Multiple bifurcation in the solution set of the von Kármán equations with S^1-symmetries

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Abstract

In this work we study bifurcation of forms of equilibrium of a thin circular elastic plate lying on an elastic base under the action of a compressive force (see Picture 1). The forms of equilibrium may be found as solutions of the von Kármán equations with two real positive parameters defined on the unit disk in $\mathbb{R}^2$ centered at the origin. These equations are equivalent to an operator equation $F(x,p)=0$ in the real Hölder spaces with a nonlinear S^1-equivariant Fredholm map of index 0. For the existence of bifurcation at a point $(0,p)$ it is necessary that $\dim \operatorname{Ker} F'_x(0,p) > 0$. The space $\operatorname{Ker} F'_x(0,p)$ can be at most four-dimensional. We apply the Crandall-Rabinowitz theorem to prove that if $\dim \operatorname{Ker} F'_x(0,p) = 3$ then there is bifurcation of radial solutions at $(0,p)$. What is more, using the Lyapunov-Schmidt finite-dimensional reduction we investigate the number of branches of radial bifurcation at $(0,p)$.

1 Introduction

Let $C^{4,\mu}_{0,0}\left(\overline{\Omega}\right)$ denote the subspace of such functions $f : \overline{\Omega} \to \mathbb{R}$ from the real Hölder space $C^{4,\mu}\left(\overline{\Omega}\right)$ that satisfy the following boundary conditions:

$$f|_{\partial\Omega} = \Delta f|_{\partial\Omega} = 0,$$

where Δ is the Laplace operator, $\Omega = \{(u,v) \in \mathbb{R}^2 : u^2+v^2 < 1\}$ and $\mu \in (0,1)$. The operators $\Delta^2 : C^4\left(\overline{\Omega}\right) \to C\left(\overline{\Omega}\right)$ and $[\cdot,\cdot] : C^2\left(\overline{\Omega}\right) \times C^2\left(\overline{\Omega}\right) \to C\left(\overline{\Omega}\right)$ are defined

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by:

$$\Delta^2 f = \frac{\partial^4 f}{\partial u^4} + 2\frac{\partial^4 f}{\partial u^2 \partial v^2} + \frac{\partial^4 f}{\partial v^4}, \quad [f,g] = \frac{\partial^2 f}{\partial u^2}\frac{\partial^2 g}{\partial v^2} - 2\frac{\partial^2 f}{\partial u \partial v}\frac{\partial^2 g}{\partial u \partial v} + \frac{\partial^2 f}{\partial v^2}\frac{\partial^2 g}{\partial u^2}.$$

Our purpose is to investigate bifurcation of forms of equilibrium of a thin circular elastic plate lying on an elastic base under the action of a compressive force. This physical phenomenon is strictly connected with the von Kármán equations (see [3]) given as follows:

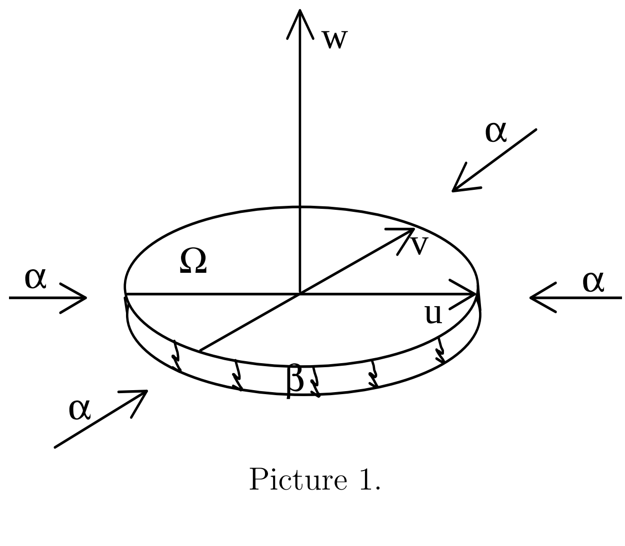
$$\begin{cases} \Delta^2 w - [w,\sigma] + 2\alpha\Delta w + \beta w - \gamma w^3 = 0 & \\ \Delta^2 \sigma + \frac{1}{2}[w,w] = 0 & \text{in } \Omega, \\ \Delta w = w = 0 & \\ \Delta\sigma = \sigma = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $w,\, \sigma \in C^{4,\mu}_{0,0}\left(\overline{\Omega}\right)$, $w(u,v)$ is a deflection function, $\sigma(u,v)$ is a stress function, $\alpha > 0$ is a value of the compressive force, $\beta > 0$ and $\gamma > 0$ are parameters of the elastic foundation. More precisely, the solutions (w,σ) of the system (1) lying in a small neighbourhood of the point $(0,0)$ are forms of equilibrium of the plate. In the remainder of this paper we assume γ to be constant.

In the last twenty years many authors have studied von Kármán equations of different types. The classical works on this subject are [1, 2, 4, 5, 6, 11, 17, 21, 23], and modern ones are [3, 7, 9, 10, 16, 19].

The studies, including the elasticity of foundation, by the use of bifurcation theory have been started by Yu. Morozov in [18]. Morozov investigated the forms of equilibrium of a homogenous finite beam clamped at the edges to the foundation. He proved that if we consider additional nonlinear terms corresponding to an elastic foundation then subcritical branches of solutions at a bifurcation point will occur. In [12] we came to the same conclusion for simple bifurcation points in the solution set of (1).

This paper is a continuation of our earlier results in [12, 13, 15]. To study bifurcation we apply methods of nonlinear analysis and representation theory.



Picture 1.

Let $X = C^{4,\mu}_{0,0}\left(\overline{\Omega}\right) \times C^{4,\mu}_{0,0}\left(\overline{\Omega}\right)$ and $Y = C^{0,\mu}\left(\overline{\Omega}\right) \times C^{0,\mu}\left(\overline{\Omega}\right)$. The system (1) is equivalent to an operator equation

$$F(x,p) = 0 \tag{2}$$

with the nonlinear map $F : X \times \mathbb{R}^2_+ \to Y$ given by

$$F(x,p) = \left(\Delta^2 w - [w,\sigma] + 2\alpha\Delta w + \beta w - \gamma w^3, -\Delta^2\sigma - \frac{1}{2}[w,w]\right), \tag{3}$$

where $x = (w,\sigma)$ and $p = (\alpha,\beta)$.

In [12] we showed that F is C^∞ and $F'_x(0,p) : X \to Y$ is a Fredholm map of index 0 for each $p \in \mathbb{R}^2_+$. We also proved that F is a variational gradient for the energy functional $E : X \times \mathbb{R}^2_+ \to \mathbb{R}$ defined by

$$\begin{aligned} E(x,p) &= \frac{1}{2\pi}\iint_\Omega \left((\Delta w)^2 - (\Delta\sigma)^2 - [w,w]\sigma\right) dudv \\ &\quad -\frac{1}{2\pi}\iint_\Omega 2\alpha\left(\left(\frac{\partial w}{\partial u}\right)^2 + \left(\frac{\partial w}{\partial v}\right)^2\right) dudv \\ &\quad +\frac{1}{2\pi}\iint_\Omega \left(\beta w^2 - \frac{1}{2}\gamma w^4\right) dudv, \end{aligned} \tag{4}$$

with respect to the standard inner product in $L^2(\Omega) \times L^2(\Omega)$. Let $\Gamma = \{(0,p) : p \in \mathbb{R}^2_+\}$ be a subset of $X \times \mathbb{R}^2_+$. Every point in Γ is said to be *a trivial solution* of the equation (2). A point $(x,p) \in X \times \mathbb{R}^2_+$ such that $F(x,p) = 0$ and $x \neq 0$ is called *a nontrivial solution* of (2). We say that $(0,p) \in \Gamma$ is *a bifurcation point* of (2) (or there is bifurcation at $(0,p)$) if in every neighbourhood of this point there exists a nontrivial solution of (2). For $(0,p) \in \Gamma$, set

