Uniform polynomial approximation and generalized growth of entire functions on arbitrary compact sets

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Abstract

For arbitrary compact set $K \subset C$, we consider the uniform polynomial approximation error $E_n(f, K)$ of an entire function f and relate it to the generalized order, generalized lower order and generalized type of f.

1 Introduction

Varga [14] had given the characterization of growth parameters of an entire function $f(z)$ in terms of the sequence of approximation errors $E_n(f)$ taken over the interval [-1, 1]. Different workers like Batyrev [1], Reddy [9], Ibragimov and Shikhaliev [6], Giroux [4] and Vakarchuk [13] and others considered the approximation on bounded domain $K \subset C$ which does not divide the complex plane. Dovgoshei [2] considered the uniform approximation on compact subsets of the complex plane.

Let K be a compact subset of the complex plane and let $u_{1,} u_2, ..., u_n \in K$. Following [5, p.285] we put

$$\begin{aligned} V(u_1, u_2, u_3, ..., u_n) &= \prod_{k,l(k<l)}^{n} (u_k - u_l), V_n \\ &= \max\{|V(u_1, u_2, u_3, ..., u_n)| : u_k \in K, 1 \le j \le n\}. \end{aligned}$$

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If f is continuous on K then the best uniform approximation by polynomials is defined by

$$E_n = E_n(f,K) = \inf\{\max_{z\in K} |f(z) - P_n(z)| : \deg P_n \leq n\}.$$

Dovgoshei [2, Theorems 1-3] obtained the necessary and sufficient conditions for the continuous function f to be extendable to an entire function of given order and type in terms of the errors E_n and V_n. To the best of our knowledge, these characterizations have not been obtained in terms of the lower order of entire function. In this paper, we have tried to fill this gap. We have also extended the results of Dovgoshei to generalized orders which will cover the cases of entire functions of fast growth (Sato order) as well as slow growth. Following Dovgoshei [2], we give some more definitions.
Let D_R denote the disk of radius R centered at the origin and Γ_R be its boundary. Further, suppose that K is an arbitrary compact subset of the plane with card $K = \infty$. Set $d = \max\{|z| : z \in K\}$. Also, let $\mu_n(z) = z^n + a_1 z^{n-1} + ... + a_n$ denote the Chebyshev polynomial for K such that all zeros of μ_n belong to K. We set

$$m_n^* = \max\{|\mu_n(z)|, z \in K\}.$$

Then we have [5, p. 287-289],

$$m_n^* \leq \frac{V_{n+1}}{V_n} \leq (n+1)m_n^*, \tag{1}$$

$$\lim_{n\to\infty} \left(\frac{V_{n+1}}{V_n}\right)^{1/n} = \tau \tag{2}$$

where τ is the transfinite diameter of K. We also have [2, Lemma 1]

Lemma A. Let $P_n(z)$ be a polynomial of degree $\leq n$ and let $M_n = \max\{|P_n(z)| : z \in K\}$ Then for $R > d$,

$$\max_{z\in\Gamma_R} |P_n(z)| \leq R^n(n+1)M_n \left(\frac{V_n}{V_{n+1}} \frac{(1+d/R)^{n+1}}{(1-d/R)}\right). \tag{3}$$

2 Generalized order and generalized type

Let $\phi : [a,\infty) \to R$ be a real valued function such that (i) $\phi(x) > 0$, (ii) $\phi(x)$ is differentiable $\forall\, x \in [a,\infty)$, (iii) $\phi(x)$ is strictly increasing, and (iv) $\phi(x) \to \infty$ as $x \to \infty$. Further, for every real valued function $\eta(x)$ such that $\eta(x) \to 0$ as $x \to \infty$, ϕ satisfies

$$\lim_{x\to\infty} \frac{\phi\ [(1+\eta(x)\,)\ x]}{\phi(x)} = 1. \tag{4}$$

Then ϕ is said to belong to the class L^0. The function $\phi(x)$ is said to belong to the class Λ if $\phi(x) \in L^0$ and in place of (4), satisfies the stronger condition

$$\lim_{x\to\infty} \frac{\phi\,(cx)}{\phi(x)} = 1, \tag{5}$$

for all c, $0 < c < \infty$. Functions ϕ satisfying (5) are also called slowly increasing functions (see [8]).

