1^2) + B(x_1x_3^2, x_2^3)$ $+(x_2^2x_3^2) + B(x_3^5)$		5	189	13
$B(x_1x_2) + B(x_0x_3^2, x_2^2x_3)$ $+(x_1x_3^3) + B(x_3^5)$		5	189	13
$B(x_1x_2) + B(x_1x_3^2, x_2^3)$ $+(x_2^2x_3^2) + B(x_3^5)$		5	189	13
$B(x_0x_3, x_1x_2) + B(x_2x_3^3) + (x_3^5)$		5	193	17

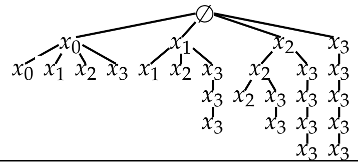
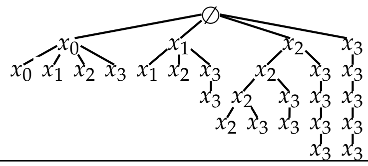
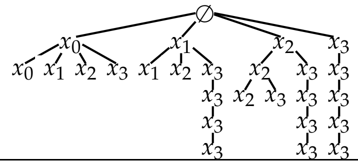
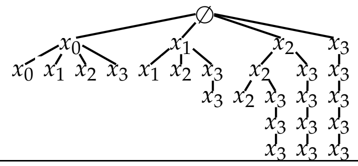
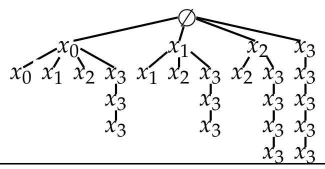
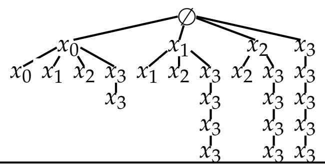
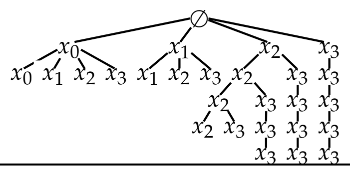
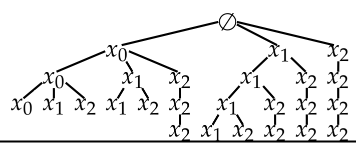
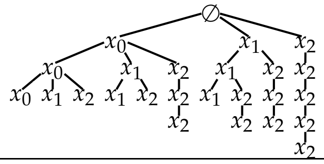
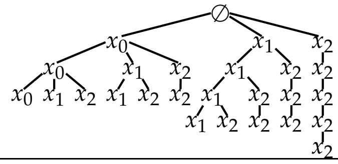
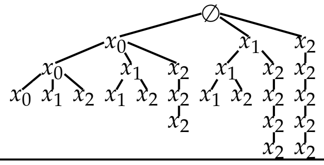
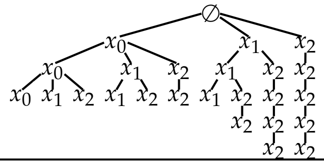
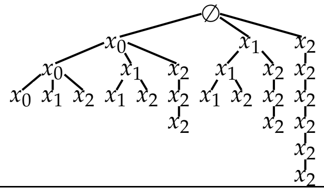
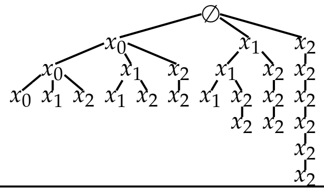
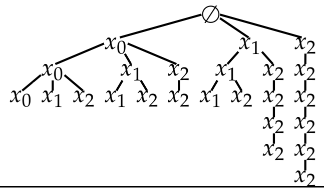
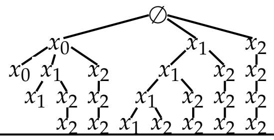
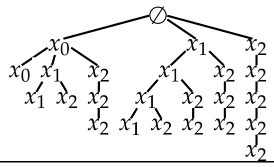
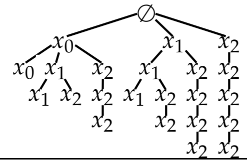
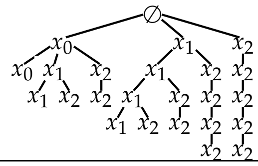
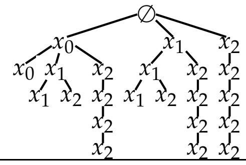
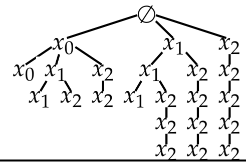
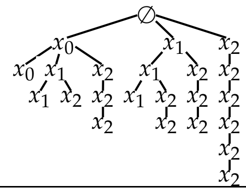
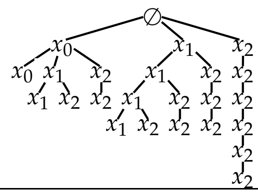
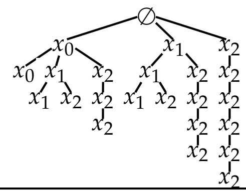
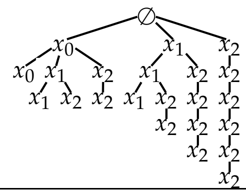
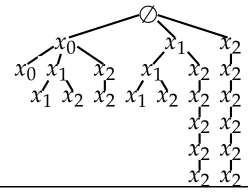
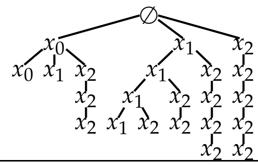
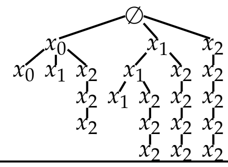
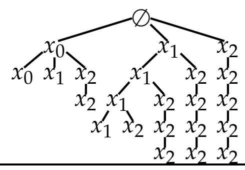
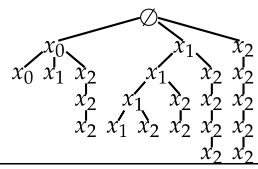
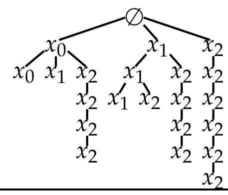
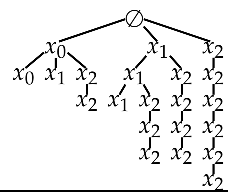
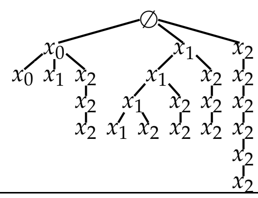
I	$T(I)$	m	h^0	g_Γ
$B(x_0x_3, x_1x_2)$ $+(x_1x_3^3, x_2^3, x_2^2x_3^2) + B(x_3^5)$		5	194	18
$B(x_0x_3, x_1x_2) + (x_1x_3^2)$ $+B(x_2^2x_3^2) + B(x_3^5)$		5	194	18
$B(x_0x_3, x_1x_2)$ $+B(x_2^2x_3) + B(x_3^5)$		5	195	19
$B(x_0x_3, x_1x_2)$ $+(x_1x_3^2, x_2^3) + B(x_3^5)$		5	195	19
$B(x_2^2) + B(x_1x_3^3) + B(x_3^5)$		5	194	18
$B(x_2^2) + (x_0x_3^2) + B(x_3^5)$		5	195	19
$B(x_1x_3) + B(x_2^3x_3) + B(x_3^5)$		5	196	20

