About spaces of ω_1-ω_2-ultradifferentiable functions

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Abstract

Let Ω_1 and Ω_2 be non empty open subsets of $\mathbb{R}^r$ and $\mathbb{R}^s$ respectively and let ω_1 and ω_2 be weights. We introduce the spaces of ultradifferentiable functions $\mathcal{E}_{(\omega_1,\omega_2)}(\Omega_1 \times \Omega_2)$, $\mathcal{D}_{(\omega_1,\omega_2)}(\Omega_1 \times \Omega_2)$, $\mathcal{E}_{\{\omega_1,\omega_2\}}(\Omega_1 \times \Omega_2)$ and $\mathcal{D}_{\{\omega_1,\omega_2\}}(\Omega_1 \times \Omega_2)$, study their locally convex properties, examine the structure of their elements and also consider their links with the tensor products $\mathcal{E}_*(\Omega_1) \otimes \mathcal{E}_*(\Omega_2)$ and $\mathcal{D}_*(\Omega_1) \otimes \mathcal{D}_*(\Omega_2)$ endowed with the ε-, π- or i-topologies. This leads to kernel theorems.

1 Introduction

Spaces of ultradifferentiable functions can be defined by use of special sequences of positive numbers or by use of weights. The first point of view has been developed in [7]. In this paper, we investigate the second point of view. The results are similar but not identical. We concentrate on the differences and refer to [7] when the methods are the same.

All functions we consider are complex-valued and all vector spaces are $\mathbb{C}$-vector spaces. The euclidean norm of $x \in \mathbb{R}^n$ is designated by $|x|$. If f is a function on $A \subset \mathbb{R}^n$, we set $\|f\|_A := \sup_{x \in A} |f(x)|$.

If E is a Hausdorff locally convex topological vector space (in short: a locally convex space), then we designate by E' its topological dual endowed with the strong topology $\beta(E', E)$. If E and F are locally convex spaces, $L_b(E, F)$ designates the

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space of the continuous linear maps from E into F equipped with the bounded convergence topology. We refer to [3] and [6] for properties of the locally convex spaces.

Unless explicitely stated, r and s are positive integers; Ω_1 and Ω_2 are non empty open subsets of $\mathbb{R}^r$ and $\mathbb{R}^s$ respectively; ω_1 and ω_2 are weights (notion defined in Paragraph 2).

Definition. Let us describe the four basic spaces we deal with:
a) $\mathcal{E}_{(\omega_1,\omega_2)}(\Omega_1 \times \Omega_2)$: its elements are the $\mathcal{C}^\infty$-functions on $\Omega_1 \times \Omega_2$ such that

$$\|f\|_{H\times K,h} := \sup_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} \frac{\|\mathrm{D}^{(\alpha,\beta)} f\|_{H\times K}}{\exp\left(\varphi_1^*(h\,|\alpha|)/h + \varphi_2^*(h\,|\beta|)/h\right)} < \infty$$

for every $h > 0$ and compact subsets H of Ω_1 and K of Ω_2.
b) $\mathcal{D}_{(\omega_1,\omega_2)}(\Omega_1 \times \Omega_2)$: its elements are those of $\mathcal{E}_{(\omega_1,\omega_2)}(\Omega_1 \times \Omega_2)$ which have a compact support contained in $\Omega_1 \times \Omega_2$.
c) $\mathcal{E}_{\{\omega_1,\omega_2\}}(\Omega_1 \times \Omega_2)$: its elements are the $\mathcal{C}^\infty$-functions on $\Omega_1 \times \Omega_2$ such that, for every compact subsets H of Ω_1 and K of Ω_2, there is $h > 0$ such that $\|f\|_{H\times K,h} < \infty$.
d) $\mathcal{D}_{\{\omega_1,\omega_2\}}(\Omega_1 \times \Omega_2)$: its elements are those of $\mathcal{E}_{\{\omega_1,\omega_2\}}(\Omega_1 \times \Omega_2)$ which have a compact support contained in $\Omega_1 \times \Omega_2$.

As usual if a statement is valid for $\mathcal{E}_{(\omega_1,\omega_2)}(\Omega_1 \times \Omega_2)$ and $\mathcal{E}_{\{\omega_1,\omega_2\}}(\Omega_1 \times \Omega_2)$ [resp. $\mathcal{D}_{(\omega_1,\omega_2)}(\Omega_1 \times \Omega_2)$ and $\mathcal{D}_{\{\omega_1,\omega_2\}}(\Omega_1 \times \Omega_2)$], we simply write that it is valid for the space $\mathcal{E}_*(\Omega_1 \times \Omega_2)$ [resp. $\mathcal{D}_*(\Omega_1 \times \Omega_2)$].

In Paragraph 5, we endow these four spaces with locally convex topologies by means of the auxiliary spaces $\mathcal{E}_{(\omega_1,\omega_2),h}(H \times K)$ and $\mathcal{D}_{(\omega_1,\omega_2),h}(H \times K)$ where H and K are compact subsets of Ω_1 and Ω_2 respectively, these compact subsets being strictly regular in the case of $\mathcal{E}_{(\omega_1,\omega_2),h}(H \times K)$. We obtain that $\mathcal{E}_{(\omega_1,\omega_2)}(\Omega_1 \times \Omega_2)$ is a Fréchet nuclear space; $\mathcal{D}_{(\omega_1,\omega_2)}(\Omega_1 \times \Omega_2)$ is a (LFN)-space; $\mathcal{E}_{\{\omega_1,\omega_2\}}(\Omega_1 \times \Omega_2)$ is complete, nuclear and (by Proposition 6.4) ultrabornological; $\mathcal{D}_{\{\omega_1,\omega_2\}}(\Omega_1 \times \Omega_2)$ is a (DFN)-space.

In the paragraphs 9 and 10, different approximation and denseness properties are developed. This leads to the study of the structure of the elements of $\mathcal{E}_*(\Omega_1 \times \Omega_2)$ and $\mathcal{D}_*(\Omega_1 \times \Omega_2)$ in Paragraph 11. We then investigate tensor product descriptions of these spaces; in particular we obtain in part d) of Theorem 13.1 that the spaces $\mathcal{D}_*(\Omega_1 \times \Omega_2)$ and $\mathcal{D}_*(\Omega_1)\widehat{\otimes}_i\mathcal{D}_*(\Omega_2)$ coincide, a result leading to kernel theorems in Paragraph 13.

2 Weights

The *Young conjugate* of a function $\psi\colon [0,\infty[\to [0,\infty[$ which is convex, increasing and such that $\psi(0) = 0$ and $\lim_{y\to\infty} \psi(y)/y = \infty$, is the function $\psi^*\colon [0,\infty[\to [0,\infty[$ defined by $\psi^*(y) := \sup_{x\geq 0}(xy - \psi(x))$. It is a convex and increasing function that verifies $\psi^*(0) = 0$ and $\lim_{y\to\infty} \psi^*(y)/y = \infty$.

Let us adopt the definition of Braun, Meise and Taylor (cf. [1]) and say that a *weight* is a continuous and increasing function $\omega\colon [0,\infty[\to [0,\infty[$ identically 0 on the interval $[0, 1]$ and verifying the following four conditions:

(α) there is $M > 1$ such that $\omega(2t) \leq M(1 + \omega(t))$ for every $t \geq 0$;
(β) $\int_0^\infty \omega(t)(1 + t^2)^{-1}\, dt < \infty$;
(γ) $\lim_{t\to\infty}(\log(1 + t))/\omega(t) = 0$;
(δ) the function $\varphi : [0, \infty[\to [0, \infty[$ defined by $\varphi(t) = \omega(e^t)$ is convex. So it has a meaning to speak about the *Young conjugate φ^* associated to ω.*

Lemma 2.1. *If ω is a weight,*
a) $\varphi(t + 1) \leq M(M + 1)(1 + \varphi(t))$ *for every $t \geq 0$;*
b) *for every $b \geq M(M + 1)$ and $h > 0$, there is $a_0 > 0$ such that*

$$a + \varphi^*(ah)/h \leq 1/h + \varphi^*(abh)/(bh), \quad \forall a \geq a_0/(bh). \tag{1}$$

Proof. a) It suffices to note that, for every $t \geq 0$, we successively have $\varphi(t + 1) \leq \omega(4e^t) \leq M(1 + M(1 + \omega(e^t))) \leq M(M + 1)(1 + \varphi(t))$.

b) So, by use of the Lemma 1.4 of [1], there is a positive number y_0 such that $\varphi^*(y) - y \geq b\varphi^*(y/b) - b$ for every $y \geq y_0$. Hence the conclusion by setting $y_0 = a_0$, replacing y by abh and dividing both members by bh. ■

In the proof of Lemma 2.2, we use the following information. Let the function $w\colon [0, +\infty[\to [0, +\infty[$ be defined by $w(t) = 0$ if $t \in [0, 1]$ and $w(t) = t - 1$ if $t \in]1, +\infty[$. Then we have $\phi(t) := w(e^t) = e^t - 1$ for every $t \in [0, \infty[$ and the function $\phi^*\colon [0, \infty[\to [0, \infty[$ defined by $\phi^*(y) := \sup_{x\geq 0}(xy-\phi(x))$ is explicitely given by $\phi^*(y) = 0$ if $y \in [0, 1]$ and $\phi^*(y) = y \log(y)-y+1$ if $y \in]1, +\infty[$. Given a weight ω, we have $\omega(t)/t \to 0$ if $t \to \infty$ hence there is $B > 1$ such that $\omega(t) \leq Bt \leq B(w(t)+1)$ for every $t \in [0, +\infty[$ hence $xy-\varphi(x) \geq B(xy/B-\phi(x))-B$ for every $x, y \in [0, +\infty[$. This leads to: *for every weight ω, there is $B > 1$ such that*

$$B\phi^*(y/B) - B \leq \varphi^*(y), \quad \forall y \in [0, +\infty[. \tag{2}$$

Lemma 2.2. *For every weight ω, there is $B > 1$ such that*

$$\alpha!(h/(4B))^{|\alpha|} \leq \exp(\varphi^*(h\,|\alpha|)/h + B/h) \tag{3}$$

for every $h > 0$, $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$.