Let $f(z)$ be an entire function, its maximum modulus function being given by $M(r,f)= \max_{|z|=r} |f(z)|$. Using the generalized functions of the class L^0 and Λ, Seremeta [10], obtained the following characterizations:

Theorem B. Let $\alpha(t) \in \Lambda$, $\beta(t) \in L^0$. Set $F(t,c) = \beta^{-1}\ [c\,\alpha(t)]$. If $d\,F(t,c)/d\ \ln t = O(1)$ as $t \to \infty$ for all c, $0 < c < \infty$, then for the entire function $f(z) = \sum_{n=0}^{\infty} c_n z^n$,

$$\limsup_{r\to\infty} \frac{\alpha\ (\ln M(r,f))}{\beta\ (\ln r)} = \limsup_{n\to\infty} \frac{\alpha\,(n)}{\beta\left(\ \ln |c_n|^{-1/n}\right)}. \tag{6}$$

Theorem C. Let $\alpha(t) \in L^0$, $\beta(t) \in L^0$, $\gamma(t) \in L^0$. Let ρ $(0 < \rho < \infty)$ be a fixed number. Set $F(t,\sigma,\rho) = \gamma^{-1}\left\{ \left[\beta^{-1}\ (\sigma\,\alpha(t))\right]^{1/\rho}\right\}$. Suppose that for all σ, $0 < \sigma < \infty$, F satisfies:

(a) If $\gamma(t) \in \Lambda$ and $\alpha(t) \in \Lambda$, then $dF(t,\sigma,\rho)/d\ln t = O(1)$ as $t \to \infty$,

(b) If $\gamma(t) \in L^0 - \Lambda$ or $\alpha(t) \in L^0 - \Lambda$, then $\lim_{t\to\infty} d\ln F(t,\sigma,\rho)/d\ln t = 1/\rho$.

Then we have

$$\limsup_{r\to\infty} \frac{\alpha\ (\ln M(r,f))}{\beta\ \left[(\gamma(r))^{\rho}\right]} = \limsup_{n\to\infty} \frac{\alpha\,(n/\rho)}{\beta\left\{\left[\gamma\left(e^{1/\rho}\,|c_n|^{-1/n}\right)\right]^{\rho}\right\}}. \tag{7}$$

Later, S.M.Shah [12] called the left hand quantity in (6) as the generalized order $\rho(\alpha,\beta,f)$ and introduced the generalized lower order $\lambda(\alpha,\beta,f)$ as

$$\lambda(\alpha,\beta,f) = \liminf_{r\to\infty} \frac{\alpha\ (\ln M(r,f))}{\beta\ (\ln r)}. \tag{8}$$

Further, Shah obtained the coefficient characterization of $\lambda(\alpha,\beta,f)$.

Theorem D [12, Theorem 2]. Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be an entire function. Set $F(t) = \beta^{-1}(\alpha(t))$. Let

(i) For some function $\psi(t)$ tending to ∞ (howsoever slowly) as $t \to \infty$,

$$\frac{\beta(t\psi(t))}{\beta(e^t)} \to 0 \text{ as } t \to \infty,$$

(ii) $\dfrac{dF(t)}{d(\log t)} = O(1)$ as $t \to \infty$,

(iii) $|c_n/c_{n+1}|$ is ultimately a non decreasing function of n.

Then

$$\lambda(\alpha,\beta,f) = \liminf_{n\to\infty} \frac{\alpha\,(n)}{\beta\left(\ \ln |c_n|^{-1/n}\right)}.$$

3 Main Results

We reiterate that the left hand expressions in (6) and (8) will be called generalized order and generalized lower order respectively. Now we prove

Theorem 1. Let $K \subseteq C$ be an arbitrary compact set with *card* $K = \infty$. Let f be an entire function. Then f has generalized order ρ if and only if

$$\limsup_{n\to\infty} \frac{\alpha\,(n)}{\beta\,[\ln\{E_n(f,K)/m^*_{n+1}\}^{-1/n}]}. = \rho. \tag{9}$$

Proof : Let $d = \max\{|z|;\, z \in K\}$. In the course of proof of Lemma 2 [2, p.923], it has been shown that for $R > d$,

$$E_n(f,K) \;\leq\; \frac{R\, m^*_{n+1}}{(R-d)^{n+2}} \frac{1}{2\pi} \int_0^{2\pi} |f(Re^{i\theta})|\, d\theta. \tag{10}$$