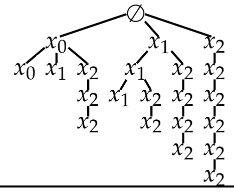
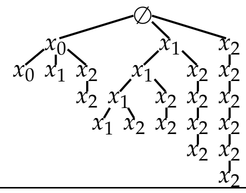
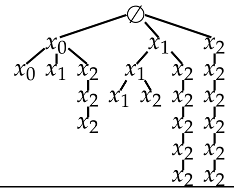
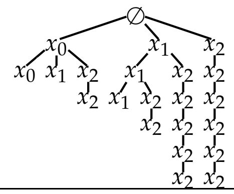
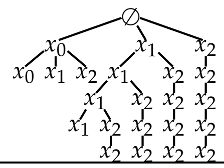
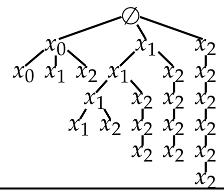
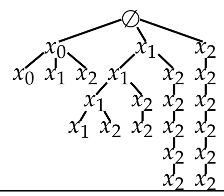
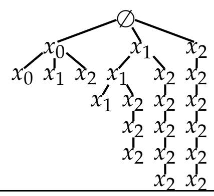
Table 3: $\deg C = 15$ and C spans $\mathbb{P}^4$

I	$T(I)$	m	h^0	g_Γ
$B(x_0x_1x_2) + B(x_2^4)$		4	25	16

I	$T(I)$	m	h^0	g_Γ
$B(x_0x_1x_2, x_1^3) + B(x_1x_2^3) + (x_2^5)$		5	67	17
$B(x_0x_2^2) + B(x_1x_2^3) + (x_2^5)$		5	67	17
$B(x_1x_2^2) + (x_0x_2^3, x_1x_2^4, x_2^5)$		5	68	18
$B(x_0x_2^2, x_1^3) + (x_1^2x_2^2, x_1x_2^4, x_2^5)$		5	68	18
$B(x_1x_2^2) + (x_0x_2^3, x_1x_2^3, x_2^6)$		6	138	19
$B(x_0x_2^2, x_1^3) + (x_1^2x_2^2, x_1x_2^3, x_2^6)$		6	138	19
$B(x_0x_2^2, x_1^2x_2) + (x_1x_2^4, x_2^6)$		6	139	20
$(x_0^2, x_0x_1^2) + B(x_2^4)$		4	26	17
$(x_0^2, x_2^5) + B(x_0x_1x_2) + B(x_1x_2^3)$		5	68	18