Proof. Let $B > 1$ verify the inequality (2). Given $h > 0$, $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$, we clearly have

$$\alpha!(h/(4B))^{|\alpha|} \leq |\alpha|^{|\alpha|}\,(h/(4B))^{|\alpha|} = \exp(|\alpha| \log(h\,|\alpha|\,/(4B))).$$

As we also have

$$|\alpha| \log\Big(\frac{h\,|\alpha|}{4B}\Big) \leq 0 \leq \frac{B}{h}\phi^*\Big(\frac{h\,|\alpha|}{B}\Big)$$

if $h\,|\alpha|\,/(4B) \leq 1$ and

$$|\alpha| \log\Big(\frac{h\,|\alpha|}{4B}\Big) \leq \frac{B}{h}\Big(\frac{h\,|\alpha|}{B}\log\Big(\frac{h\,|\alpha|}{eB}\Big) + 1\Big) = \frac{B}{h}\phi^*\Big(\frac{h\,|\alpha|}{B}\Big)$$

if $h\,|\alpha|\,/(4B) > 1$, we conclude at once by use of the inequality (2). ■

Notation. From now on, unless explicitely stated,
$M > 1$ is fixed so that ω_1 and ω_2 verify condition (α);
b is an integer such that $b \geq M(M+1)$.

Therefore there are $a_0 > 0$ and $B > 1$ such that the inequalities (1) and (3) are valid for $\omega = \omega_1$ and $\omega = \omega_2$. There also is $C > 0$ such that

$$a + \varphi_j^*(ah)/h \leq C + 1/h + \varphi_j^*(abh)/(bh), \quad \forall a \in \mathbb{N}_0, j \in \{1, 2\}. \tag{4}$$

3 The auxiliary space $\mathcal{E}_p(K)$

Definition. A compact subset of $\mathbb{R}^n$ is *strictly regular* if it has a finite number of connected components and if each of these connected components B verifies the following two properties:
a) B is *regular*, i.e. $B = B^{\circ -}$;
b) there is a constant $C > 0$ such that, for every x, $y \in B^\circ$, there is a polygonal path joining x to y in B°, of length $L \leq C\,|x - y|$.

It is immediate that a finite union $\cup_{j=1}^p B_j$ of closed balls in $\mathbb{R}^n$ is a strictly regular compact set if, whenever B_j meets B_k, $B_j \cap B_k$ has non empty interior. Therefore every non empty open subset of $\mathbb{R}^n$ has a cover $(K_n)_{n\in\mathbb{N}}$ by means of a sequence of strictly regular compact subsets such that $K_n \subset K_{n+1}^\circ$ for every $n \in \mathbb{N}$. Let us also remark that, if the compact subsets K of $\mathbb{R}^r$ and K' of $\mathbb{R}^s$ are strictly regular, then $K \times K'$ is a strictly regular compact subset of $\mathbb{R}^{r+s}$.

Notation. Let K be a strictly regular compact subset of $\mathbb{R}^n$ and let f be a function defined on K°. If, for some $\alpha \in \mathbb{N}_0^n$, the derivative $\mathrm{D}^\alpha f$ exists on K° and has a continuous extension on K, $\mathrm{D}^\alpha f$ will also designate this extension.

Definition. a) The notation $\mathcal{E}_p(K)$ requires that p is a non negative integer and that K is a strictly regular compact subset of some euclidean space $\mathbb{R}^n$. It designates the following Banach space: its elements are those of $\mathcal{C}^p(K^\circ)$ whose derivatives of order $\leq p$ have a continuous extension on K and its norm is $|\cdot|_{(K,p)}$ defined by

$$|f|_{(K,p)} := \sup_{|\alpha| \leq p} \|\mathrm{D}^\alpha f\|_K\,, \quad \forall f \in \mathcal{E}_p(K).$$

b) The notation $\mathcal{D}_p(K)$ requires that p is a non negative integer and that K is a compact subset of some euclidean space $\mathbb{R}^n$. It designates as usual the Banach space of the $\mathcal{C}^\infty$-functions on $\mathbb{R}^n$ with support contained in K and is equiped with the norm $|\cdot|_{(K,p)}$.

Construction. Let K be a strictly regular compact subset of $\mathbb{R}^n$ and let us proceed as in ([4], p. 42). We first choose $l > 0$ so that K is contained in the interior of $L = [-l, l]^n$. Next we apply successively results of [9] and [10] and obtain a continuous linear extension map $E\colon \mathcal{E}_{n+1}(K) \to \mathcal{D}_{n+1}(\pi L)$, i.e. a map such that $(Ef)|_K = f$ for every $f \in \mathcal{E}_{n+1}(K)$.

For every $m \in \mathbb{Z}^n$, we introduce the linear functional u_m on $\mathcal{D}_{n+1}(\pi L)$ by

$$\langle u_m, f\rangle := \int_{\pi L} f(y) e^{-i\sum_{k=1}^n m_k y_k / l}\, dy, \quad \forall f \in \mathcal{D}_{n+1}(\pi L).$$

If $m = 0$, we have $|\langle u_m, f\rangle| \leq (2\pi l)^n |f|_{(\pi L,n+1)}$. If $m \neq 0$, we proceed as follows: we choose $j \in \{1, \dots, n\}$ such that $|m_j| \geq |m_k|$ for every $k = 1, \dots, n$ and note that this implies $|m_j| \geq (1 + |m|)/(1 + n)$. Integrating $n + 1$ times by parts with respect to y_j leads directly to the existence of some $C > 1$ such that

$$|\langle u_m, f\rangle| \leq C(1 + |m|)^{-n-1} |f|_{(\pi L,n+1)}, \quad \forall f \in \mathcal{D}_{n+1}(\pi L), m \in \mathbb{Z}^n.$$

Therefore for every $m \in \mathbb{Z}^n$, $w_m := (2\pi l)^{-n} u_m \circ E$ is a continuous linear functional on $\mathcal{E}_{n+1}(K)$, of norm $|w_m|_{(K,n+1)}$ such that

$$|w_m|_{(K,n+1)} \leq C(2\pi l)^{-n}(1 + |m|)^{-n-1} \|E\| .$$

So, if we enumerate the set $\{w_m : m \in \mathbb{Z}^n\}$ as a sequence $(v_j)_{j\in\mathbb{N}}$, we have obtained the following information: *there is a sequence $(v_j)_{j\in\mathbb{N}}$ in $\mathcal{E}_{n+1}(K)'$ such that*

$$\sum_{j=1}^{\infty} |v_j|_{(K,n+1)} < \infty \tag{5}$$

and

$$|g(x)| \leq \sum_{j=1}^{\infty} |\langle v_j, g\rangle| , \quad \forall g \in \mathcal{E}_{n+1}(K), x \in K. \tag{6}$$

4 The auxiliary spaces $\mathcal{E}_{(\omega_1,\omega_2),h}(H \times K)$ and $\mathcal{D}_{(\omega_1,\omega_2),h}(H \times K)$

Definition. a) The notation $\mathcal{D}_{(\omega_1,\omega_2),h}(H \times K)$ requires that H and K are compact subsets of $\mathbb{R}^r$ and $\mathbb{R}^s$ respectively and that h is a positive number. It designates the vector space of the $\mathcal{C}^\infty$-functions f on $\mathbb{R}^r \times \mathbb{R}^s$ with compact support contained in $H \times K$ and such that $\|f\|_{H\times K,h} < \infty$, endowed with the norm $\|\cdot\|_{H\times K,h}$. It is a Banach space.

b) The notation $\mathcal{E}_{(\omega_1,\omega_2),h}(H \times K)$ requires that H and K are strictly regular compact subsets of $\mathbb{R}^r$ and $\mathbb{R}^s$ respectively and that h is a positive number. It designates the vector space of the $\mathcal{C}^\infty$-functions f on $H^\circ \times K^\circ$, the derivatives of which all have a continuous extension on $H \times K$ and such that $\|f\|_{H\times K,h} < \infty$, endowed with the norm $\|\cdot\|_{H\times K,h}$. It is a Banach space.

Proposition 4.1. *The map*

$$\Lambda_h : \mathcal{E}_{(\omega_1,\omega_2),h}(H \times K) \times \mathcal{E}_{(\omega_1,\omega_2),h}(H \times K) \to \mathcal{E}_{(\omega_1,\omega_2),bh}(H \times K)$$

(with b as in the Notation following Lemma 2.2) defined by $\Lambda_h(f, g) = fg$ is well defined, continuous and bilinear.

Proof. Given f, $g \in \mathcal{E}_{(\omega_1,\omega_2),h}(H \times K)$, $(\alpha, \beta) \in \mathbb{N}_0^r \times \mathbb{N}_0^s$ and $(x, y) \in \mathbb{R}^r \times \mathbb{R}^s$, let us evaluate $|\mathrm{D}^{(\alpha,\beta)}(fg)(x, y)|$ as follows. We use the Leibniz formula, we majorize the absolute value of the derivatives of f and g by means of $\|f\|_{H\times K,h}$ and $\|g\|_{H\times K,h}$ respectively, we group the exponentials in φ_1^* and φ_2^* separately, we use the properties of φ_1^* and φ_2^* as well as the inequalities $2^a \leq e^a$ for every $a \in \mathbb{N}$ and the inequalities (4).This procedure leads to

$$\left|\mathrm{D}^{(\alpha,\beta)}(fg)(x, y)\right| e^{-\varphi_1^*(bh|\alpha|)/(bh)-\varphi_2^*(bh|\beta|)/(bh)} \leq e^{2C+2/h} \|f\|_{H\times K,h} \|g\|_{H\times K,h}$$

and permits to conclude at once. ■

Proposition 4.2. *If h, $k > 0$ verify $2bh < k$, then the canonical injection*

$$J\colon \mathcal{E}_{(\omega_1,\omega_2),h}(H \times K) \to \mathcal{E}_{(\omega_1,\omega_2),k}(H \times K)$$

is a well defined quasi-nuclear linear map.