Hence we get,

$$E_n(f,K)/m^*_{n+1} \leq M(R)\,(R-d)^{-(n+1)}. \tag{11}$$

By the definition of ρ, we have for any given $\varepsilon > 0$ and $R > R_0(\varepsilon)$,

$$\alpha[M(R)] \;<\; \beta(\ln R)\,(\rho+\varepsilon)$$

i.e.; $E_n(f,K)/m^*_{n+1} < \quad (R-d)^{-(n+1)} \exp[\alpha^{-1}\{\overline{\rho}\,\beta(\ln R)\}],\ \overline{\rho} = \rho+\varepsilon,$
Since d is finite and fixed and the above inequality holds for all $R > R_0(\varepsilon)$, we can choose $R = R(n) = \exp\left[\beta^{-1}\{\frac{\alpha(n)}{\overline{\rho}}\}\right] = \exp\left[F(n, \frac{1}{\overline{\rho}})\right]$ where $F(x,c)$ is as defined in the statement of Theorem B. Substituting this value of R in the last inequality above, we get

$$\begin{aligned} E_n(f,K)/m^*_{n+1} \;&<\; \exp\left[-(n+1)F(n,\frac{1}{\overline{\rho}})\right] \exp\left[\alpha^{-1}\left\{\overline{\rho}\,\frac{\alpha(n)}{\overline{\rho}}\right\}\right]. \\ &<\; \exp\left[-n\left\{F(n,\frac{1}{\overline{\rho}}) - 1\right\}\right], \end{aligned}$$

since $F(n, \frac{1}{\overline{\rho}}) \;\to \infty\, for\, n \to \infty$.
Hence $\ln[\{E_n(f,K)/m^*_{n+1}\}^{-1/n}] \;>\; F(n,\frac{1}{\overline{\rho}}) - 1 = \beta^{-1}\left\{\frac{\alpha(n)}{\overline{\rho}}\right\}\left\{1 - (F(n,\frac{1}{\overline{\rho}}))^{-1}\right\}.$
i.e.; $\beta\left[\ln[\{E_n(f,K)/m^*_{n+1}\}^{-1/n}]\ \left\{1-(F(n,\frac{1}{\overline{\rho}}))^{-1}\right\}^{-1}\right] \;>\; \frac{\alpha(n)}{\overline{\rho}}.$
Since $\beta \in L^0$ and $(F(n,\frac{1}{\overline{\rho}}))^{-1} \to 0$ as $n \to \infty$, we get on proceeding to limits,

$$\limsup_{n\to\infty} \frac{\alpha\,(n)}{\beta\,[\ln\{E_n(f,K)/m^*_{n+1}\}^{-1/n}]} \leq \overline{\rho} = \rho+\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have

$$\limsup_{n\to\infty} \frac{\alpha\,(n)}{\beta\,[\ln\{E_n(f,K)/m^*_{n+1}\}^{-1/n}]} \leq\; \rho. \tag{12}$$

To prove the reverse inequality, let us put

$$\limsup_{n\to\infty} \frac{\alpha\,(n)}{\beta\,[\ln\{E_n(f,K)/m^*_{n+1}\}^{-1/n}]} = \rho'.$$

We assume that $0 \le \rho' < \infty$.. Then for a given $\varepsilon > 0$ and all $n > n_0(\varepsilon)$, we have

$$E_n(f,K) \;<\; m^*_{n+1}\,\exp\left[-n\,F(n,\frac{1}{\rho'+\varepsilon})\right].$$

We now consider the function

$$h(z) \;=\; \sum_{n=n_o}^{\infty} a^{n+1}_{n+1,R_0} z^{n+1} \exp\left[-n\,F(n,\frac{1}{\rho'+\varepsilon})\right]$$

where $R_0 > d$ and $a_{n+1,R_0} \;=\; \left[(1+\frac{d}{R_0})^{n+2}\{\frac{2(n+2)}{1-d/R_0}\}\right]^{1/(n+1)}$. Let $\{P_n(z)\}_0^\infty$ be the best approximating polynomials for the function f on K. Let D_R denote the disk of radius R centered at the origin and Γ_R be the boundary of D_R. Let

$$S(z) \;=\; \sum_{n=0}^{\infty}\{P_{n+1}(z)-P_n(z)\} \;+\; P_0(z). \tag{13}$$

In the course of proof of Theorem 1[2 , p.924], it has been shown that the series (13) is uniformly convergent on Γ_R for any arbitrary $R > 0$. Thus the sum represents an entire function. Now

$$\begin{aligned} S(z) \;&=\; \lim_{n\to\infty}\{\sum_{m=0}^{n}\{P_{m+1}(z)-P_m(z)\}\} + P_0(z). \\ &= \lim_{n\to\infty} P_{n+1}(z) = f(z). \end{aligned}$$