$$N(p) = \operatorname{Ker} F'_x(0,p).$$

A bifurcation point $(0,p) \in \Gamma$ is called either *simple* if $\dim N(p) = 1$ or *multiple* if $\dim N(p) \geq 2$. Applying the implicit function theorem we conclude that for bifurcation at a point $(0,p) \in \Gamma$ it is necessary that $\dim N(p) > 0$. In [12] we proved that $\dim N(p)$ is no greater than 4. We showed that if $\dim N(p) = 1$ then there exists bifurcation of the Crandall-Rabinowitz type at $(0,p)$. The proof was based on the Crandall-Rabinowitz theorem (see [8, 20]). In [13] we proved that a sufficient condition for bifurcation at $(0,p)$ is $\dim N(p) > 0$. In [15] we described the solution set of (1) in a small neighbourhood of a simple bifurcation point.

In this paper we discuss the case $\dim N(p) = 3$. Our investigations are based on S^1-symmetries. We notice that the subspace of S^1-equivariant functions in $N(p)$ is one-dimensional. It implies that $(0,p) \in \Gamma$ is a simple degeneracy point of the restriction of F to the subspace of S^1-equivariant functions in X. By the use of the Crandall-Rabinowitz theorem we prove that there is bifurcation of radial solutions at $(0,p)$. Next, applying the Lyapunov-Schmidt finite-dimensional reduction we study the number of branches of radial bifurcation at $(0,p)$.

In case $\dim N(p)$ is 2 or 4 this method breaks down, because the subspace of S^1-equivariant functions in $N(p)$ is not one-dimensional.

2 S^1-invariant subspaces in the space $N(p)$

At the beginning we introduce some notations. We will denote by S^1 the set $\{e^{i\Theta} : 0 \leq \Theta < 2\pi\}$. Obviously, S^1 with the multiplication of complex numbers is an abelian group. Define $G = \{T_\Theta : 0 \leq \Theta < 2\pi\}$, where $T_\Theta : \mathbb{R}^2 \to \mathbb{R}^2$ is a rotation through Θ. The group G is a linear representation of S^1 in $GL(\mathbb{R}^2)$.

Definition 2.1. *A set $U \subset \mathbb{R}^2$ is called S^1-invariant if $T_\Theta(u,v) \in U$ for all $(u,v) \in U$ and $\Theta \in [0,2\pi)$.*

Definition 2.2. *Let $U \subset \mathbb{R}^2$ be S^1-invariant. A map $f\colon U \to \mathbb{R}^n$ is said to be S^1-equivariant if $f(T_\Theta(u,v)) = f(u,v)$ for all $\Theta \in [0,2\pi)$ and $(u,v) \in U$.*

Property 2.1. *Let $U \subset \mathbb{R}^2$ be an S^1-invariant set. The following conditions are equivalent.*

(i) $f : U \to \mathbb{R}^n$ is an S^1-equivariant map.

(ii) There exists a map $g : \mathbb{R} \to \mathbb{R}^n$ such that $f(u,v) = g(\sqrt{u^2+v^2})$ for each $(u,v) \in U$.

Let $Z \subset \{f : U \to \mathbb{R}^n\}$ be a linear space, where $U \subset \mathbb{R}^2$ is S^1-invariant. We will denote by Z^{S^1} the subspace of all S^1-equivariant functions in Z, i.e.

$$Z^{S^1} = \{f \in Z : f \circ T_\Theta = f \text{ for each } \Theta \in [0,2\pi)\}.$$

Clearly, the unit ball Ω, its boundary $\partial\Omega$ and closure $\overline{\Omega}$ are S^1-invariant sets. Define

$$C_0^{m,\mu}\left(\overline{\Omega}\right) = \{f \in C^{m,\mu}\left(\overline{\Omega}\right) : f|_{\partial\Omega} = 0\}.$$

Let (r,φ) denote the polar coordinates of a point $(u,v) \in \overline{\Omega}$. It is well known that λ is an eigenvalue of $\Delta : C_0^{m,\mu}\left(\overline{\Omega}\right) \to C^{m-2,\mu}\left(\overline{\Omega}\right)$, $m \geq 2$, iff $\lambda < 0$ and $\sqrt{-\lambda}$ is zero of one of the Bessel functions

$$J_k(s) = \frac{1}{\pi}\int_0^\pi \cos(s\sin t - kt)\,dt, \quad k \in \mathbb{N} \cup \{0\}.$$

If $J_0(\sqrt{-\lambda}) = 0$ then $\dim \operatorname{Ker}(\Delta - \lambda I) = 1$ and $\operatorname{Ker}(\Delta - \lambda I) = \operatorname{span}\{J_0(\sqrt{-\lambda}r)\}$. If $J_k(\sqrt{-\lambda}) = 0$ and $k \neq 0$ then $\dim \operatorname{Ker}(\Delta - \lambda I) = 2$ and $\operatorname{Ker}(\Delta - \lambda I) = \operatorname{span}\{J_k(\sqrt{-\lambda}r)\cos(k\varphi), J_k(\sqrt{-\lambda}r)\sin(k\varphi)\}$. Here and subsequently, I stands for the natural embedding of $C^{m,\mu}\left(\overline{\Omega}\right)$ into $C^{m-2,\mu}\left(\overline{\Omega}\right)$ for $m \geq 2$, i.e. $I(x) = x$.

We now turn our attention to the space $N(p)$. It was computed in [12] that

$$F'_x(x,p)(z,\eta) = \left(\Delta^2 z - [z,\sigma] - [w,\eta] + 2\alpha\Delta z + \beta z - 3\gamma w^2 z, -\Delta^2\eta - [w,z]\right), \quad (5)$$

and so

$$F'_x(0,p)(z,\eta) = \left(\Delta^2 z + 2\alpha\Delta z + \beta z, -\Delta^2\eta\right) \quad (6)$$

for $z,\eta \in C_{0,0}^{4,\mu}(\overline{\Omega})$. One knows that $\Delta\colon C_0^{m,\mu}(\overline{\Omega}) \to C^{m-2,\mu}(\overline{\Omega})$, $m \geq 2$, is an isomorphism. Hence $\Delta^2\colon C_{0,0}^{4,\mu}(\overline{\Omega}) \to C^{0,\mu}(\overline{\Omega})$ is an isomorphism and, in consequence,

$$N(p) = \operatorname{Ker}(\Delta^2 + 2\alpha\Delta + \beta I) \times \{0\}, \quad (7)$$

where $\Delta^2 + 2\alpha\Delta + \beta I : C_{0,0}^{4,\mu}(\overline{\Omega}) \to C^{0,\mu}(\overline{\Omega})$. Fix $p = (\alpha,\beta) \in \mathbb{R}_+^2$. Set $\delta = \alpha^2 - \beta$. If $\delta \geq 0$ then a and b are defined as follows: $a = -\alpha - \sqrt{\delta}$, $b = -\alpha + \sqrt{\delta}$.