Proof. It is immediate that J is a well defined continuous linear map.

For the sake of clear notations, let us write $\|.\|$ for the norm in $\mathcal{E}_{(\omega_1,\omega_2),h}(H \times K)'$ and $|.|$ for the norm in $\mathcal{E}_{(\omega_1,\omega_2),k}(H \times K)'$.

The construction made in Paragraph 3 provides a sequence $(v_j)_{j\in\mathbb{N}}$ in the space $\mathcal{E}_{r+s+1}(H \times K)'$ such that $\sum_{j=1}^{\infty} |v_j|_{(H\times K,r+s+1)} < \infty$ and

$$\|g\|_{H\times K} \le \sum_{j=1}^{\infty} |\langle v_j, g\rangle|\,, \quad \forall g \in \mathcal{E}_{r+s+1}(H \times K). \tag{7}$$

For every $j \in \mathbb{N}$ and $(\alpha, \beta) \in \mathbb{N}_0^r \times \mathbb{N}_0^s$, let us define the continuous linear functional $u_{(\alpha,\beta),j}$ on $\mathcal{E}_{(\omega_1,\omega_2),k}(H \times K)$ by

$$\langle u_{(\alpha,\beta),j}, f\rangle := \langle v_j, \mathrm{D}^{(\alpha,\beta)} f\rangle \exp(-\varphi_1^*(k\,|\alpha|)/k - \varphi_2^*(k\,|\beta|)/k).$$

Then developing the functionals and using the inequality (7) provides

$$\|f\|_{H\times K,k} \le \sum_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} \sum_{j\in\mathbb{N}} \left|\langle u_{(\alpha,\beta),j}, f\rangle\right|, \quad \forall f \in \mathcal{E}_{(\omega_1,\omega_2),k}(H \times K).$$

Therefore, as every $u_{(\alpha,\beta),j}$ also belongs to $\mathcal{E}_{(\omega_1,\omega_2),h}(H \times K)'$, to conclude we just have to prove that we also have $\sum_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} \sum_{j\in\mathbb{N}} \left\|u_{(\alpha,\beta),j}\right\| < \infty$.

For every $(\alpha, \beta) \in \mathbb{N}_0^r \times \mathbb{N}_0^s$ and $j \in \mathbb{N}$, let us evaluate $\left\|u_{(\alpha,\beta),j}\right\|$. For this purpose, let f be any element of $\mathcal{E}_{(\omega_1,\omega_2),h}(H \times K)$. As f belongs to $\mathcal{E}_{(\omega_1,\omega_2),k}(H \times K)$, we have

$$\left|\langle u_{(\alpha,\beta),j}, f\rangle\right| \le |v_j|\, |\mathrm{D}^{(\alpha,\beta)} f|_{(H\times K,r+s+1)} e^{-\varphi_1^*(k|\alpha|)/k-\varphi_2^*(k|\beta|)/k}$$

with

$$|\mathrm{D}^{(\alpha,\beta)} f|_{(H\times K,r+s+1)} \le \|f\|_{H\times K,h} \sup_{|(\gamma,\delta)|\le r+s+1} e^{\varphi_1^*(h|\alpha+\gamma|)/h+\varphi_2^*(h|\beta+\delta|)/h}.$$

Now we note that for $j \in \{1, 2\}$ and p, $q \in \mathbb{N}_0$ such that $q \le r+s+1$, the properties of φ_j^* provide

$$\varphi_j^*(h(p + q)) \le \varphi_j^*(2hp)/2 + \varphi_j^*(2h(r + s + 1))/2.$$

So, if we set $A(h) := \exp(\varphi_1^*(2h(r + s + 1))/(2h) + \varphi_2^*(2h(r + s + 1))/(2h))$, we end up with

$$\left\|u_{(\alpha,\beta),j}\right\| \le A(h)\,|v_j| \exp\Big(\frac{\varphi_1^*(2h\,|\alpha|)}{2h} - \frac{\varphi_1^*(k\,|\alpha|)}{k} + \frac{\varphi_2^*(2h\,|\beta|)}{2h} - \frac{\varphi_2^*(k\,|\beta|)}{k}\Big).$$

Now we note that part b) of Lemma 2.1 provides

$$\frac{1}{2h}\varphi_j^*(2ah) \le \frac{1}{2bh}\varphi_j^*(2abh) + \frac{1}{2h} - a \le \frac{1}{k}\varphi_j^*(ak) + \frac{1}{2h} - a$$

for every $j \in \{1,2\}$ and $a \geq a_0/(2bh)$. So, setting $d := a_0/(2bh)$ and

$$B(h,k,d) := \sup_{|\alpha| \leq d} \exp(\varphi_1^*(2h\,|\alpha|)/(2h) - \varphi_1^*(k\,|\alpha|)/k),$$

we get

$$\sum_{\substack{|\alpha|\leq d \\ |\beta|\geq d}} \sum_{j\in\mathbb{N}} \left\| u_{(\alpha,\beta),j} \right\| \leq A(h)B(h,k,d)e^{1/(2h)} \sum_{j\in\mathbb{N}} |v_j| \sum_{\substack{|\alpha|\leq d \\ |\beta|\geq d}} e^{-|\beta|} < \infty$$

and similarly $\sum_{|\alpha|\geq d} \sum_{|\beta|\leq d} \sum_{j\in\mathbb{N}} \left\| u_{(\alpha,\beta),j} \right\| < \infty$.

Hence the conclusion since we also have

$$\sum_{\substack{|\alpha|\geq d \\ |\beta|\geq d}} \left\| u_{(\alpha,\beta),j} \right\| \leq A(h)e^{1/h} \sum_{j\in\mathbb{N}} |v_j| \sum_{\substack{|\alpha|\geq d \\ |\beta|\geq d}} e^{-|\alpha|-|\beta|} < \infty. \qquad \blacksquare$$

As the same proof establishes that *if h, $k > 0$ verify $2bh < k$, then the canonical injection from $\mathcal{D}_{(\omega_1,\omega_2),h}(H \times K)$ into $\mathcal{D}_{(\omega_1,\omega_2),k}(H \times K)$ is a well defined quasi-nuclear map*, we get the following result (cf. [5]).

Proposition 4.3. *If h, $k > 0$ verify $4b^2h < k$, the canonical injections*

$$J\colon \mathcal{E}_{(\omega_1,\omega_2),h}(H \times K) \to \mathcal{E}_{(\omega_1,\omega_2),k}(H \times K)$$
$$J\colon \mathcal{D}_{(\omega_1,\omega_2),h}(H \times K) \to \mathcal{D}_{(\omega_1,\omega_2),k}(H \times K)$$

are well defined nuclear linear maps. ■

5 The spaces $\mathcal{E}_*(\Omega_1 \times \Omega_2)$ and $\mathcal{D}_*(\Omega_1 \times \Omega_2)$

Definition. a) The notation $\mathcal{D}_{(\omega_1,\omega_2)}(H \times K)$ requires that H and K are compact subsets of $\mathbb{R}^r$ and $\mathbb{R}^s$ respectively. It is defined by

$$\mathcal{D}_{(\omega_1,\omega_2)}(H \times K) := \varprojlim_{m\in\mathbb{N}} \mathcal{D}_{(\omega_1,\omega_2),1/m}(H \times K).$$

b) The notation $\mathcal{E}_{(\omega_1,\omega_2)}(H \times K)$ requires that H and K are strictly regular compact subsets of $\mathbb{R}^r$ and $\mathbb{R}^s$ respectively. It is defined by

$$\mathcal{E}_{(\omega_1,\omega_2)}(H \times K) := \varprojlim_{m\in\mathbb{N}} \mathcal{E}_{(\omega_1,\omega_2),1/m}(H \times K).$$

By the results of Paragraph 4, $\mathcal{D}_{(\omega_1,\omega_2)}(H \times K)$ and $\mathcal{E}_{(\omega_1,\omega_2)}(H \times K)$ are Fréchet nuclear spaces and, if H and K are strictly regular, $\mathcal{D}_{(\omega_1,\omega_2)}(H \times K)$ is a closed subspace of $\mathcal{E}_{(\omega_1,\omega_2)}(H \times K)$.

Definition. Under analogous restrictions on H and K, we also introduce the locally convex spaces

$$\mathcal{D}_{\{\omega_1,\omega_2\}}(H \times K) := \varinjlim_{m\in\mathbb{N}} \mathcal{D}_{(\omega_1,\omega_2),m}(H \times K)$$
$$\mathcal{E}_{\{\omega_1,\omega_2\}}(H \times K) := \varinjlim_{m\in\mathbb{N}} \mathcal{E}_{(\omega_1,\omega_2),m}(H \times K).$$

They are regular countable inductive limits and (DFN)-spaces; if H and K are strictly regular, $\mathcal{D}_{\{\omega_1,\omega_2\}}(H \times K)$ is a closed subspace of $\mathcal{E}_{\{\omega_1,\omega_2\}}(H \times K)$.