We also have [2 , p.924],

$$\max_{z\in\Gamma_R}|P_{n+1}(z)-P_n(z)|^{1/n} \;\le\; [R^{n+1}\,E_n(f,K)/m^*_{n+1}]^{1/n}\left[2(n+2)\frac{(1+(d/R))^{n+2}}{1-d/R}\right] \tag{14}$$

leading to the relation

$$\max_{z\in\Gamma_R}|h(z)\,| = \sum_{n=n'}^{\infty} a^{n+1}_{n+1,R_0} R^{n+1}\exp\left[-n\,F(n,\frac{1}{\rho'+\varepsilon})\right] \ge\; \max_{z\in\Gamma_R}|S(z)-P_{n'}(z)|.$$

From the last inequality, we observe that the generalized order of $h(z)$ $\ge$ the generalized order of f. If ρ_1 denotes the generalized order of $h(z)$ then by (6) we have

$$\rho_1 = \limsup_{n\to\infty} \frac{\alpha(n+1)}{\beta\left[\ln\left\{|a^{n+1}_{n+1,R_0}|\,\exp\{-nF(n,\frac{1}{\rho'+\varepsilon})\right\}^{-1/(n+1)}\right]}.$$

Now $\ln(a_{n+1.R_0}) \;=\; \frac{n+2}{n+1}\ln(1+\frac{d}{R_0}) \;+\; \frac{1}{n+1}\ln\left(\frac{2(n+2)}{1-d/R_0}\right) = O(1)$. Hence

$$-\frac{1}{n+1}\ln\left[\exp\left(-n\,F(n,\frac{1}{\rho'+\varepsilon}\right)\right] \;=\; \frac{n}{n+1}F(n,\frac{1}{\rho'+\varepsilon})\approx\beta^{-1}\left[\frac{\alpha(n)}{\rho'+\varepsilon}\right].$$

Since $\alpha(x)\in\Lambda$, we finally get

$$\rho_1 \;=\; \limsup_{n\to\infty}\frac{(\rho'+\varepsilon)\,\alpha(n+1)}{\alpha(n)} = \rho'+\varepsilon\,.$$

Since $\varepsilon>0$ was arbitrary, we get $\rho'=\rho_1\;\geq\;\rho$. Combining this with (11), we get (9).This completes the proof of Theorem 1.

Next we characterize the generalized type. We prove

Theorem 2. Let $0<\rho<\infty$ and functions $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ satisfy the conditions of Theorem C. Denote by $F(x,\sigma,\rho) \;=\; \gamma^{-1}\left\{\left[\beta^{-1}\left(\sigma\,\alpha(x)\right)\right]^{1/\rho}\right\}$. For $0<\sigma<\infty$, let us suppose that the function F satisfies the conditions

(a) If $\gamma(x)\in\Lambda$ and $\alpha(x)\in\Lambda$, then $dF(x,\sigma,\rho)/d\ln x=O(1)$ as $x\to\infty$;

(b) If $\gamma(x)\in L^0-\Lambda$ or $\alpha(x)\in L^0-\Lambda$, then $\lim\limits_{x\to\infty}\,d\ln F(x,\sigma,\rho)/d\ln x=\;1/\rho$.

Then the entire function f is of generalized type σ if and only if

$$\limsup_{n\to\infty}\frac{\alpha\left(n/\rho\right)}{\beta\left\{\left[\gamma\left(e^{1/\rho}\{E_n(f,K)/m^*_{n+1}\}^{-1/n}\right)\right]^{\rho}\right\}} \;=\; \sigma. \tag{15}$$

Proof . First we assume that f is of generalized type σ with respect to the finite number ρ i.e., $\limsup\limits_{r\to\infty}\frac{\alpha\,(\ln M(R))}{\beta\,[(\gamma(R))^{\rho}]} \;=\; \sigma$.