	Assumptions		Results		
	δ	a and b	$\dim N(p)$	base of $N(p)$	$\dim N(p)^{S^1}$
1.	$-$	not defined	0	$\emptyset$	0
2.	$+$ or 0	$\forall_{k\geq 0}$ $J_k\left(\sqrt{-a}\right)\neq 0$ $J_k\left(\sqrt{-b}\right)\neq 0$	0	$\emptyset$	0
3.	0	$J_0\left(\sqrt{-a}\right)=0$	1	$e_1(u,v)=\left(J_0\left(\sqrt{-a}r\right),0\right)$	1
4.	$+$	$\forall_{k\geq 0}$ $J_k\left(\sqrt{-b}\right)\neq 0$ $J_0\left(\sqrt{-a}\right)=0$	1	$e_1(u,v)=\left(J_0\left(\sqrt{-a}r\right),0\right)$	1
5.	$+$	$\forall_{k\geq 0}$ $J_k\left(\sqrt{-a}\right)\neq 0$ $J_0\left(\sqrt{-b}\right)=0$	1	$e_1(u,v)=\left(J_0\left(\sqrt{-b}r\right),0\right)$	1
6.	0	$\exists_{k>0}$ $J_k\left(\sqrt{-a}\right)=0$	2	$e_1(u,v)=\left(J_k\left(\sqrt{-a}r\right)\cos(k\varphi),0\right)$ $e_2(u,v)=\left(J_k\left(\sqrt{-a}r\right)\sin(k\varphi),0\right)$	0
7.	$+$	$J_0\left(\sqrt{-a}\right)=0$ $J_0\left(\sqrt{-b}\right)=0$	2	$e_1(u,v)=\left(J_0\left(\sqrt{-a}r\right),0\right)$ $e_2(u,v)=\left(J_0\left(\sqrt{-b}r\right),0\right)$	2
8.	$+$	$\forall_{l\geq 0}$ $J_l\left(\sqrt{-b}\right)\neq 0$ $\exists_{k>0}$ $J_k\left(\sqrt{-a}\right)=0$	2	$e_1(u,v)=\left(J_k\left(\sqrt{-a}r\right)\cos(k\varphi),0\right)$ $e_2(u,v)=\left(J_k\left(\sqrt{-a}r\right)\sin(k\varphi),0\right)$	0
9.	$+$	$\forall_{k\geq 0}$ $J_k\left(\sqrt{-a}\right)\neq 0$ $\exists_{l>0}$ $J_l\left(\sqrt{-b}\right)=0$	2	$e_1(u,v)=\left(J_l\left(\sqrt{-b}r\right)\cos(l\varphi),0\right)$ $e_2(u,v)=\left(J_l\left(\sqrt{-b}r\right)\sin(l\varphi),0\right)$	0
10.	$+$	$\exists_{k>0}$ $J_k\left(\sqrt{-a}\right)=0$ $J_0\left(\sqrt{-b}\right)=0$	3	$e_1(u,v)=\left(J_k\left(\sqrt{-a}r\right)\cos(k\varphi),0\right)$ $e_2(u,v)=\left(J_k\left(\sqrt{-a}r\right)\sin(k\varphi),0\right)$ $e_3(u,v)=\left(J_0\left(\sqrt{-b}r\right),0\right)$	1
11.	$+$	$\exists_{k>0}$ $J_k\left(\sqrt{-b}\right)=0$ $J_0\left(\sqrt{-a}\right)=0$	3	$e_1(u,v)=\left(J_0\left(\sqrt{-a}r\right),0\right)$ $e_2(u,v)=\left(J_k\left(\sqrt{-b}r\right)\cos(k\varphi),0\right)$ $e_3(u,v)=\left(J_k\left(\sqrt{-b}r\right)\sin(k\varphi),0\right)$	1
12.	$+$	$\exists_{k,l>0}$ $J_k\left(\sqrt{-a}\right)=0$ $J_l\left(\sqrt{-b}\right)=0$	4	$e_1(u,v)=\left(J_k\left(\sqrt{-a}r\right)\cos(k\varphi),0\right)$ $e_2(u,v)=\left(J_k\left(\sqrt{-a}r\right)\sin(k\varphi),0\right)$ $e_3(u,v)=\left(J_l\left(\sqrt{-b}r\right)\cos(l\varphi),0\right)$ $e_4(u,v)=\left(J_l\left(\sqrt{-b}r\right)\sin(l\varphi),0\right)$	0

Table 1.

Lemma 2.2 (see [12]). *Let* $\Delta - aI, \Delta - bI : C_0^{2,\mu}\left(\overline{\Omega}\right) \to C^{0,\mu}\left(\overline{\Omega}\right)$. *The following implications hold.*

(*i*) *If* $\delta < 0$ *then* $\mathrm{Ker}(\Delta^2 + 2\alpha\Delta + \beta I) = \{0\}$.

(*ii*) *If* $\delta = 0$ *then* $\mathrm{Ker}(\Delta^2 + 2\alpha\Delta + \beta I) = \mathrm{Ker}(\Delta - aI)$.

(*iii*) *If* $\delta > 0$ *then* $\mathrm{Ker}(\Delta^2 + 2\alpha\Delta + \beta I) = \mathrm{Ker}(\Delta - aI) \oplus \mathrm{Ker}(\Delta - bI)$.

Applying Lemma 2.2 and the description of eigenspaces of Δ on $\overline{\Omega}$ we receive the dimension and the base of $N(p)$. The results are announced in Table 1.

In Table 1 the character '$+$' means positive and the character '$-$' means negative. Combining the results of Table 1 with Property 2.1 we can determine the base of $N(p)^{S^1}$.

3 The properties of F and $F|_{X^{S^1}\times\mathbb{R}^2_+}$

Let $U \subset \mathbb{R}^2$ be S^1-invariant. Assume that $\Lambda \subset \mathbb{R}^k$ and $E_1, E_2 \subset \{f : U \to \mathbb{R}^n\}$ are real linear subspaces such that if $f \in E_i$ and $\Theta \in [0, 2\pi)$ then $f \circ T_\Theta \in E_i$ for $i = 1, 2$. We will say that

(*i*) $P : E_1 \to E_2$ is S^1-equivariant if $P(f \circ T_\Theta) = P(f) \circ T_\Theta$ for $\Theta \in [0, 2\pi)$ and $f \in E_1$;

(*ii*) $T : E_1 \times \Lambda \to E_2$ is S^1-equivariant if $T(\cdot, \lambda) : E_1 \to E_2$ is S^1-equivariant for each $\lambda \in \Lambda$.

Let F^{S^1} denote the restriction of F given by (3) to the space $X^{S^1} \times \mathbb{R}^2_+$. In this section we will look more closely at the operators F and F^{S^1}. Let us remark that if f belongs to a Hölder space then for each $\Theta \in [0, 2\pi)$ a function $f \circ T_\Theta$ lies in this space, too. It follows from the fact that $\overline{\Omega}$ and $\partial\Omega$ are S^1-invariant sets.

Theorem 3.1. *The operator* $F : X \times \mathbb{R}^2_+ \to Y$ *defined by* (3) *is* S^1*-equivariant, i.e.*

$$F(x \circ T_\Theta, p) = F(x, p) \circ T_\Theta$$

for all $x \in X$, $p \in \mathbb{R}^2_+$ *and* $\Theta \in [0, 2\pi)$.

Proof. It is known that the Laplace operator on $\overline{\Omega}$ is S^1-equivariant. Therefore it suffices to show that $[w \circ T_\Theta, \sigma \circ T_\Theta] = [w, \sigma] \circ T_\Theta$ for all $w,\ \sigma \in C_{0,0}^{4,\mu}\left(\overline{\Omega}\right)$ and $\Theta \in [0, 2\pi)$.