Definition. The notations $\mathcal{E}_*(\Omega_1 \times \Omega_2)$ and $\mathcal{D}_*(\Omega_1 \times \Omega_2)$ require that the sequences $(H_n)_{n\in\mathbb{N}}$ and $(K_n)_{n\in\mathbb{N}}$ are compact exhaustions of Ω_1 and Ω_2 respectively, by means of sequences of strictly regular compact sets such that $H_n \subset H_{n+1}^\circ$ and $K_n \subset K_{n+1}^\circ$ for every $n \in \mathbb{N}$. They are the locally convex spaces

$$\mathcal{E}_*(\Omega_1 \times \Omega_2) := \varprojlim_{m\in\mathbb{N}} \mathcal{E}_*(H_n \times K_n) \quad \text{and} \quad \mathcal{D}_*(\Omega_1 \times \Omega_2) := \varinjlim_{m\in\mathbb{N}} \mathcal{D}_*(H_n \times K_n).$$

So $\mathcal{E}_{(\omega_1,\omega_2)}(\Omega_1 \times \Omega_2)$ is a Fréchet nuclear space and $\mathcal{D}_{(\omega_1,\omega_2)}(\Omega_1 \times \Omega_2)$ is a strict countable inductive limit of Fréchet nuclear spaces, it is a (LFN)-space.

The space $\mathcal{E}_{\{\omega_1,\omega_2\}}(\Omega_1 \times \Omega_2)$ carries a complicated locally convex structure but certainly is complete and nuclear. In Proposition 6.4, we prove that it also is ultrabornological. The space $\mathcal{D}_{\{\omega_1,\omega_2\}}(\Omega_1 \times \Omega_2)$ is a strict countable inductive limit of (DFN)-spaces hence is a (DFN)-space.

6 Elementary properties

Acting as in the proof of ([7], Proposition 3.1) leads to the following result.

Proposition 6.1. *For every $n \in \mathbb{N}$,*

$$\Lambda_{(n)} \colon \mathcal{E}_*(H_n \times K_n) \times \mathcal{E}_*(H_n \times K_n) \to \mathcal{E}_*(H_n \times K_n); \quad (f, g) \mapsto fg$$

is a well defined continuous bilinear map. Therefore

$$\Lambda_* \colon \mathcal{E}_*(\Omega_1 \times \Omega_2) \times \mathcal{E}_*(\Omega_1 \times \Omega_2) \to \mathcal{E}_*(\Omega_1 \times \Omega_2); \quad (f, g) \mapsto fg$$

also is a well defined continuous bilinear map. ■

In [1], one finds that, for every $\varepsilon > 0$, there are non-zero and positive functions $f \in \mathcal{D}_{(\omega_1)}(\mathbb{R}^r)$ and $g \in \mathcal{D}_{(\omega_2)}(\mathbb{R}^s)$ with support contained in the closed ball of center 0 and radius $\varepsilon/2$.

So, using $f \otimes g$ and acting as in ([4], p. 61), one obtains that
a) for every non empty compact subset K of an open subset A of $\mathbb{R}^r \times \mathbb{R}^s$, there is a positive function in $\mathcal{D}_{(\omega_1,\omega_2)}(\mathbb{R}^r \times \mathbb{R}^s)$, identically 1 on a neighbourhood of K and support contained in A;
b) for every finite open cover $\{A_j : j = 1, \ldots, q\}$ of a compact subset K of $\Omega_1 \times \Omega_2$, there are positive function $f_j \in \mathcal{D}_{(\omega_1,\omega_2)}(\Omega_1 \times \Omega_2)$ such that $\mathrm{supp}(\varphi_j) \subset A_j$ and $\sum_{j=1}^q \varphi_j \equiv 1$ on a neighbourhood of K;
c) for every open cover $\{A_j : j \in \mathbb{N}\}$ of a non empty open subset A of $\mathbb{R}^r \times \mathbb{R}^s$, there is a $\mathcal{D}_{(\omega_1,\omega_2)}(\mathbb{R}^r \times \mathbb{R}^s)$-partition of unity subordinate to the cover.

As a consequence, we note that every continuous linear map from the space $\mathcal{D}_*(\Omega_1 \times \Omega_2)$ into a locally convex space has a support. In fact, a lot more can be said: acting as in [7] leads directly to the following results.

Proposition 6.2. *The set $\mathcal{D}_*(\Omega_1 \times \Omega_2)$ is a sequentially dense vector subspace of $\mathcal{E}_*(\Omega_1 \times \Omega_2)$.* ■

Proposition 6.3. *Let G be a locally convex space.*

a) *If A is a bounded subset of $L_s(\mathcal{D}_*(\Omega_1 \times \Omega_2), G)$ and if there are compact subsets H of Ω_1 and K of Ω_2 such that* $\mathrm{supp}(S) \subset H \times K$ *for every $S \in A$, then every $S \in A$ has a unique continuous linear extension $T(S)$ from $\mathcal{E}_*(\Omega_1 \times \Omega_2)$ into G and $\{T(S) : S \in A\}$ is an equicontinuous subset of $L(\mathcal{E}_*(\Omega_1 \times \Omega_2), G)$.*

b) *If the topology of G comes from a system of norms and if B is a simply bounded set of sequentially continuous linear maps from $\mathcal{E}_*(\Omega_1 \times \Omega_2)$ into G, then there are compact subsets H of Ω_1 and K of Ω_2 such that, for every $T \in B$, the support of the restriction of T to $\mathcal{D}_*(\Omega_1 \times \Omega_2)$ is contained in $H \times K$.*

c) *Every simply bounded set of sequentially continuous linear maps from the space $\mathcal{E}_{\{\omega_1,\omega_2\}}(\Omega_1 \times \Omega_2)$ into G is equicontinuous.* ■

Theorem 6.4. *The space $\mathcal{E}_{\{\omega_1,\omega_2\}}(\Omega_1 \times \Omega_2)$ is ultrabornological.* ■

7 The spaces $\mathcal{E}_*(\Omega)$ and $\mathcal{D}_*(\Omega)$

In this paragraph, given a weight ω and a non void open subset Ω of $\mathbb{R}^n$, we make precise the definition of the spaces $\mathcal{E}_{(\omega)}(\Omega)$, $\mathcal{E}_{\{\omega\}}(\Omega)$, $\mathcal{D}_{(\omega)}(\Omega)$ and $\mathcal{D}_{\{\omega\}}(\Omega)$ by use of strictly regular compact subsets of Ω.

Definition. Given a weight ω, a strictly regular compact subset K of $\mathbb{R}^n$ and a positive number h, the Banach space $\mathcal{E}_{(\omega),h}(K)$ is defined as follows: its elements are the $\mathcal{C}^\infty$-functions f on K° such that, for every $\alpha \in \mathbb{N}_0^n$, $\mathrm{D}^\alpha f$ has a continuous extension on K and such that

$$\|f\|_{K,h} := \sup_{\alpha \in \mathbb{N}_0^n} \|\mathrm{D}^\alpha f\|_K \exp(-\varphi^*(h\,|\alpha|)/h) < \infty;$$

its norm is $\|\cdot\|_{K,h}$.

We then introduce the Fréchet space $\mathcal{E}_{(\omega)}(K) := \lim_{\longleftarrow} \mathcal{E}_{(\omega),1/m}(K)$ and the countable inductive limit of Banach spaces $\mathcal{E}_{\{\omega\}}(K) := \lim_{\longrightarrow} \mathcal{E}_{(\omega),m}(K)$.

In a second step, we consider a non void open subset Ω of $\mathbb{R}^n$ and a countable cover $(K_n)_{n\in\mathbb{N}}$ of Ω by means of strictly regular compact sets such that $K_n \subset K_{n+1}^\circ$ for every $n \in \mathbb{N}$ and set $\mathcal{E}_{(\omega)}(\Omega) := \lim_{\longleftarrow} \mathcal{E}_{(\omega)}(K_n)$ and $\mathcal{E}_{\{\omega\}}(\Omega) := \lim_{\longleftarrow} \mathcal{E}_{\{\omega\}}(K_n)$.

Moreover $\mathcal{D}_{(\omega)}(K)$ and $\mathcal{D}_{\{\omega\}}(K)$ denote respectively the subspaces of $\mathcal{E}_{(\omega)}(K)$ and $\mathcal{E}_{\{\omega\}}(K)$, the elements of which have a compact support contained in K. Finally we set $\mathcal{D}_{(\omega)}(\Omega) := \lim_{\longrightarrow} \mathcal{D}_{(\omega)}(K_n)$ and $\mathcal{D}_{\{\omega\}}(\Omega) := \lim_{\longrightarrow} \mathcal{E}_{\{\omega\}}(K_n)$.

From now on, let us agree on the following use of the notations: if the notation $\mathcal{E}_*(\Omega_1 \times \Omega_2)$ [resp. $\mathcal{D}_*(\Omega_1 \times \Omega_2)$] appears in a statement as well as $\mathcal{E}_*(\Omega_1)$, $\mathcal{E}_*(\Omega_2)$, $\mathcal{D}_*(\Omega_1)$ or $\mathcal{D}_*(\Omega_2)$, it means that two statements are valid:
a) one with $\mathcal{E}_{(\omega_1,\omega_2)}(\Omega_1 \times \Omega_2)$ [resp. $\mathcal{D}_{(\omega_1,\omega_2)}(\Omega_1 \times \Omega_2)$]; in this case, the notations become $\mathcal{E}_{(\omega_1)}(\Omega_1)$, $\mathcal{E}_{(\omega_2)}(\Omega_2)$, $\mathcal{D}_{(\omega_1)}(\Omega_1)$ and $\mathcal{D}_{(\omega_2)}(\Omega_2)$ respectively;
b) one with $\mathcal{E}_{\{\omega_1,\omega_2\}}(\Omega_1 \times \Omega_2)$ [resp. $\mathcal{D}_{\{\omega_1,\omega_2\}}(\Omega_1 \times \Omega_2)$]; in this case, the notations become $\mathcal{E}_{\{\omega_1\}}(\Omega_1)$, $\mathcal{E}_{\{\omega_2\}}(\Omega_2)$, $\mathcal{D}_{\{\omega_1\}}(\Omega_1)$ and $\mathcal{D}_{\{\omega_2\}}(\Omega_2)$ respectively.