Let $\sigma<\infty$. Then for arbitrary $\varepsilon>0$ and $R>R'(\varepsilon)$,

$$M(R) \;<\; \exp\left[\alpha^{-1}\left\{(\sigma+\varepsilon)\beta((\gamma(R))^{\rho})\right\}\right].$$

Using (11), we get

$$E_n(f,K)/m^*_{n+1} < \;(R-d)^{-(n+1)}\exp\left[\alpha^{-1}\left\{(\sigma+\varepsilon)\beta((\gamma(R)^{\rho})\right\}\right]$$

The above inequality holds for all n and $R\;>\;R'(\varepsilon)$. Hence we can choose $R\;=\;R(n)\;=\;F(\frac{n}{\rho},\frac{1}{\sigma+\varepsilon},\rho)$ where function F is as defined above. Then for all large values of n, we have $\exp\left[\alpha^{-1}\left\{(\sigma+\varepsilon)\beta((\gamma(R)^{\rho}))\right\}\right] \;=\; \exp\left[\alpha^{-1}\left\{\alpha(\frac{n}{\rho})\right\}\right]$ and $(R-d)^{-n}\cong R^{-n} \;=\; \left\{F\left[\frac{n}{\rho},\frac{1}{\sigma+\varepsilon},\rho\right]\right\}^{-n}$.

Hence $E_n(f,K)/m^*_{n+1}<\exp(\frac{n}{\rho})\;\left\{F\left[\frac{n}{\rho},\frac{1}{\sigma+\varepsilon},\rho\right]\right\}^{-n}$,

or, $F\left[\frac{n}{\rho},\frac{1}{\sigma+\varepsilon},\rho\right]<\left\{E_n(f,K)/m^*_{n+1}\right\}^{-1/n}e^{1/\rho}$,

or $\frac{1}{\sigma+\varepsilon}\,\alpha\left(\frac{n}{\rho}\right) \;<\; \beta\left[\left\{\gamma\left(\left\{E_n(f,K)/m^*_{n+1}\right\}^{-1/n}e^{1/\rho}\right)\right\}^{\rho}\right]$.

Since the above inequality holds for all large values of n and $\varepsilon>0$ is arbitrary, we obtain on proceeding to limits,

$$\limsup_{n\to\infty}\frac{\alpha\left(n/\rho\right)}{\beta\left\{\left[\gamma\left(e^{1/\rho}\{E_n(f,K)/m^*_{n+1}\}^{-1/n}\right)\right]^{\rho}\right\}} \leq \;\sigma. \tag{16}$$

To prove the reverse inequality, we follow the method of proof of Theorem 1. Hence let

$$\limsup_{n\to\infty} \frac{\alpha\,(n/\rho)}{\beta\left\{\left[\gamma\left(e^{1/\rho}\{E_n(f,K)/m^*_{n+1}\}^{-1/n}\right)\right]^{\rho}\right\}} = \sigma_1.$$

Then for a given $\varepsilon > 0$ and all $n > N_1$,

$$E_n(f,K) < m^*_{n+1} \exp(\frac{n}{\rho}) \left\{F\left[\frac{n}{\rho}, \frac{1}{\sigma_1+\varepsilon}, \rho\right]\right\}^{-n}.$$

Now we consider the function $g(z)$ defined by the infinite series $g(z) = \sum_{n=N_1}^{\infty} a^{n+1}_{n+1,R_0} z^{n+1} \exp\left(\frac{n}{\rho}\right) \left\{F\left[\frac{n}{\rho}, \frac{1}{\sigma_1+\varepsilon}, \rho\right]\right\}^{-n} = \sum_{n=N_1}^{\infty} b_{n+1}\, z^{n+1}$, say where the sequence $\left\{a^{n+1}_{n+1,R_0}\right\}$ is as defined before. Since $F(t,\sigma,\rho) \to \infty$ as $t \to \infty$, we get

$$\limsup_{n\to\infty} \left[\mathrm{a}^{\mathrm{n+1}}_{\mathrm{n+1,R_0}} \exp(\frac{n}{\rho}) \left\{F\left[\frac{n}{\rho}, \frac{1}{\sigma_1+\varepsilon}, \rho\right]\right\}^{-n}\right]^{1/\mathrm{n}} = 0$$