Fix $w,\ \sigma \in C_{0,0}^{4,\mu}\left(\overline{\Omega}\right)$ and $\Theta \in [0, 2\pi)$. Applying twice the theorem on the derivative of superposition we get

$$\begin{aligned}\frac{\partial^2 (w \circ T_\Theta)}{\partial u^2}(u, v) = {} & \frac{\partial^2 w}{\partial u^2}(T_\Theta(u, v)) \cos^2\Theta + 2\frac{\partial^2 w}{\partial u \partial v}(T_\Theta(u, v)) \sin\Theta\cos\Theta \\ & + \frac{\partial^2 w}{\partial v^2}(T_\Theta(u, v)) \sin^2\Theta,\end{aligned} \tag{8}$$

$$\frac{\partial^2 (w \circ T_\Theta)}{\partial v^2}(u,v) = \frac{\partial^2 w}{\partial u^2}(T_\Theta(u,v))\sin^2\Theta - 2\frac{\partial^2 w}{\partial u \partial v}(T_\Theta(u,v))\sin\Theta\cos\Theta + \frac{\partial^2 w}{\partial v^2}(T_\Theta(u,v))\cos^2\Theta \tag{9}$$

and

$$\frac{\partial^2 (w \circ T_\Theta)}{\partial u \partial v}(u,v) = -\frac{\partial^2 w}{\partial u^2}(T_\Theta(u,v))\sin\Theta\cos\Theta + 2\frac{\partial^2 w}{\partial u \partial v}(T_\Theta(u,v))\cos^2\Theta - 2\frac{\partial^2 w}{\partial u \partial v}(T_\Theta(u,v))\sin^2\Theta + \frac{\partial^2 w}{\partial v^2}(T_\Theta(u,v))\sin\Theta\cos\Theta \tag{10}$$

for $(u,v) \in \overline{\Omega}$. Combining (8), (9) and (10) we have $[w \circ T_\Theta, \sigma \circ T_\Theta](u,v) = [w,\sigma] \circ T_\Theta(u,v)$. ■

It is clear that subspaces of S^1-equivariant functions in the Hölder spaces are closed linear ones. Furthermore, they are mapped into spaces of S^1-equivariant functions by any S^1-equivariant operator. Since F is S^1-equivariant, we have $F^{S^1} : X^{S^1} \times \mathbb{R}^2_+ \to Y^{S^1}$.

The remainder of this section is devoted to the study of the Fréchet derivative of F with respect to x at a point $(0,p) \in X \times \mathbb{R}^2_+$.

Theorem 3.2. *The map $F^{S^1} : X^{S^1} \times \mathbb{R}^2_+ \to Y^{S^1}$ given by* (3) *is C^∞ with respect to all variables. Moreover, for each $p \in \mathbb{R}^2_+$, $(F^{S^1})'_x(0,p) : X^{S^1} \to Y^{S^1}$ is a linear Fredholm map of index* 0.

The proof of Theorem 3.2 is similar in spirit to the proof of Theorem 2.2 of [12].

Proof. Since F is C^∞, its restriction F^{S^1} is also C^∞. The task is now to check the second part of the claim. Fix $p \in \mathbb{R}^2_+$. We can write $F'_x(0,p)$ in the form

$$F'_x(0,p)(z,\eta) = A(z,\eta) + B(z,\eta), \tag{11}$$

where the operators A, $B : X \to Y$ are given as follows:

$$A(z,\eta) = \left(\Delta^2 z, -\Delta^2 \eta\right), \quad B(z,\eta) = (2\alpha\Delta z + \beta z, 0).$$

Define $A^{S^1} = A|_{X^{S^1}}$ and $B^{S^1} = B|_{X^{S^1}}$. In the proof of Theorem 2.2 of [12] we showed that A is a linear Fredholm map of index 0 and B is completely continuous. Since $\Delta^2 : C^{4,\mu}_{0,0}\left(\overline{\Omega}\right) \to C^{0,\mu}\left(\overline{\Omega}\right)$ is an S^1-equivariant isomorphism, we receive that $\Delta^2 : C^{4,\mu}_{0,0}\left(\overline{\Omega}\right)^{S^1} \to C^{0,\mu}\left(\overline{\Omega}\right)^{S^1}$ is one-to-one. We show that the restriction of Δ^2 to $C^{4,\mu}_{0,0}\left(\overline{\Omega}\right)^{S^1}$ is onto $C^{0,\mu}\left(\overline{\Omega}\right)^{S^1}$. Take $f \in C^{0,\mu}\left(\overline{\Omega}\right)^{S^1}$. There exists $g \in C^{4,\mu}_{0,0}\left(\overline{\Omega}\right)$ such that $\Delta^2 g = f$. We have

$$\Delta^2 g = \left(\Delta^2 g\right) \circ T_\Theta = \Delta^2 (g \circ T_\Theta)$$

for each $\Theta \in [0, 2\pi)$ and, in consequence, $g = g \circ T_\Theta$. From this $g \in C^{4,\mu}_{0,0}\left(\overline{\Omega}\right)^{S^1}$.

Summarizing, we have just proved that $A : X^{S^1} \to Y^{S^1}$ is an isomorphism. Thus $A : X^{S^1} \to Y^{S^1}$ is a Fredholm map of index 0. Additionally, $B : X^{S^1} \to Y^{S^1}$ is completely continuous, which follows immediately from two facts: (1) $B : X \to Y$ is completely continuous; (2) Y^{S^1} is a closed linear subspace of Y. From the above we deduce that

$$(F^{S^1})'_x(0,p) = A^{S^1} + B^{S^1}. \tag{12}$$

Therefore $(F^{S^1})'_x(0,p)$ is a linear Fredholm map of index 0. ■

4 Theorems on existence of radial bifurcation

We say that there is *radial bifurcation* at $(0,p) \in \Gamma$ if in every neighbourhood of this point there is a nontrivial radial solution of (2). A curve of nontrivial solutions starting with a bifurcation point is said to be *a branch of bifurcation*.

In Section 4 we formulate a sufficient condition for radial bifurcation at a point $(0,p) \in \Gamma$ such that $\dim N(p) = 3$. We also investigate the number of branches of nontrivial radial solutions bifurcating from such a point. Our proof is based on the Crandall-Rabinowitz theorem on simple bifurcation points (see [8, 20]) and the Lyapunov-Schmidt finite-dimensional reduction (see [20]).

Here and subsequently, $M_\epsilon(x)$ denotes an open ball of radius ϵ centered at x in a metric space M. For simplicity of notation, in general theorems we use the same letters F, E and X, Y for maps and spaces, respectively, as in von Kármán's problem.

Theorem 4.1 (Crandall, Rabinowitz)**.** *Let X, Y be real Banach spaces and F be a C^q map from a neighbourhood of $(x_0, \lambda_0) \in X \times \mathbb{R}$ into Y, where $q \geq 2$. Assume that*

(*i*) $F(x_0, \lambda_0) = 0$,

(*ii*) $F'_\lambda(x_0, \lambda_0) = 0$,

(*iii*) $\dim \operatorname{Ker} F'_x(x_0, \lambda_0) = 1$, $F'_x(x_0, \lambda_0)e = 0$, $e \neq 0$,

(*iv*) $\operatorname{codim} \operatorname{Im} F'_x(x_0, \lambda_0) = 1$,

(*v*) $F''_{\lambda\lambda}(x_0, \lambda_0) \in \operatorname{Im} F'_x(x_0, \lambda_0)$,

(*vi*) $F''_{x\lambda}(x_0, \lambda_0)e \notin \operatorname{Im} F'_x(x_0, \lambda_0)$.