Proposition 7.1. *The bilinear map*

$$\lambda_* : \mathcal{E}_*(\Omega_1) \times \mathcal{E}_*(\Omega_2) \to \mathcal{E}_*(\Omega_1 \times \Omega_2); \quad (f, g) \mapsto f \otimes g$$

and the canonical injection from $\mathcal{E}_*(\Omega_1) \otimes_\pi \mathcal{E}_*(\Omega_2)$ into $\mathcal{E}_*(\Omega_1 \times \Omega_2)$ are continuous.

■

8 The space $\mathcal{E}^{(p!q!)}(\mathbb{R}^r \times \mathbb{R}^s)$

Definition. By $\mathcal{E}^{(p!q!)}(\mathbb{R}^r \times \mathbb{R}^s)$, we designate the space of the $\mathcal{C}^\infty$-functions f on $\mathbb{R}^s \times \mathbb{R}^s$ such that, for every $h > 0$ and compact subsets H of $\mathbb{R}^r$ and K of $\mathbb{R}^s$,

$$|f|_{H\times K,h} := \sup_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} \frac{\|\mathrm{D}^{(\alpha,\beta)}f\|_{H\times K}}{h^{|\alpha|+|\beta|}\alpha!\beta!} < \infty,$$

endowed with the system $\{|\cdot|_{H\times K,h} : H \Subset \mathbb{R}^r, K \Subset \mathbb{R}^s, h > 0\}$ of semi-norms. It clearly is a Fréchet space.

We also denote by $\mathcal{H}(\mathbb{C}^n)$ the Fréchet space of the holomorphic functions on $\mathbb{C}^n$ endowed with the topology of uniform convergence on the compact sets. Classical holomorphy arguments easily provide that *the restriction map*

$$\Gamma\colon \mathcal{H}(\mathbb{C}^{r+s}) \to \mathcal{E}^{(p!q!)}(\mathbb{R}^r \times \mathbb{R}^s); \quad f \mapsto f|_{\mathbb{R}^r\times\mathbb{R}^s}$$

is a well defined isomorphism.

Proposition 8.1. *The restriction map*

$$R_{\Omega_1\times\Omega_2}\colon \mathcal{E}^{(p!q!)}(\mathbb{R}^r \times \mathbb{R}^s) \to \mathcal{E}_*(\Omega_1 \times \Omega_2); \quad f \mapsto f|_{\Omega_1\times\Omega_2}$$

is well defined, continuous and linear.

In fact, for every $h > 0$, there is $B > 1$ such that

$$\|R_{\Omega_1\times\Omega_2} f\|_{H\times K,h} \le e^{2B/h}\,|f|_{H\times K,h/(4B)}$$

for every $f \in \mathcal{E}^{(p!q!)}(\mathbb{R}^r \times \mathbb{R}^s)$ and strictly regular compact subsets H of Ω_1 and K of Ω_2.

Proof. Let us establish first the second part of the statement.

Given $h > 0$, we choose $B > 1$ such that the inequalities (3) hold for $\varphi^* = \varphi_1^*$ and every $\alpha \in \mathbb{N}_0^r$ as well as for $\varphi^* = \varphi_2^*$ and every $\beta \in \mathbb{N}_0^s$.

Then we note that, for every $f \in \mathcal{E}^{(p!q!)}(\mathbb{R}^r \times \mathbb{R}^s)$ and strictly compact subsets H of $\mathbb{R}^r$ and K of $\mathbb{R}^s$, we have

$$\begin{aligned}\|\mathrm{D}^{(\alpha,\beta)}f\|_{H\times K} &\le |f|_{H\times K,h/(4B)}\,(h/(4B))^{|\alpha|+|\beta|}\alpha!\beta!\\ &\le |f|_{H\times K,h/(4B)}\,e^{2B/h}\exp(\varphi_1^*(h\,|\alpha|)/h + \varphi_2^*(h\,|\beta|)/h)\end{aligned}$$

hence the announced inequality and the fact that $R|_{\Omega_1\times\Omega_2}$ is a well defined linear map.

At this point, the case $* = (\omega_1, \omega_2)$ is clear.

In the case $* = \{\omega_1, \omega_2\}$, we note that, for every $n \in \mathbb{N}$, the inequality we just established implies the continuity of the linear map $(R_{\Omega_1\times\Omega_2}\cdot)|_{H_n\times K_n}$ from $\mathcal{E}^{(p!q!)}(\mathbb{R}^r \times \mathbb{R}^s)$ into $\mathcal{E}_{(\omega_1,\omega_2),1}(H_n \times K_n)$ hence into $\mathcal{E}_{\{\omega_1,\omega_2\}}(H_n \times K_n)$. The conclusion then follows at once. ■

9 Approximation

Notation. For every $m \in \mathbb{N}$, the function ψ_m is defined on $\mathbb{R}^r \times \mathbb{R}^s$ by

$$\psi_m(u,v) := m^{r+s}\pi^{-(r+s)/2}e^{-m^2|u|^2}e^{-m^2|v|^2}, \quad \forall (u,v) \in \mathbb{R}^r \times \mathbb{R}^s.$$

Proposition 9.1. *For every $m \in \mathbb{N}$ and $f \in \mathcal{D}_*(\mathbb{R}^r \times \mathbb{R}^s)$, the function $f \star \psi_m$ has a holomorphic extension on $\mathbb{C}^{r+s}$ hence belongs to $\mathcal{E}^{(p!q!)}(\mathbb{R}^r \times \mathbb{R}^s)$.* ■

Proposition 9.2. *For every $f \in \mathcal{D}_*(\Omega_1 \times \Omega_2)$, $(R_{\Omega_1\times\Omega_2}(f \star \psi_m))_{m\in\mathbb{N}}$ is a sequence in $\mathcal{E}_*(\Omega_1 \times \Omega_2)$ converging to f.*

Proof. There is $n \in \mathbb{N}$ such that $f \in \mathcal{D}_*(H_n \times K_n)$. So, in the case $* = (\omega_1, \omega_2)$, f belongs to $\mathcal{D}_{(\omega_1,\omega_2),1/m}(H_n \times K_n)$ for every $m \in \mathbb{N}$ and, in the case $* = \{\omega_1, \omega_2\}$, there is $m \in \mathbb{N}$ such that f belongs to $\mathcal{D}_{(\omega_1,\omega_2),m}(H_n \times K_n)$.

Let f belong to $\mathcal{D}_{(\omega_1,\omega_2),h}(H_n \times K_n)$ for some $h > 0$. For every $m \in \mathbb{N}$, we just proved that $f \star \psi_m$ belongs to $\mathcal{E}^{(p!q!)}(\mathbb{R}^r \times \mathbb{R}^s)$ hence, by Proposition 8.1, that its restriction to $H_n \times K_n$ belongs to $\mathcal{E}_{(\omega_1,\omega_2),h}(H_n \times K_n)$. We are going to prove that the sequence $((f \star \psi_m)|_{H_n\times K_n})_{m\in\mathbb{N}}$ converges to f in $\mathcal{E}_{(\omega_1,\omega_2),bh}(H_n \times K_n)$, which allows to conclude at once.

Let $\varepsilon > 0$ be given.

We first choose $C > 0$ for which the inequalities (4) hold and then fix $q \in \mathbb{N}$ such that $2^{-q}e^{2/h+2C}\,\|f\|_{H_n\times K_n,h} \le \varepsilon/2$.

Now we evaluate $\|\mathrm{D}^{(\alpha,\beta)}(f \star \psi_m) - \mathrm{D}^{(\alpha,\beta)}f\|_{H_n\times K_n}$.

If $|\alpha| + |\beta| \ge q$, we write down the convolution product and easily get

$$\begin{aligned}&\|\mathrm{D}^{(\alpha,\beta)}(f \star \psi_m) - \mathrm{D}^{(\alpha,\beta)}f\|_{H_n\times K_n}\\ &\qquad \le 2\,\|f\|_{H_n\times K_n,h}\,2^{-|\alpha|-|\beta|}\exp(|\alpha| + \varphi_1^*(h\,|\alpha|)/h + |\beta| + \varphi_2^*(h\,|\beta|)/h)).\end{aligned}$$

So the inequalities (4) and the choice of q provide

$$\|\mathrm{D}^{(\alpha,\beta)}(f \star \psi_m) - \mathrm{D}^{(\alpha,\beta)}f\|_{H_n\times K_n} \le \varepsilon\exp(\varphi_1^*(bh\,|\alpha|)/(bh) + \varphi_2^*(bh\,|\beta|)/(bh)).$$

If $|\alpha| + |\beta| < q$, we note that $\{\mathrm{D}^{(\alpha,\beta)}f\colon\ |\alpha| + |\beta| < q\}$ is a finite set of continuous functions on $\mathbb{R}^r \times \mathbb{R}^s$ with compact supports hence is a uniformly equicontinuous set. Therefore there is $\delta > 0$ such that

$$\frac{|\mathrm{D}^{(\alpha,\beta)}f(x-u, y-v) - \mathrm{D}^{(\alpha,\beta)}f(x,y)|}{\exp(\varphi_1^*(bh\,|\alpha|)/(bh) + \varphi_2^*(bh\,|\beta|)/(bh))} \le \frac{\varepsilon}{2}$$

for every (x,y), $(u,v) \in \mathbb{R}^r \times \mathbb{R}^s$ and $(\alpha,\beta) \in \mathbb{N}_0^r \times \mathbb{N}_0^s$ such that $|(u,v)| \le \delta$ and $|\alpha| + |\beta| < q$. Now we set

$$M := 2 \sup_{|\gamma|+|\delta|<q} \frac{\|\mathrm{D}^{(\gamma,\delta)}f\|_{\mathbb{R}^r\times\mathbb{R}^s}}{\exp(\varphi_1^*(bh\,|\gamma|)/(bh) + \varphi_2^*(bh\,|\delta|)/(bh))}$$

and fix $m_0 \in \mathbb{N}$ such that $M\int_{|(u,v)|\ge\delta}\psi_m\,du\,dv \le \varepsilon/2$ for every $m \ge m_0$. Therefore by writing down the convolution product and by splitting the integral on $\{(u,v)\colon\ |(u,v)| \le \delta\}$ and $\{(u,v)\colon\ |(u,v)| \ge \delta\}$, we easily obtain

$$\frac{\|\mathrm{D}^{(\alpha,\beta)}(f \star \psi_m) - \mathrm{D}^{(\alpha,\beta)}f\|_{H_n\times K_n}}{\exp(\varphi_1^*(bh\,|\alpha|)/(bh) + \varphi_2^*(bh\,|\beta|)/bh))} \le \varepsilon, \quad \forall m \ge m_0.$$