I	$T(I)$	m	h^0	g_Γ
$(x_0^2, x_0x_2^3, x_1^2x_2^2)$ $+B(x_0x_1x_2, x_1^3) + +B(x_2^5)$		5	69	19
$(x_0^2) + B(x_0x_2^2)$ $+B(x_1^2x_2^2) + B(x_2^5)$		5	69	19
$(x_0^2) + B(x_1^2x_2) + B(x_2^5)$		5	70	20
$(x_0^2) + B(x_0x_2^2, x_1^3) + B(x_2^5)$		5	70	20
$(x_0^2) + B(x_0x_1x_2, x_1^3) + B(x_1x_2^3) + (x_2^6)$		6	139	20
$(x_0^2) + B(x_0x_2^2)$ $+B(x_1x_2^3) + (x_2^6)$		6	139	20
$(x_0^2, x_0x_2^3, x_1x_2^4, x_2^6) + B(x_1^2x_2)$		6	140	21
$(x_0^2) + B(x_0x_2^2, x_1^3)$ $+(x_1^2x_2^2, x_1x_2^4, x_2^6)$		6	140	21

I	$T(I)$	m	h^0	g_Γ
$(x_0^2) + B(x_0x_2^2, x_1^2x_2) + B(x_2^6)$		6	142	23
$B(x_0x_1) + B(x_0x_2^3, x_1^2x_2^2) + B(x_2^5)$		5	70	20
$B(x_0x_1) + (x_0x_2^3, x_1^3) + B(x_2^5)$		5	71	21
$B(x_0x_1) + (x_0x_2^2)$ $+B(x_1^3x_2) + B(x_2^5)$		5	71	21
$B(x_0x_1) + B(x_0x_2^3, x_1^2x_2^2) + B(x_2^5)$		5	71	21
$B(x_0x_1) + B(x_1^2x_2)$ $+B(x_1x_2^4) + (x_2^6)$		6	141	22
$B(x_0x_1) + (x_0x_2^2, x_1^3, x_2^6)$ $+B(x_1x_2^4)$		6	141	22
$B(x_0x_1) + B(x_1x_2^3) + (x_2^6)$		6	140	21

I	$T(I)$	m	h^0	g_Γ
$B(x_0x_1) + (x_0x_2^3, x_1^3, x_1^2x_2^2, x_1x_2^4, x_2^6)$		6	141	22
$B(x_0x_1) + B(x_1^2x_2^2)$ $+(x_0x_2^2, x_1x_2^4, x_2^6)$		6	141	22
$B(x_0x_1) + B(x_1^2x_2)$ $+(x_0x_2^3) + B(x_2^6)$		6	143	24
$B(x_0x_1) + (x_0x_2^2, x_1^3, x_1^2x_2^2) + B(x_2^6)$		6	143	24
$B(x_0x_2) + (x_1^4) + B(x_2^5)$		5	72	22
$B(x_0x_2) + B(x_1^3x_2)$ $+B(x_1x_2^4) + (x_2^6)$		6	143	24
$B(x_0x_2) + B(x_1^2x_2^2) + B(x_2^6)$		6	144	25
$B(x_0x_2) + (x_1^3, x_1^2x_2^3) + B(x_2^6)$		6	145	26

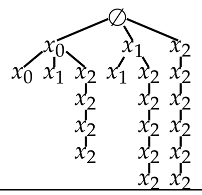
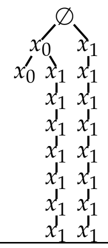
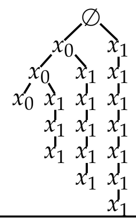
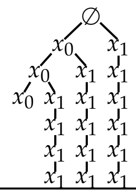
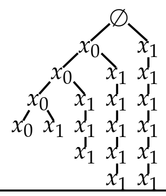
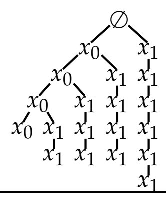
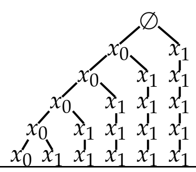
I	$T(I)$	m	h^0	g_Γ
$B(x_1^2)+(x_0x_2^3)+B(x_2^6)$		6	145	26

Table 4: $\deg C = 15$ and C spans $\mathbb{P}^3$

I	$T(I)$	m	h^0	g_Γ
$(x_0^2, x_0x_1^7, x_1^8)$		8	86	42
$(x_0^3, x_0^2x_1^3, x_0x_1^5, x_1^7)$		7	45	31
$(x_0^3, x_0^2x_1^4, x_0x_1^5, x_1^6)$		6	23	30
$(x_0^4, x_0^3x_1, x_0^2x_1^3, x_0x_1^5, x_1^6)$		6	21	28
$(x_0^4, x_0^3x_1^2, x_0^2x_1^3, x_0x_1^4, x_1^6)$		6	21	28
$B(x_1^5)$		5	6	26

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