Then the solution set of the equation $F(x, \lambda) = 0$ in a certain neighbourhood of (x_0, λ_0) is the union of two C^{q-2} curves Γ_1 and Γ_2 that intersect at (x_0, λ_0) only. Moreover, if $q \geq 3$ then

$$\Gamma_1 = \{(x_1(\lambda), \lambda) : \lambda \in \mathbb{R}_\epsilon(\lambda_0)\},\ x_1(\lambda_0) = x_0,\ x'_1(\lambda_0) = 0,$$

and

$$\Gamma_2 = \{(x_2(t), \lambda(t)) : t \in \mathbb{R}_\epsilon(0)\},\ x_2(0) = x_0,\ x'_2(0) = e,\ \lambda(0) = \lambda_0.$$

Theorem 4.2. *Let X, Y be real Banach spaces continuously embedded in a real Hilbert space H with scalar product $(\cdot,\cdot)_H : H \times H \to \mathbb{R}$ and let $E : X_\varrho(x_0) \times \mathbb{R}_\varrho(\lambda_0) \to \mathbb{R}$ be a C^{q+1} functional, where $q \geq 2$. Consider the equation*

$$F(x, \lambda) = 0 \tag{13}$$

with a real parameter λ, *where* $F : X_\varrho(x_0) \times \mathbb{R}_\varrho(\lambda_0) \to Y$ *belongs to the class* C^q. *Assume that*

(C_1) $F(x_0, \lambda) = 0$ *for every* $\lambda \in \mathbb{R}_\varrho(\lambda_0)$,

(C_2) $\dim \operatorname{Ker} F'_x(x_0, \lambda_0) = 1$, $F'_x(x_0, \lambda_0)e = 0$, $(e, e)_H = 1$,

(C_3) $\operatorname{codim} \operatorname{Im} F'_x(x_0, \lambda_0) = 1$,

(C_4) $E'_x(x, \lambda)h = (\, F(x, \lambda)\, ,\, h\,)_H$ *for all* $(x, \lambda) \in X_\varrho(x_0) \times \mathbb{R}_\varrho(\lambda_0)$ *and for each* $h \in X$,

(C_5) $E'''_{xx\lambda}(x_0, \lambda_0)\, ee \neq 0$.

Then the solution set of (13) *in a small neighbourhood of* (x_0, λ_0) *is the union of*

$$\Gamma_1 = \{(x_0, \lambda) : \lambda \in \mathbb{R}_\varrho(\lambda_0)\}$$

and the C^{q-2} *curve* Γ_2. Γ_1 *and* Γ_2 *intersect at* (x_0, λ_0) *only. Moreover, if* $q \geq 3$ *then* Γ_2 *is parametrized as follows:*

$$\Gamma_2 = \{(x(t), \lambda(t)) : t \in \mathbb{R}_\epsilon(0)\},$$

where $x(0) = x_0$, $\lambda(0) = \lambda_0$ *and* $x'(0) = e$.

Proof. It is sufficient to show that conditions $(C_1) - (C_5)$ imply conditions (i)-(vi). First we prove that

$$\operatorname{Ker} F'_x(x, \lambda) \perp \operatorname{Im} F'_x(x, \lambda) \tag{14}$$

for all $(x, \lambda) \in X_\varrho(x_0) \times \mathbb{R}_\varrho(\lambda_0)$. From (C_4) it follows that

$$E''_{xx}(x, \lambda)hg = (F'_x(x, \lambda)h, g)_H = (F'_x(x, \lambda)g, h)_H$$

for all $h, g \in X$. Hence for $h \in X$ and $g \in \operatorname{Ker} F'_x(x, \lambda)$ we get $(F'_x(x, \lambda)h, g)_H = (F'_x(x, \lambda)g, h)_H = (0, h)_H = 0$. Differentiating $E''_{xx}(x, \lambda)$ with respect to λ we receive

$$E'''_{xx\lambda}(x, \lambda)hg = (F''_{x\lambda}(x, \lambda)h, g)_H$$

for all $h, g \in X$. Thus $E'''_{xx\lambda}(x_0, \lambda_0)ee = (F''_{x\lambda}(x_0, \lambda_0)e, e)_H$. By (C_5) we have $(F''_{x\lambda}(x_0, \lambda_0)e, e)_H \neq 0$. From this and (14) we get $F''_{x\lambda}(x_0, \lambda_0)e \notin \operatorname{Im} F'_x(x_0, \lambda_0)$. Finally, (C_1) implies (i), (ii) and (v). ■

Let $H = L^2(\Omega) \times L^2(\Omega)$. The function $(\cdot, \cdot)_H : H \times H \to \mathbb{R}$ given by the formula

$$((z, \eta), (z_1, \eta_1))_H = \frac{1}{\pi} \iint_\Omega (zz_1 + \eta\eta_1)\, dudv \tag{15}$$

is an inner product in H. Furthermore, the pair $(H, (\cdot, \cdot)_H)$ is a Hilbert space. The Banach spaces $X = C^{4,\mu}_{0,0}\left(\overline{\Omega}\right) \times C^{4,\mu}_{0,0}\left(\overline{\Omega}\right)$ and $Y = C^{0,\mu}\left(\overline{\Omega}\right) \times C^{0,\mu}\left(\overline{\Omega}\right)$ are easily checked to be continuously embedded in H. Hence their closed linear subspaces X^{S^1} and Y^{S^1} are also Banach spaces continuously embedded in H. In [12] we showed that for each $p \in \mathbb{R}^2_+$ the map $F(\cdot, p) : X \to Y$ defined by (3) is a variational gradient of the functional $E(\cdot,p) : X \to \mathbb{R}$ defined by (4) with respect to the scalar product in H, i.e.

$$E'_x(x,p)h = (F(x,p),h)_H \tag{16}$$

for x, $h \in X$ (see Theorem 2.4 of [12]). From now on, we will denote by E^{S^1} the restriction of E to the space $X^{S^1} \times \mathbb{R}^2_+$. Let us note the important consequence of the above fact.

Conclusion 4.3. *For each $p \in \mathbb{R}^2_+$ the map $F^{S^1}(\cdot,p) : X^{S^1} \to Y^{S^1}$ is a variational gradient of the functional $E^{S^1}(\cdot,p) : X^{S^1} \to \mathbb{R}$ with respect to the inner product in H, i.e.*

$$(E^{S^1})'_x(x,p)h = \Big(F^{S^1}(x,p),h\Big)_H \tag{17}$$

for x, $h \in X^{S^1}$.

Theorem 4.4. *Let $p_0 = (\alpha_0,\beta_0) \in \mathbb{R}^2_+$ satisfy the following condition*

$$\dim N(p_0) = 3, \quad (F^{S^1})'_x(0,p_0)e = 0, \quad (e,e)_H = 1, \quad e = (e_1,0). \tag{18}$$

Then $(0,\alpha_0) \in X^{S^1} \times \mathbb{R}_+$ is a bifurcation point of the equation

$$F^{S^1}(x,\alpha,\beta_0) = 0. \tag{19}$$

The solution set of (19) *in a small neighbourhood of* $(0,\alpha_0)$ *is the union of the curve of trivial solutions*

$$\Gamma_{1,\alpha} = \{(0,\alpha) : \alpha \in \mathbb{R}_+\}$$

and the C^∞ curve $\Gamma_{2,\alpha}$. $\Gamma_{1,\alpha}$ and $\Gamma_{2,\alpha}$ intersect at $(0,\alpha_0)$ only. Moreover, $\Gamma_{2,\alpha}$ is parametrized as follows:

$$\Gamma_{2,\alpha} = \{(x(t),\alpha(t)) : t \in \mathbb{R}_\epsilon(0)\},$$

where $x(0) = 0$, $\alpha(0) = \alpha_0$ and $x'(0) = e$.