Hence the conclusion by putting these two informations together. ■

Proposition 9.3. *For every $f \in \mathcal{D}_*(\Omega_1 \times \Omega_2)$, there is a sequence of polynomials on $\mathbb{R}^{r+s}$ converging to f in $\mathcal{E}_*(\Omega_1 \times \Omega_2)$.*

Therefore the set of the restrictions to $\Omega_1 \times \Omega_2$ of the polynomials on $\mathbb{R}^r \times \mathbb{R}^s$ is a dense vector subspace of $\mathcal{E}_*(\Omega_1 \times \Omega_2)$.

Proof. The first statement can be established as Proposition 5.3 of [7] The second is then a direct consequence of Proposition 6.2. ∎

Proposition 9.4. *The vector space $\mathcal{D}_*(\Omega_1) \otimes \mathcal{D}_*(\Omega_2)$ is a dense vector subspace of $\mathcal{D}_*(\Omega_1 \times \Omega_2)$.*

Therefore $\mathcal{E}_*(\Omega_1) \otimes \mathcal{E}_*(\Omega_2)$ is a dense vector subspace of $\mathcal{E}_*(\Omega_1 \times \Omega_2)$.

Proof. The first statement can be established as Proposition 7.1 of [7] The second is then a direct consequence of Proposition 6.2. ∎

10 Denseness of $\mathcal{D}_*(H) \otimes \mathcal{D}_*(K)$ in $\mathcal{D}_*(H \times K)$

Notation. Given $b \in \mathbb{R}^n$ and a function f on $\mathbb{R}^n$, $\tau_b f$ designates the function defined on $\mathbb{R}^n$ by $(\tau_b f)(\cdot) = f(\cdot - b)$.

Proposition 10.1. *For every $b \in \mathbb{R}^n$, the map τ_b is a well defined continuous linear map from $\mathcal{E}_*(\mathbb{R}^r \times \mathbb{R}^s)$ into itself.*

Moreover we have $\lim_{b\to 0} \tau_b f = f$ for every $f \in \mathcal{E}_*(\mathbb{R}^r \times \mathbb{R}^s)$.

Proof. The first part of the statement is immediate.

Now for any strictly regular compact subsets H of $\mathbb{R}^r$ and K of $\mathbb{R}^s$, we first choose strictly regular compact subsets H' of $\mathbb{R}^r$ and K' of $\mathbb{R}^s$ such that $H \subset H'^\circ$ and $K \subset K'^\circ$. We next choose a positive number $\delta < d(H \times K, (\mathbb{R}^r \times \mathbb{R}^s) \setminus (H'^\circ \times K'^\circ)))$. So for every $b = (b_1, b_2) \in \mathbb{R}^r \times \mathbb{R}^s$ such that $|b| \le \delta$ and $(x, y) \in H \times K$, $(x-b_1, y-b_2)$ belongs to $H' \times K'$.

In the case $* = (\omega_1, \omega_2)$, $f|_{H'\times K'}$ belongs to $\mathcal{E}_{(\omega_1,\omega_2),1/m}(H' \times K')$ for every $m \in \mathbb{N}$; in the case $* = \{\omega_1, \omega_2\}$, $f|_{H'\times K'}$ belongs to $\mathcal{E}_{(\omega_1,\omega_2),m}(H' \times K')$ for some $m \in \mathbb{N}$.

We are going to prove that, if $f|_{H'\times K'}$ belongs to $\mathcal{E}_{(\omega_1,\omega_2),h}(H' \times K')$ for some $h > 0$, then, for every $\varepsilon > 0$, there is $\eta > 0$ such that $\|\tau_b f - f\|_{H\times K,bh} \le \varepsilon$ for every $b \in \mathbb{R}^r \times \mathbb{R}^s$ such that $|b| \le \eta$. The conclusion then follows at once.

Clearly the functions $f|_{H\times K}$ and $(\tau_b f)|_{H\times K}$ belong to $\mathcal{E}_{(\omega_1,\omega_2),h}(H \times K)$. We first choose $C > 0$ for which the inequalities (4) hold and then fix $q \in \mathbb{N}$ such that $2^{-q} e^{2C+2/h} \|f\|_{H'\times K',h} \le \varepsilon/2$.

On one hand, for every $(x, y) \in H \times K$, $b = (b_1, b_2) \in \mathbb{R}^r \times \mathbb{R}^s$ and $(\alpha, \beta) \in \mathbb{N}_0^r \times \mathbb{N}_0^s$ such that $|b| \le \delta$ and $|\alpha| + |\beta| \ge q$, the inequality (4) directly leads to

$$\begin{aligned}&|\mathrm{D}^{(\alpha,\beta)} f(x, y) - \mathrm{D}^{(\alpha,\beta)} f(x - b_1, y - b_2)| \\ &\qquad \le 2e^{2C+2/h} 2^{-q} \|f\|_{H'\times K',h} \exp(\varphi_1^*(bh\,|\alpha|)/(bh) + \varphi_2^*(bh\,|\beta|)/(bh)).\end{aligned}$$

On the other hand, $\{\mathrm{D}^{(\alpha,\beta)} f \colon |\alpha| + |\beta| < q\}$ is a finite set of continuous functions on the compact set $H' \times K'$. Therefore there is $\eta > 0$ such that

$$\sup_{|\alpha|+|\beta|<q} \sup_{(x,y)\in H\times K} \frac{|\mathrm{D}^{\alpha,\beta)} f(x, y) - \mathrm{D}^{(\alpha,\beta)} f(x - b_1, y - b_2)|}{\exp(\varphi_1^*(bh\,|\alpha|)/(bh) + \varphi_2^*(bh\,|\beta|)/(bh))} \le \varepsilon$$

for every $b = (b_1, b_2) \in \mathbb{R}^r \times \mathbb{R}^s$ such that $|b| \leq \eta$.

These two informations put together provide the conclusion. ∎

Definition. A subset B of $\mathbb{R}^n$ *has the local displacement property* if every $x \in B$ has a neighbourhood W such that, for every $\varepsilon > 0$, there is $a \in \mathbb{R}^n$ such that $|a| \leq \varepsilon$ and $a + (B \cap W) \subset B^\circ$.

If $B_1, \ldots, B_q$ are closed balls in $\mathbb{R}^n$ in finite number and such that $B_j \cap B_k \neq \emptyset$ implies that $B_j^\circ \cap B_k^\circ \neq \emptyset$, one can check that their union has the local displacement property. Moreover if the compact subsets H of $\mathbb{R}^r$ and K of $\mathbb{R}^s$ have the local displacement property, it is clear that $H \times K$ also has this property.

Therefore, from now on, we agree that the covers $(H_n)_{n\in\mathbb{N}}$ of Ω_1 and $(K_n)_{n\in\mathbb{N}}$ of Ω_2 consist of strictly regular compact sets having the local displacement property and such that $H_n \subset H_{n+1}^\circ$ and $K_n \subset K_{n+1}^\circ$ for every $n \in \mathbb{N}$.

An argument analogous to the one of the proof of ([7], Proposition 8.1) then establishes the following result.

Proposition 10.2. *If the compact subsets H of $\mathbb{R}^r$ and K of $\mathbb{R}^s$ have the local displacement property, then the vector space $\mathcal{D}_*(H) \otimes \mathcal{D}_*(K)$ is a dense subspace of $\mathcal{D}_*(H \times K)$.* ∎

11 Structure of the elements of $\mathcal{E}_*(\Omega_1 \times \Omega_2)$ and $\mathcal{D}_*(\Omega_1 \times \Omega_2)$

If f belongs to $\mathcal{E}_*(\Omega_1 \times \Omega_2)$, it is clear that, for every $y \in \Omega_2$, $f(.,y)$ belongs to $\mathcal{E}_*(\Omega_1)$. Let us investigate this property.

Proposition 11.1. *For every $f \in \mathcal{E}_*(\Omega_1 \times \Omega_2)$, the function*

$$g\colon \Omega_2 \to \mathcal{E}_*(\Omega_1); \quad y \mapsto f(.,y)$$

is $\mathcal{C}^\infty$ and such that $[D^\beta f(y)](.) = D^{(0,\beta)} f(.,y)$ for every $\beta \in \mathbb{N}_0^s$ and $y \in \Omega_2$.