and therefore $g(z)$ represents an entire function. Now

$$\begin{aligned}
\max_{z\in\Gamma_R} |g(z)| &= \sum_{n=N_1}^{\infty} a^{n+1}_{n+1,R_0} R^{n+1}\; e\left(\frac{n}{\rho}\right) \left\{F\left[\frac{n}{\rho}, \frac{1}{\sigma_1+\varepsilon}, \rho\right]\right\}^{-n} \\
&\geq \sum_{n=N_1}^{\infty} a^{n+1}_{n+1.R_0} R^{n+1} \left(E_n(f,K)/m^*_{n+1}\right) \\
&\geq \max_{z\in\Gamma_R} |S(z) - P_{N_1}|.
\end{aligned}$$

Hence if $g(z)$ is an entire function of generalized type σ' with respect to the finite number ρ then from above inequality we get $\sigma' \geq \sigma$. Now applying Theorem C to the entire function $g(z)$, we have

$$\sigma' = \limsup_{n\to\infty} \frac{\alpha\,(n/\rho)}{\beta\left\{\left[\gamma\left(e^{1/\rho}\,|b_n|^{-1/n}\right)\right]^{\rho}\right\}}. \tag{17}$$

$$\begin{aligned}
\text{Now } |b_n|^{-1/n} &\simeq (a_{n+1,R_0}) e^{-1/\rho} F\left(\frac{n}{\rho}, \frac{1}{\sigma_1+\varepsilon}, \rho\right) \\
\text{or } e^{1/\rho}\,|b_n|^{-1/n} &\simeq \left(1+\frac{d}{R_0}\right)^{(n+2)/(n+1)} \left(\frac{2(n+2)}{1-d/R_0}\right) F\left(\frac{n}{\rho}, \frac{1}{\sigma_1+\varepsilon}, \rho\right) \\
&= (1+o(1))\; F\left(\frac{n}{\rho}, \frac{1}{\sigma_1+\varepsilon}, \rho\right).
\end{aligned}$$

Since $\gamma(x) \in L^o$, $\gamma\left(e^{1/\rho}(b_n)^{-1/n}\right) \simeq \gamma\left[F\left(\frac{n}{\rho}, \frac{1}{\sigma_1+\varepsilon}, \rho\right)\right]$

$$= \beta^{-1}\left[\left(\frac{1}{\sigma_1+\varepsilon}\,\alpha\left(\frac{n}{\rho}\right)\right)^{1/\rho}\right].$$

Putting these estimates in (17) , we get $\sigma' = \sigma_1 + \varepsilon$. As stated above we have $\sigma' \geq \sigma$. Hence we get for arbitrary $\varepsilon > 0, \sigma \leq \sigma_1 + \varepsilon$, i.e. $\sigma \leq \sigma_1$. Combining this with (15), we get (16) which proves Theorem 2.

Remark: If we choose $\alpha(x) = \beta)x) = \gamma(x) = x$ in the above result, we get Theorem 3 of Dovgoshei [2].
In his paper, Dovgoshei did not consider the growth in terms of the lower order. Presently, we give a characterization of the generalized lower order as defined by (8). We prove
Theorem 3: Let $K \subseteq C$ be an arbitrary compact set with *card* $K = \infty$. Let f be an entire function. Further, suppose that the sequence

$$\left\{E_n(f,K)\, m^*_{n+2} / E_{n+1}(f,K)\, m^*_{n+1}\right\}$$

forms a non decreasing function of n. Then f has generalized lower order λ if and only if

$$\liminf_{n\to\infty} \frac{\alpha\,(n)}{\beta\,[\ln\{E_n(f,K)/m^*_{n+1}\}^{-1/n}]}. = \lambda. \tag{18}$$

Proof: From the definition of generalized lower order λ, given $\in > 0$, there exists a sequence $\{R_n\}, R_n \to \infty$ as $n \to \infty$ such that

$$M(R) < \exp[\alpha^{-1}\{(\lambda + \in)\,\beta(\log R)\}], R = R_n.$$

Using (11), and proceeding as in the proof of Theorem 1 we can easily show that

$$\liminf_{k\to\infty} \frac{\alpha(k)}{\beta\left[\frac{1}{k}\,\log(E_k(f,K)/m^*_{k+1})^{-1}\right]} \leq \lambda. \tag{19}$$