From (7) it follows that if $e \in N(p_0)$ then $e = (e_1,0)$ and $e_1 \in \mathrm{Ker}(\Delta + 2\alpha_0\Delta + \beta_0 I)$.

Proof. As α_0 is positive, there exists $\varrho > 0$ such that $\mathbb{R}_\varrho(\alpha_0) \subset \mathbb{R}_+$. We verify that the operator $F^{S^1}(\cdot,\cdot,\beta_0) : X^{S^1}_\varrho(0) \times \mathbb{R}_\varrho(\alpha_0) \to Y^{S^1}$ satisfies the assumptions of Theorem 4.2. Substituting $p = (\alpha,\beta_0)$ and $x = 0$ into (3) we get

$$F^{S^1}(0,\alpha,\beta_0) = 0$$

for each $\alpha \in \mathbb{R}_\varrho(\alpha_0)$. By Theorem 3.2, the map $F^{S^1}(\cdot,\alpha_0,\beta_0) : X^{S^1} \to Y^{S^1}$ is C^∞ and $(F^{S^1})'_x(0,\alpha_0,\beta_0) : X^{S^1} \to Y^{S^1}$ is a Fredholm map of index 0. Therefore

$$\dim N(p_0)^{S^1} = \operatorname{codim} \mathrm{Im}(F^{S^1})'_x(0,\alpha_0,\beta_0). \tag{20}$$

From Table 1 it follows that

$$\dim N(p_0)^{S^1} = 1. \tag{21}$$

Combining (21) with (20) we have

$$\operatorname{codim} \mathrm{Im}(F^{S^1})'_x(0,\alpha_0,\beta_0) = 1.$$

Conclusion 4.3 says that for each $\alpha \in \mathbb{R}_+$ and for all $x,\ h \in X^{S^1}$

$$(E^{S^1})'_x(x,\alpha,\beta_0)h = \Big(F^{S^1}(x,\alpha,\beta_0),\ h\Big)_H . \tag{22}$$

Notice that we have just proved that assumptions $(C_1) - (C_4)$ of Theorem 4.2 are fulfilled. To finish the proof we have to show that assumption (C_5) of Theorem 4.2 holds. Since the spaces X^{S^1}, Y^{S^1} are continuously embedded in H, differentiating both sides of the equality (22) with respect to x we obtain

$$(E^{S^1})''_{xx}(x,\alpha,\beta_0)hg = \Big((F^{S^1})'_x(x,\alpha,\beta_0)h,\ g\Big)_H \tag{23}$$

for $x,\ h,\ g \in X^{S^1}$ and $\alpha \in \mathbb{R}_+$. Applying (5), (15) and (23) we have

$$\begin{aligned}(E^{S^1})''_{xx}(x,\alpha,\beta_0)hg = {} & \frac{1}{\pi}\iint_\Omega \Big(\Delta^2 z - [z,\sigma] - [w,\eta] + 2\alpha\Delta z + \beta_0 z - 3\gamma w^2 z\Big)\, z_1 dudv \\ & + \frac{1}{\pi}\iint_\Omega \Big(-\Delta^2\eta - [w,z]\Big)\,\eta_1 dudv,\end{aligned}$$

where $x = (w,\sigma)$, $h = (z,\eta)$, $g = (z_1,\eta_1)$. Hence

$$(E^{S^1})'''_{xx\alpha}(x,\alpha,\beta_0)hg = \frac{1}{\pi}\iint_\Omega 2\,(\Delta z)\, z_1 dudv.$$

Taking $x = 0$, $\alpha = \alpha_0$ and $h = g = e$, we get

$$(E^{S^1})'''_{xx\alpha}(0,\alpha_0,\beta_0)ee = \frac{1}{\pi}\iint_\Omega 2\,(\Delta e_1)\, e_1 dudv.$$

By the assumption $\delta_0 = \alpha_0^2 - \beta_0 > 0$ (see Table 1). From Lemma 2.2

$$\operatorname{Ker}\Big(\Delta^2 + 2\alpha_0\Delta + \beta_0 I\Big) = \operatorname{Ker}(\Delta - a_0 I) \oplus \operatorname{Ker}(\Delta - b_0 I),$$

where $\Delta^2 + 2\alpha_0\Delta + \beta_0 I : C^{4,\mu}_{0,0}(\overline{\Omega}) \to C^{0,\mu}(\overline{\Omega})$, $\Delta - a_0 I$, $\Delta - b_0 I : C^{2,\mu}_0(\overline{\Omega}) \to C^{0,\mu}(\overline{\Omega})$, $a_0 = -\alpha_0 - \sqrt{\delta_0}$ and $b_0 = -\alpha_0 + \sqrt{\delta_0}$. We can choose e so that $\Delta e_1 - a_0 e_1 = 0$ or $\Delta e_1 - b_0 e_1 = 0$ (see Table 1). If $\Delta e_1 - a_0 e_1 = 0$ then

$$(E^{S^1})'''_{xx\alpha}(0,\alpha_0,\beta_0)ee = \frac{2a_0}{\pi}\iint_\Omega e_1^2 dudv = 2a_0\,(e,e)_H = 2a_0 < 0. \tag{24}$$

If $\Delta e_1 - b_0 e_1 = 0$ then

$$(E^{S^1})'''_{xx\alpha}(0,\alpha_0,\beta_0)ee = 2b_0 < 0, \tag{25}$$

which completes the proof. ∎

Let $(0,p_0) \in \Gamma$ satisfy (18). From Theorem 4.4 it follows that $(0,p_0)$ is a bifurcation point of the equation (2). What is more, at least two C^∞ branches of nontrivial radial solutions bifurcate from this point. The union of this branches is the curve $\Gamma_{2,\alpha}$.

Theorem 4.4 refers to bifurcation with respect to α. Our purpose now is to prove an analogical theorem on bifurcation with respect to β.

Theorem 4.5. *Let $p_0 = (\alpha_0, \beta_0) \in \mathbb{R}^2_+$ satisfy the condition* (18). *Then* $(0, \beta_0) \in X^{S^1} \times \mathbb{R}_+$ *is a bifurcation point of the equation*

$$F^{S^1}(x, \alpha_0, \beta) = 0. \tag{26}$$

The solution set of (26) *in a small neighbourhood of* $(0, \beta_0)$ *is the union of the curve of trivial solutions*

$$\widehat{\Gamma}_{1,\beta} = \{(0, \beta) : \beta \in \mathbb{R}_+\}$$

and the C^∞ *curve* $\widehat{\Gamma}_{2,\beta}$. $\widehat{\Gamma}_{1,\beta}$ *and* $\widehat{\Gamma}_{2,\beta}$ *intersect at* $(0, \beta_0)$ *only. Moreover,* $\widehat{\Gamma}_{2,\beta}$ *is parametrized as follows:*

$$\widehat{\Gamma}_{2,\beta} = \{(\widehat{x}(t), \beta(t)) : t \in \mathbb{R}_\epsilon(0)\},$$

where $\widehat{x}(0) = 0$, $\beta(0) = \beta_0$ *and* $\widehat{x}'(0) = e$.