Proof. For every $\beta \in \mathbb{N}_0^s$, since $\mathrm{D}^{(0,\beta)} f$ belongs to $\mathcal{E}_*(\Omega_1 \times \Omega_2)$, Proposition 10.1 provides the continuity of the function $g_\beta\colon \Omega_2 \to \mathcal{E}_*(\Omega_1)$ defined by $g_\beta(y) = \mathrm{D}^{(0,\beta)} f(.,y)$. Therefore to conclude, it is enough to establish the formula in the case $|\beta| = 1$. Let us do this for $\beta = (1, 0, \ldots, 0)$. So we only have to prove that

$$\lim_{k\to 0} (g(c + ke_1) - g(c))/h = g_\beta(c) \text{ in } \mathcal{E}_*(\Omega_1), \quad \forall c \in \Omega_2.$$

Given $c \in \Omega_2$, we have $c \in K_n^\circ$ for n large enough. For such an integer n, $f|_{H_n \times K_n}$ belongs to $\mathcal{E}_{(\omega_1,\omega_2),1/m}(H_n \times K_n)$ for every $m \in \mathbb{N}$ if $* = (\omega_1, \omega_2)$ and to $\mathcal{E}_{(\omega_1,\omega_2),m}(H_n \times K_n)$ for some $m \in \mathbb{N}$ if $* = \{\omega_1, \omega_2\}$.

If c belongs to K_n° and f to $\mathcal{E}_{(\omega_1,\omega_2),h}(H_n \times K_n)$ for some $h > 0$, we are going to prove that, for every $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|(g(c + ke_1) - g(c))/k - g_\beta(c)\|_{H_n,bh} \leq \varepsilon$$

if $0 < |k| \leq \delta$. The conclusion then follows at once.

Up to considering the real and imaginary parts of f separately, we may assume f real valued. As c belongs to K_n°, there is $\eta > 0$ such that $c + tke_1$ belongs to K_n° for every $t \in [0,1]$ if $|k| \leq \eta$. Let us consider $k \in \mathbb{R}$ such that $0 < |k| \leq \eta$. For every $x \in \Omega_1$ and $\alpha \in \mathbb{N}_0^r$, the limited Taylor formula provides $\theta(k,x,\alpha) \in]0,1[$ such that

$$\mathrm{D}^{(\alpha,0)} f(x, c + ke_1) - \mathrm{D}^{(\alpha,0)} f(x,c) = k\mathrm{D}^{(\alpha,\beta)} f(x, c + \theta(k,x,\alpha)ke_1).$$

Let a_0 be a positive number such that the inequalities (1) hold for every $a \geq a_0/(bh)$ and let $q \geq a_0/(bh)$ be an integer such that

$$2^{-q} \|f\|_{H_n \times K_n, h} \exp(1/h + \varphi_2^*(h)/h) \leq \varepsilon/2.$$

As $\{\mathrm{D}^{(\alpha,\beta)} f \colon |\alpha| \leq q\}$ is a finite set of continuous functions on the compact set $H_n \times K_n$, we can also choose $\delta \in]0,\eta[$ such that

$$|\mathrm{D}^{(\alpha,\beta)} f(x,y) - \mathrm{D}^{(\alpha,\beta)} f(x,y')| \leq \varepsilon \exp(\varphi_1^*(bh\,|\alpha|)/(bh))$$

for every $x \in H_n$; y, $y' \in K_n$ and $\alpha \in \mathbb{N}_0^r$ such that $|y - y'| \leq \delta$ and $|\alpha| \leq q$.

So for $k \in \mathbb{R}$ such that $0 < |k| \leq \delta$, we arrive directly at

$$\begin{aligned} &\|(g(c+ke_1) - g(c))/k - g_\beta(c)\|_{H_n, bh} \\ &\qquad \leq \sup\left\{\varepsilon, 2 \sup_{|\alpha|>q} \|\mathrm{D}^{(\alpha,\beta)} f\|_{H_n \times K_n} \exp(-\varphi_1^*(bh\,|\alpha|)/(bh))\right\} \end{aligned}$$

with

$$\begin{aligned} &\sup_{|\alpha|>q} \|\mathrm{D}^{(\alpha,\beta)} f\|_{H_n \times K_n} \exp(-\varphi_1^*(bh\,|\alpha|)/(bh)) \\ &\quad \leq \|f\|_{H_n \times K_n, h} \sup_{|\alpha|>q} \exp(\varphi_1^*(h\,|\alpha|)/h - \varphi_1^*(bh\,|\alpha|)/(bh) + \varphi_2^*(h\,|\beta|)/h) \\ &\quad \leq 2^{-q} \|f\|_{H_n \times K_n, h} \exp(1/h + \varphi_2^*(h)/h) \end{aligned}$$

[to obtain the last inequality, we use the inequalities (1)].

Hence the conclusion. ∎

Then one can proceed as in [7] and get the following properties.

Proposition 11.2. a) *For every $f \in \mathcal{E}_*(\Omega_1 \times \Omega_2)$ and $S \in \mathcal{E}_*(\Omega_1)'$, the function $\langle S, f(.,y)\rangle$ belongs to $\mathcal{E}_*(\Omega_2)$ and verifies*

$$D^\beta \langle S, f(.,y)\rangle = \langle S, D^{(0,\beta)} f(.,y)\rangle, \quad \forall \beta \in \mathbb{N}_0^s, y \in \Omega_2.$$

b) *The bilinear map $\Delta_* \colon \mathcal{E}_*(\Omega_1 \times \Omega_2) \times \mathcal{E}_*(\Omega_1)' \to \mathcal{E}_*(\Omega_2)$ defined by $\Delta_*(f,S) = \langle S, f(.,y)\rangle$ is well defined and hypocontinuous.* ∎

Permuting the roles of Ω_1 and Ω_2 as well as those of ω_1 and ω_2 leads to analogous results and in particular to a hypocontinuous bilinear map ${}_*\Delta$.

Proposition 11.3. *For every $f \in \mathcal{D}_*(\Omega_1 \times \Omega_2)$, the function*

$$g_\beta \colon \Omega_2 \to \mathcal{D}_*(\Omega_1); \quad y \mapsto f(.,y)$$

is $\mathcal{C}^\infty$ and such that $[D^\beta g(y)](.) = D^{(0,\beta)} f(.,y)$ for every $\beta \in \mathbb{N}_0^s$ and $y \in \Omega_2$.

Proposition 11.4. a) *For every $f \in \mathcal{D}_*(\Omega_1 \times \Omega_2)$ and $S \in \mathcal{D}_*(\Omega_1)'$, the function $\langle S, f(.,y)\rangle$ belongs to $\mathcal{D}_*(\Omega_2)$ and verifies*

$$D^\beta \langle S, f(.,y)\rangle = \langle S, D^{(0,\beta)} f(.,y)\rangle, \quad \forall \beta \in \mathbb{N}_0^s, y \in \Omega_2.$$

b) *The bilinear map $\Gamma_* \colon \mathcal{D}_*(\Omega_1, \Omega_2) \times \mathcal{D}_*(\Omega_1)' \to \mathcal{D}_*(\Omega_2)$ defined by $\Gamma_*(f,S) = \langle S, f(.,y)\rangle$ is well defined and hypocontinuous.* ∎

Permuting the roles of Ω_1 and Ω_2 as well as those of ω_1 and ω_2 leads to analogous results and in particular to a hypocontinuous bilinear map ${}_*\Gamma$.

12 The tensor product $\otimes$ on the duals

Definition. Given $S \in \mathcal{D}_*(\Omega_1)'$ and $T \in \mathcal{D}_*(\Omega_2)'$, we know that
a) $\langle S, f(.,y)\rangle$ belongs to $\mathcal{D}_*(\Omega_1)$ for every $f \in \mathcal{D}_*(\Omega_1 \times \Omega_2)$ and $y \in \Omega_2$,
b) $S \otimes T \colon \mathcal{D}_*(\Omega_1 \times \Omega_2) \to \mathbb{C}$ defined by

$$\langle S \otimes T, f\rangle = \langle T, \Gamma_*(f,S)\rangle = \langle T, \langle S, f(.,y)\rangle\rangle, \quad \forall f \in \mathcal{D}_*(\Omega_1 \times \Omega_2)$$

is a continuous linear functional.
a') $\langle T, f(x,.)\rangle$ belongs to $\mathcal{D}_*(\Omega_2)$ for every $f \in \mathcal{D}_*(\Omega_1 \times \Omega_2)$ and $x \in \Omega_1$,
b') $T \otimes S \colon \mathcal{D}_*(\Omega_1 \times \Omega_2) \to \mathbb{C}$ defined by

$$\langle T \otimes S, f\rangle = \langle {}_*\Gamma(f,T)\rangle = \langle S, \langle T, f(x,.)\rangle\rangle, \quad \forall f \in \mathcal{D}_*(\Omega_1 \times \Omega_2)$$

is a continuous linear functional.

Since we have $\langle S \otimes T, f_1 \otimes f_2\rangle = \langle S, f_1\rangle\langle T, f_2\rangle = \langle T \otimes S, f_1 \otimes f_2\rangle$ for every $f \in \mathcal{D}_*(\Omega_1)$ and $g \in \mathcal{D}_*(\Omega_2)$, Proposition 9.4 implies that these two continuous linear functionals coincide. We call $S \otimes T = T \otimes S$ the *tensor product of S and T*. In fact the restriction of $S \otimes T = T \otimes S$ to $\mathcal{D}_*(\Omega_1) \otimes \mathcal{D}_*(\Omega_2)$ coincides with the tensor product of S and T considered as a continuous linear functional on $\mathcal{D}_*(\Omega_1) \otimes_\varepsilon \mathcal{D}_*(\Omega_2)$.

It is clear that if S and T have compact support, then $S \otimes T$ also has a compact support.

Proposition 12.1. *The map*

$$\chi_* \colon \mathcal{E}_*(\Omega_1)' \times \mathcal{E}_*(\Omega_2)' \to \mathcal{E}_*(\Omega_1 \times \Omega_2)'; \quad (S,T) \mapsto S \otimes T$$

is well defined, hypocontinuous and bilinear.

In the case $* = (\omega_1, \omega_2)$, χ_* is continuous.

Proof. Clearly χ_* is a well defined bilinear map.