To obtain the reverse inequality we assume that the sequence

$$\left\{E_n(f,K)\, m^*_{n+2} / E_{n+1}(f,K)\, m^*_{n+1}\right\}$$

forms a non decreasing function of n. We consider the function

$$H(z) = \sum_{n=1}^{\infty} \frac{E_n(f,K)}{m^*_{n+1}} \left\{2(n+2)\frac{(1+d/R)^{n+2}}{1-d/R}\right\} z^{n+1}.$$

Since $f(z)$ is an entire function, from Theorem 1 [2], we have

$$\lim_{n\to\infty} \{E_n(f,K)\,/m^*_{n+1}\}^{1/n} = 0.$$

Since $\lim_{n\to\infty}\left\{2(n+2)\frac{(1+d/R)^{n+2}}{1-d/R}\right\}^{1/n} = O(1)$, it follows that H(z) represents an entire function of the complex variable z. Further from (13) and (14), we get $M(R,f) \leq M(R,H)$ and consequently

$$\lambda = \liminf_{R\to\infty} \frac{\alpha\ (\ln M(R,f))}{\beta\ (\ln R)} \leq \liminf_{R\to\infty} \frac{\alpha\ (\ln M(R,H))}{\beta\ (\ln R)}. \tag{20}$$

If we denote the coefficients of the Taylor series of $H(z)$ by

$$c_n = \frac{E_n(f,K)}{m^*_{n+1}} \left\{ 2(n+2)\frac{(1+d/R)^{n+2}}{1-d/R} \right\}$$

then

$$\frac{c_n}{c_{n+1}} = \frac{E_n(f,K)m^*_{n+2}}{E_{n+1}(f,K)m^*_{n+1}} \left(\frac{n+2}{n+3}\right) \left(\frac{1}{1+d/R}\right).$$

Hence, under the assumption of the theorem, $|c_n/c_{n+1}|$ forms a non decreasing function of n. Using Theorem D stated earlier, we obtain

$$\liminf_{R\to\infty} \frac{\alpha\ (\ln M(R,H))}{\beta\ (\ln R)} = \liminf_{n\to\infty} \frac{\alpha\,(n)}{\beta\left(\ \ln|c_n|^{-1/n}\right)}.$$

Now $\ln|c_n|^{-1/n} = \ln(E_n(f,K)/m^*_{n+1})^{-1/n} + O(1)$. Since $\beta \in L^o$, we finally get from (20),

$$\lambda \le \liminf_{n\to\infty} \frac{\alpha\,(n)}{\beta\left(\ \ln\left|E_n(f,K)/m^*_{n+1})\right|^{-1/n}\right)}.$$

Combining this with the inequality (19), we get (18) and proof of Theorem 3 is complete.
The generalized order and lower generalized order studied above leave an important case, that is, when $\alpha(t) = \beta(t)$. This represents the class of entire functions of slow growth and the coefficient formulae derived above are not valid in this case as the assumptions made in Theorem B and Theorem D on the functions $F(t,c)$ or $F(t)$ can not hold. To overcome this difficulty, Kapoor and Nautiyal [7] introduced a new class of functions. Thus a function $\phi(t) \in \Omega$ if $\phi(t)$ satisfies (4) and :

1. (a) There exists a function $\delta(t) \in \Lambda$ and t_0, K_1 and K_2 such that for all $t > t_0$

$$0 < K_1 \le \frac{d(\phi(t))}{d(\delta(\ln t))} \le K_2 < \infty.$$

Further a function $\phi(t) \in \overline{\Omega}$ if $\phi(t)$ satisfies (4) and

$$\lim_{t\to\infty} \frac{d(\phi(t))}{d(\ln(t))} = K,\ 0 < K < \infty. \tag{21}$$

Kapoor and Nautiyal [7, p66] showed that Ω, $\overline{\Omega} \subseteq \Lambda$ and $\Omega \bigcap \overline{\Omega} = \Phi$.
Let $\alpha(t) \in \Omega$ or $\overline{\Omega}$. Then following Kapoor & Nautiyal [7, p.66], for the entire function $f(z)$ we define the generalized order and generalized lower order λ^* as

$$\rho^* = \rho(\alpha,\alpha,f) = \lim_{r\to\infty} \sup \frac{\alpha(\ln M(r))}{\alpha(\ln r)},$$
$$\lambda^* = \lambda(\alpha,\alpha,f) = \lim_{r\to\infty} \inf \frac{\alpha(\ln M(r))}{\alpha(\ln r)}.$$