Proof. The proof is also based on Theorem 4.2. Take $\varrho > 0$ such that $\mathbb{R}_\varrho(\beta_0) \subset \mathbb{R}_+$. Considerations similar to those in the proof of Theorem 4.4 show that the map $F^{S^1}(\cdot, \alpha_0, \cdot) : X^{S^1}_\varrho(0) \times \mathbb{R}_\varrho(\beta_0) \to Y^{S^1}$ satisfies assumptions $(C_1) - (C_4)$ of Theorem 4.2. The details are left to the reader. The task is now to check assumption (C_5). From Conclusion 4.3 we get that for each $\beta \in \mathbb{R}_+$ and for all x, $h \in X^{S^1}$

$$(E^{S^1})'_x(x, \alpha_0, \beta)h = \left(F^{S^1}(x, \alpha_0, \beta),\ h\right)_H.$$

Hence

$$(E^{S^1})''_{xx}(x, \alpha_0, \beta)hg = \left((F^{S^1})'_x(x, \alpha_0, \beta)h,\ g\right)_H. \tag{27}$$

Using (5), (15) and (27) we obtain

$$\begin{aligned}(E^{S^1})''_{xx}(x, \alpha_0, \beta)hg = {} & \frac{1}{\pi} \iint_\Omega \left(\Delta^2 z - [z, \sigma] - [w, \eta] + 2\alpha_0 \Delta z + \beta z - 3\gamma w^2 z\right) z_1 dudv \\ & + \frac{1}{\pi} \iint_\Omega \left(-\Delta^2 \eta - [w, z]\right) \eta_1 dudv, \end{aligned} \tag{28}$$

where $x = (w, \sigma)$, $h = (z, \eta)$, $g = (z_1, \eta_1)$. Differentiating (28) with respect to β we have

$$(E^{S^1})'''_{xx\beta}(x, \alpha_0, \beta)hg = \frac{1}{\pi} \iint_\Omega z z_1 dudv.$$

In particular,

$$(E^{S^1})'''_{xx\beta}(0, \alpha_0, \beta_0)ee = \frac{1}{\pi} \iint_\Omega e_1^2 dudv = (e, e)_H = 1 > 0, \tag{29}$$

which completes the proof. ■

Fix $(0, p_0) \in \Gamma$ such that $\dim N(p_0) = 3$, $p_0 = (\alpha_0, \beta_0)$. Let us remark that if $\Gamma_{2,\alpha} \cap \widehat{\Gamma}_{2,\beta} = \{(0, p_0)\}$ then at least four C^∞ branches of nontrivial radial solutions bifurcate from $(0, p_0)$. Therefore the next question is whether the curves $\Gamma_{2,\alpha}$ and $\widehat{\Gamma}_{2,\beta}$ intersect at $(0, p_0)$ only.

In order to answer this question we apply a finite-dimensional reduction of the Lyapunov-Schmidt type with the key function due to Sapronov (see [14, 22]).

Let $G : X^{S^1} \times \mathbb{R} \times \mathbb{R}_+ \to Y^{S^1}$ be given by

$$G(x, \xi, \alpha) = F^{S^1}(x, \alpha, \beta_0) + (\xi - (x, e)_H)\, e.$$

It is easy to check that $G'_x(0, 0, \alpha_0) : X^{S^1} \to Y^{S^1}$ is an isomorphism. By the implicit function theorem there exist $\epsilon > 0$ and a map $\tilde{x} : \mathbb{R}_\varepsilon(0) \times \mathbb{R}_\varepsilon(\alpha_0) \to X^{S^1}_\varepsilon(0)$ such that $\tilde{x}(0, \alpha_0) = 0$ and for every $(x, \xi, \alpha) \in X^{S^1}_\varepsilon(0) \times \mathbb{R}_\varepsilon(0) \times \mathbb{R}_\varepsilon(\alpha_0)$ we have $G(x, \xi, \alpha) = 0$ iff $x = \tilde{x}(\xi, \alpha)$. Hence

$$G(\tilde{x}(\xi, \alpha), \xi, \alpha) = 0. \tag{30}$$

Furthermore, $\tilde{x}'_\xi(0, \alpha_0) = e$ and $\tilde{x}(0, \alpha) = 0$ for all $|\alpha - \alpha_0| < \varepsilon$. Thus

$$\tilde{x}(\xi, \alpha) = \xi e + o(\sqrt{\xi^2 + (\alpha - \alpha_0)^2}). \tag{31}$$

Let us define $\varphi, \Phi : \mathbb{R}_\varepsilon(0) \times \mathbb{R}_\varepsilon(\alpha_0) \to \mathbb{R}$ as follows:

$$\varphi(\xi, \alpha) = \xi - (\tilde{x}(\xi, \alpha), e)_H$$

and

$$\Phi(\xi, \alpha) = -E^{S^1}(\tilde{x}(\xi, \alpha), \alpha, \beta_0) + \frac{1}{2}\varphi^2(\xi, \alpha).$$

$\Phi(\xi, \alpha)$ is called *a key function*. Both Φ and φ are C^∞-smooth. We also have

$$G(\tilde{x}(\xi, \alpha), \xi, \alpha) = F^{S^1}(\tilde{x}(\xi, \alpha), \alpha, \beta_0) + \varphi(\xi, \alpha)e. \tag{32}$$

Differentiating φ and Φ with respect to ξ we receive

$$\varphi'_\xi(\xi, \alpha) = 1 - \left(\tilde{x}'_\xi(\xi, \alpha), e\right)_H$$

and

$$\begin{aligned}
\Phi'_\xi(\xi, \alpha) &= -(E^{S^1})'_x(\tilde{x}(\xi, \alpha), \alpha, \beta_0)\tilde{x}'_\xi(\xi, \alpha) + \varphi(\xi, \alpha)\varphi'_\xi(\xi, \alpha) \\
&= -\left(F^{S^1}(\tilde{x}(\xi, \alpha), \alpha, \beta_0), \tilde{x}'_\xi(\xi, \alpha)\right)_H + \varphi(\xi, \alpha) - \left(\varphi(\xi, \alpha)e, \tilde{x}'_\xi(\xi, \alpha)\right)_H \\
&= -\left(G(\tilde{x}(\xi, \alpha), \xi, \alpha), \tilde{x}'_\xi(\xi, \alpha)\right)_H + \varphi(\xi, \alpha) \\
&= \varphi(\xi, \alpha),
\end{aligned}$$

by (17) and (30). From (30) and (32) we conclude that all solutions of the equation (19) in a small neighbourhood of $(0, \alpha_0)$ in $X^{S^1} \times \mathbb{R}_+$ are of the form $(\tilde{x}(\xi, \alpha), \alpha)$ and

$$F^{S^1}(\tilde{x}(\xi, \alpha), \alpha, \beta_0) = 0 \iff \Phi'_\xi(\xi, \alpha) = 0 \iff \varphi(\xi, \alpha) = 0. \tag{33}$$

We describe now the solution set of the equation

$$\varphi(\xi, \alpha) = 0$$

in a small neighbourhood of $(0, \alpha_0)$ in $\mathbb{R} \times \mathbb{R}_+$. For this purpose we use the Taylor formula of φ at $(0, \alpha_0)$. From (32) it follows that

$$\left(F^{S^1}(\tilde{x}(\xi, \alpha), \alpha, \beta_0) + \varphi(\xi, \alpha)e, e\right)_H = 0,$$