As $\mathcal{E}_*(\Omega_1)'$ and $\mathcal{E}_*(\Omega_2)'$ are ultrabornological spaces, χ_* is hypocontinuous if it is separately continuous (cf. [6], III.5.2). Given $S \in \mathcal{E}_*(\Omega_1)'$, the continuity of $\chi_*(S,.)$ can be established as in the proof of ([7],Proposition 11.1). As the same proof implies the continuity of $\chi_*(.,T)$ for every $T \in \mathcal{E}_*(\Omega_2)$, we conclude at once.

To obtain the improvement of the case $* = (\omega_1, \omega_2)$, it suffices to note that $\mathcal{E}_*(\Omega_1)'$ and $\mathcal{E}_*(\Omega_2)'$ are strong duals of Fréchet nuclear spaces. ∎

Proposition 12.2. *The map*

$$\chi_* : \mathcal{D}_*(\Omega_1)' \times \mathcal{D}_*(\Omega_2)' \to \mathcal{D}_*(\Omega_1 \times \Omega_2)'; \quad (S,T) \mapsto S \otimes T$$

is well defined, hypocontinuous and bilinear.

In the case $* = \{\omega_1, \omega_2\}$, χ_* *is continuous.*

Proof. For the general case, one can proceed as in the proof of Proposition 12.1. The improvement in the case $* = \{\omega_1, \omega_2\}$ is immediate if one notes that $\mathcal{D}_*(\Omega_1)'$ and $\mathcal{D}_*(\Omega_2)'$ are Fréchet spaces. ∎

13 Tensor properties and kernel theorems

Proceeding as in ([7], Paragraph 12) provides the following results. In d), given two locally convex spaces E and F, $E \otimes_i F$ designates their tensor product endowed with the inductive topology (cf. [2]) and of course $E \widehat{\otimes}_i F$ its completion.

Theorem 13.1. a) *The canonical algebraic isomorphism from* $\mathcal{E}_*(\Omega_1) \otimes \mathcal{E}_*(\Omega_2)$ *as a subspace of* $\mathcal{E}_*(\Omega_1 \times \Omega_2)$ *onto* $\mathcal{E}_*(\Omega_1) \otimes_\epsilon \mathcal{E}_*(\Omega_2)$ *is continuous.*

b) *The spaces* $\mathcal{E}_*(\Omega_1 \times \Omega_2)$ *and* $\mathcal{E}_*(\Omega_1) \widehat{\otimes}_\pi \mathcal{E}_*(\Omega_2)$ *coincide.*

c) *If the compact subsets* H *of* $\mathbb{R}^r$ *and* K *of* $\mathbb{R}^s$ *have the local displacement property, then the spaces* $\mathcal{D}_*(H \times K)$ *and* $\mathcal{D}_*(H) \widehat{\otimes}_\pi \mathcal{D}_*(K)$ *coincide.*

d) *The spaces* $\mathcal{D}_*(\Omega_1 \times \Omega_2)$ *and* $\mathcal{D}_*(\Omega_1) \widehat{\otimes}_i \mathcal{D}_*(\Omega_2)$ *coincide.* ∎

Definition. A $*$-*kernel on* $\Omega_1 \times \Omega_2$ is an element of $\mathcal{D}_*(\Omega_1 \times \Omega_2)'$.

Given a $*$-kernel N on $\Omega_1 \times \Omega_2$,

$$B_N : \mathcal{D}_*(\Omega_1) \times \mathcal{D}_*(\Omega_2) \to \mathbb{C}; \quad (f,g) \mapsto N(f \otimes g)$$

clearly is a bilinear functional. By Theorem 13.1.d), B_N is separately continuous, the functional $\mathcal{N}(f) := B_N(f, .)$ belongs to $\mathcal{D}_*(\Omega_2)'$ for every $f \in \mathcal{D}_*(\Omega_1)$ and the map $\mathcal{N}$ to $L(\mathcal{D}_*(\Omega_1), \mathcal{D}_*(\Omega_2)')$. Similarly if g belongs to $\mathcal{D}_*(\Omega_2)$, then $B_N(., g)$ belongs to $\mathcal{D}_*(\Omega_1)'$; in fact, $B_N(., g) = {}^t\mathcal{N}(g)$ where ${}^t\mathcal{N}$ is the transpose of $\mathcal{N}$.

Conversely the Theorem 13.1.d) also provides the following kernel theorems.

Theorem 13.2. a) *If* T *is a continuous linear map from* $\mathcal{D}_*(\Omega_1)$ *into* $\mathcal{D}_*(\Omega_2)'$, *then there is a* $*$-*kernel* N *on* $\Omega_1 \times \Omega_2$ *such that* $\mathcal{N} = T$.

b) *If* S *is a continuous linear map from* $\mathcal{D}_*(\Omega_2)$ *into* $\mathcal{D}_*(\Omega_1)'$, *then there is a* $*$-*kernel* N *on* $\Omega_1 \times \Omega_2$ *such that* ${}^t\mathcal{N} = S$. ∎

Proceeding as in ([7], Paragraph 13) leads to the following results.

Theorem 13.3. a) *The following spaces coincide*

$$\mathcal{D}_{\{\omega_1,\omega_2\}}(\Omega_1 \times \Omega_2) \text{ and } \mathcal{D}_{\{\omega_1\}}(\Omega_1) \widehat{\otimes}_\pi \mathcal{D}_{\{\omega_2\}}(\Omega_2).$$

b) *The spaces* $\mathcal{D}_*(\Omega_1 \times \Omega_2)'$, $\mathcal{B}_b(\mathcal{D}_*(\Omega_1), \mathcal{D}_*(\Omega_2))$ *and* $L_b(\mathcal{D}_*(\Omega_1), \mathcal{D}_*(\Omega_2)')$ *coincide.*

c) *The set of the finite rank elements with compact support is a dense vector subspace of* $L_b(\mathcal{D}_*(\Omega_1), \mathcal{D}_*(\Omega_2))$. ∎

d) *The spaces* $\mathcal{D}_*(\Omega_1 \times \Omega_2)'$ *and* $\mathcal{D}_*(\Omega_1)' \widehat{\otimes}_\varepsilon \mathcal{D}_*(\Omega_2)'$ *coincide.* ∎

A way to state the classical kernel theorem of Schwartz (cf. [8]) is given by the equality $\mathcal{D}(\Omega_1 \times \Omega_2)' = \mathcal{D}(\Omega_1)' \widehat{\otimes}_\varepsilon \mathcal{D}(\Omega_2)'$. Therefore the part d) also appears as a refinement of the kernel theorem.

14 Case $\omega_1 = \omega_2$

Proposition 14.1. *Let the compact subsets H of $\mathbb{R}^r$ and K of $\mathbb{R}^s$ be strictly regular and set $L = H \times K$. Then, for every $h > 0$, the spaces $\mathcal{E}_{(\omega),h}(L)$ and $\mathcal{E}_{(\omega,\omega),h}(H \times K)$ coincide.*

Proof. On one hand, every $f \in \mathcal{E}_{(\omega,\omega),h}(H \times K)$ verifies

$$\begin{aligned}\|\mathrm{D}^{(\alpha,\beta)} f\|_L &\leq \|f\|_{H\times K,h} \exp(\varphi^*(h\,|\alpha|)/h + \varphi^*(h\,|\beta|)/h) \\ &\leq \|f\|_{H\times K,h} \exp(\varphi^*(h\,|(\alpha,\beta)|)/h)\end{aligned}$$

for every $(\alpha, \beta) \in \mathbb{N}_0^{r+s}$.

On the other hand, if f belongs to $\mathcal{E}_{\omega,h}(L)$, then, for every $(\alpha, \beta) \in \mathbb{N}_0^r \times \mathbb{N}_0^s$, we successively have

$$\begin{aligned}\|\mathrm{D}^{(\alpha,\beta)} f\|_{H\times K} &\leq \|f\|_L \exp(\varphi(h(|(\alpha,\beta)|))/h) \\ &\leq \|f\|_{L,h} \exp(\varphi^*(2h\,|\alpha|)/(2h) + \varphi^*(2h\,|\beta|)/(2h)).\end{aligned}$$

Hence the conclusion. ■

Therefore we have the following properties.

Proposition 14.2. *If the compact subsets H of $\mathbb{R}^r$ and K of $\mathbb{R}^s$ are strictly regular and if we set $L = H \times K$,*
a) *the spaces $\mathcal{E}_{(\omega)}(L)$ and $\mathcal{E}_{(\omega,\omega)}(H \times K)$ coincide.*
b) *the spaces $\mathcal{E}_{\{\omega\}}(L)$ and $\mathcal{E}_{\{\omega,\omega\}}(H \times K)$ coincide.*
c) *the spaces $\mathcal{D}_{(\omega)}(L)$ and $\mathcal{D}_{(\omega,\omega)}(H \times K)$ coincide.*
d) *the spaces $\mathcal{D}_{\{\omega\}}(L)$ and $\mathcal{D}_{\{\omega,\omega\}}(H \times K)$ coincide.*
If we set $\Omega = \Omega_1 \times \Omega_2$,
a) *the spaces $\mathcal{E}_{(\omega)}(\Omega)$ and $\mathcal{E}_{(\omega,\omega)}(\Omega_1 \times \Omega_2)$ coincide.*
b) *the spaces $\mathcal{E}_{\{\omega\}}(\Omega)$ and $\mathcal{D}_{\{\omega,\omega\}}(\Omega_1 \times \Omega_2)$ coincide.*
c) *the spaces $\mathcal{D}_{(\omega)}(\Omega)$ and $\mathcal{D}_{(\omega,\omega)}(\Omega_1 \times \Omega_2)$ coincide.*
d) *the spaces $\mathcal{D}_{\{\omega\}}(\Omega)$ and $\mathcal{D}_{\{\omega,\omega\}}(\Omega_1 \times \Omega_2)$ coincide.* ■

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