It is to be noted that if the function $\alpha(t) \in \overline{\Omega}$ then ρ^* and λ^* reduce to the ordinary case of functions of slow growth i.e. $\rho(2,2)$ and $\lambda(2,2)$, (see[11]).
Let $f(z) = \sum_{k=0}^{\infty} c_k z^k$ be an entire function. Then we have [7, Theorem 4]

$$\rho(\alpha,\alpha,f) = \begin{cases} \max\{1, L^*\} \; if\, \alpha(t) \in \Omega \\ 1 + L^* \; if\, \alpha(t) \in \overline{\Omega} \end{cases} \tag{22}$$

where $L^* = \limsup_{n\to\infty} \frac{\alpha(k)}{\alpha(\ln |c_k|^{-1/k})}$.
Further

$$\lambda(\alpha,\alpha,f) = \begin{cases} \max\{1, l^*\} \; if\, \alpha(t) \in \Omega \\ 1 + l^* \; if\, \alpha(t) \in \overline{\Omega} \end{cases} \tag{23}$$

where $l^* = \liminf_{n\to\infty} \frac{\alpha(k)}{\alpha(\ln |c_k|^{-1/k})}$
and the sequence $|c_k/c_{k+1}|$ is ultimately a non decreasing function of k.
Kapoor and Nautiyal did not consider the generalized type for functions of slow growth. Recently, Ganti and Srivastava [3] defined the type and obtained following coefficient characterization [3, Theorem 1]:
Theorem E.:Let $\alpha(x) \in \overline{\Omega}$, then the entire function $f(z)$ of generalized order $\rho, 1 < \rho < \infty$, is of generalized type τ if and only if

$$(3.17)\tau = \limsup_{r\to\infty} \frac{\alpha\ (\ln M(r,f))}{[(\alpha(\ln r))^{\rho}]} = \limsup_{n\to\infty} \frac{\alpha\,(n/\rho)}{\left\{\left[\alpha\left(\frac{\rho}{\rho-1}\ln |c_n|^{-1/n}\right)\right]^{\rho-1}\right\}}.$$

Using the estimates obtained in the course of proof of Theorems 1 to 3, the results stated in (22) , (23) and Theorem E, we can easily obtain following results for functions of slow growth. We have
Theorem 4. Let $K \subseteq C$ be an arbitrary compact set with $card\, K = \infty$. Let f be an entire function. Then f has generalized order ρ^* if and only if

$$\rho^* = \begin{cases} \max\{1, L^{**}\}\, if\, \alpha(t) \in \Omega, \\ 1 + L^{**} \; if\, \alpha(t) \in \overline{\Omega}, \end{cases} \tag{24}$$

where

$$L^{**} = \limsup_{k\to\infty} \frac{\alpha(k)}{\alpha(\ln |E_k(f,K)/m^*_{k+1}|^{-1/k})}.$$

Further, if the sequence

$$\left\{E_n(f,K)\, m^*_{n+2}/E_{n+1}(f,K)\, m^*_{n+1}\right\}$$

is ultimately a non -decreasing function of n, then the generalized lower order λ^* is given by

$$\lambda^* = \begin{cases} \max\{1, l^{**}\}\; if\, \alpha(t) \in \Omega \\ 1 + l^{**} \qquad if\, \alpha(t) \in \overline{\Omega} \end{cases} \tag{25}$$

where

$$l^{**} = \liminf_{k\to\infty} \frac{\alpha(k)}{\alpha(\ln |E_k(f,K)/m^*_{k+1}|^{-1/k})}.$$

Theorem 5. Let $K \subseteq C$ be an arbitrary compact set with $card\, K = \infty$. Let f be an entire function of generalized order ρ^* $(1 < \rho^* < \infty)$ and let $\alpha(x) \in \overline{\Omega}$. Then f is of generalized type τ if and only if

$$\tau = \limsup_{n\to\infty} \frac{\alpha\left(n/\rho^*\right)}{\left\{\left[\alpha\left(\frac{\rho^*}{\rho^*-1}\ln\left|E_n(f,K)/m^*_{n+1}\right|^{-1/n}\right)\right]^{\rho^*-1}\right\}}.$$

The proofs of above results are omitted.

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