hence
$$\varphi(\xi,\alpha) = -\left(F^{S^1}(\tilde{x}(\xi,\alpha),\alpha,\beta_0),e\right)_H,$$
and by (17)
$$\varphi(\xi,\alpha) = -(E^{S^1})'_x(\tilde{x}(\xi,\alpha),\alpha,\beta_0)e. \tag{34}$$
Applying (4) and (34) we get
$$\begin{aligned}
C_1 &:= \varphi'_\xi(0,\alpha_0) = -(E^{S^1})''_{xx}(0,p_0)ee = 0,\\
C_{11} &:= \varphi''_{\xi\xi}(0,\alpha_0) = -(E^{S^1})'''_{xxx}(0,p_0)eee = 0,\\
C_{12} &:= \varphi''_{\xi\alpha}(0,\alpha_0) = -(E^{S^1})'''_{xx\alpha}(0,p_0)ee,\\
C_{111} &:= \varphi'''_{\xi\xi\xi}(0,\alpha_0) = -(E^{S^1})^{(4)}_{xxxx}(0,p_0)eeee - 3(E^{S^1})'''_{xxx}(0,p_0)yee,
\end{aligned}$$
where $y=(y_1,y_2)=\tilde{x}''_{\xi\xi}(0,\alpha_0)$ is a solution of the equation
$$(F^{S^1})''_{xx}(0,p_0)ee + (F^{S^1})'_x(0,p_0)y = 0.$$
By (24) and (25) we have $C_{12}>0$. An easy calculation shows that
$$C_{111} = \frac{6}{\pi}\iint_\Omega \gamma e_1^4 dudv - \frac{3}{\pi}\iint_\Omega (\Delta y_2)^2 dudv.$$
Set
$$\begin{aligned}
C_{112} &:= \varphi'''_{\xi\xi\alpha}(0,\alpha_0),\\
C_{122} &:= \varphi'''_{\xi\alpha\alpha}(0,\alpha_0).
\end{aligned}$$
Since $\varphi(0,\alpha)=0$ for all $|\alpha-\alpha_0|<\epsilon$, we have
$$\varphi^{(k)}_{\alpha\ldots\alpha}(0,\alpha_0) = 0$$
for every $k\in\mathbb{N}$. In consequence,
$$\begin{aligned}
\varphi(\xi,\alpha) &= C_{12}\xi(\alpha-\alpha_0) + \frac{1}{6}C_{111}\xi^3 + \frac{1}{2}C_{112}\xi^2(\alpha-\alpha_0) + \frac{1}{2}C_{122}\xi(\alpha-\alpha_0)^2\\
&\quad + o\left(\sqrt{\xi^2+(\alpha-\alpha_0)^2}^3\right)\\
&= C_{12}\xi(\alpha-\alpha_0) + \frac{1}{6}C_{111}\xi^3 + \frac{1}{2}C_{112}\xi^2(\alpha-\alpha_0) + \frac{1}{2}C_{122}\xi(\alpha-\alpha_0)^2\\
&\quad + \xi f(\xi,\alpha)
\end{aligned}$$
where $f:\mathbb{R}_\epsilon(0)\times\mathbb{R}_\varepsilon(\alpha_0)\to\mathbb{R}$ is a C^∞ function such that $f(0,\alpha_0)=0$, $f'_\alpha(0,\alpha_0)=0$ and $f'_\xi(0,\alpha_0)=f''_{\xi\xi}(0,\alpha_0)=0$. Let $g:\mathbb{R}_\epsilon(0)\times\mathbb{R}_\varepsilon(\alpha_0)\to\mathbb{R}$ be given by
$$g(\xi,\alpha) = C_{12}(\alpha-\alpha_0) + \frac{1}{6}C_{111}\xi^2 + \frac{1}{2}C_{112}\xi(\alpha-\alpha_0) + \frac{1}{2}C_{122}(\alpha-\alpha_0)^2 + f(\xi,\alpha).$$
Then
$$\varphi(\xi,\alpha)=0 \iff \xi=0 \ \vee\ g(\xi,\alpha)=0.$$
We check at once that $g(0,\alpha_0)=0$, $g'_\xi(0,\alpha_0)=0$ and $g'_\alpha(0,\alpha_0)=C_{12}>0$. By the implicit function theorem there exists a C^∞ function $\tilde{\alpha}:\mathbb{R}_\rho(0)\to\mathbb{R}_\rho(\alpha_0)$, $0<\rho<\epsilon$ such that $\tilde{\alpha}(0)=\alpha_0$ and for all $(\xi,\alpha)\in\mathbb{R}_\rho(0)\times\mathbb{R}_\rho(\alpha_0)$ we have
$$g(\xi,\alpha)=0 \iff \alpha=\tilde{\alpha}(\xi).$$

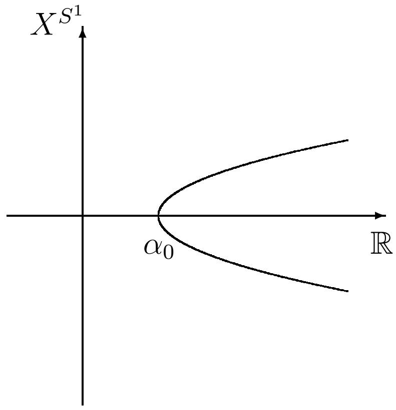


Figure 1: The scheme of postcritical bifurcation

Differentiating the equality $g(\xi, \tilde{\alpha}(\xi)) = 0$ with respect to ξ we get

$$\tilde{\alpha}'(0) = -\frac{g'_\xi(0, \alpha_0)}{g'_\alpha(0, \alpha_0)} = 0$$

and

$$\tilde{\alpha}''(0) = -\frac{g''_{\xi\xi}(0, \alpha_0)}{g'_\alpha(0, \alpha_0)} = -\frac{C_{111}}{3C_{12}}.$$

Remark that

$$\tilde{\alpha}''(0) \neq 0 \quad \Longleftrightarrow \quad C_{111} \neq 0 \quad \Longleftrightarrow \quad \gamma \neq \frac{\iint_\Omega (\Delta y_2)^2 dudv}{2 \iint_\Omega e_1^4 dudv}.$$

If $C_{111} < 0$ then $\tilde{\alpha}''(0) > 0$ and $\tilde{\alpha}$ achieves the minimum at 0. Moreover, there exists $0 < \rho_1 < \rho$ such that $\tilde{\alpha}$ is strictly decreasing for $\xi \in (-\rho_1, 0]$ and it is strictly increasing for $\xi \in [0, \rho_1)$. Hence there is $0 < \rho_2 < \rho$ and there are C^∞ functions $\xi_1 : [\alpha_0, \alpha_0 + \rho_2) \to (-\rho_1, 0]$ and $\xi_2 : [\alpha_0, \alpha_0 + \rho_2) \to [0, \rho_1)$ such that $\xi_i = \tilde{\alpha}^{-1}$ for $i = 1, 2$. From this, (31) and (33) we conclude that if $C_{111} < 0$ then there is postcritical bifurcation in the solution set of (19) at the point $(0, \alpha_0)$ (see Figure 1). All nontrivial solutions of (19) in a small neighbourhood of $(0, \alpha_0)$ lie on the curve

$$x = \tilde{x}(\xi_i(\alpha), \alpha), \quad \alpha \in [\alpha_0, \alpha_0 + \rho_2).$$

If $C_{111} > 0$ then $\tilde{\alpha}''(0) < 0$ and $\tilde{\alpha}$ achieves the maximum at 0. Moreover, there exists $0 < \rho_1 < \rho$ such that $\tilde{\alpha}$ is strictly increasing for $\xi \in (-\rho_1, 0]$ and it is strictly decreasing for $\xi \in [0, \rho_1)$. Hence there is $0 < \rho_2 < \rho$ and there are C^∞ functions $\xi_1 : (\alpha_0 - \rho_2, \alpha_0] \to (-\rho_1, 0]$ and $\xi_2 : (\alpha_0 - \rho_2, \alpha_0] \to [0, \rho_1)$ such that $\xi_i = \tilde{\alpha}^{-1}$ for $i = 1, 2$. Consequently, if $C_{111} > 0$ then there is subcritical bifurcation in the solution set of (19) at the point $(0, \alpha_0)$ (see Figure 2). All nontrivial solutions of (19) in a small neighbourhood of $(0, \alpha_0)$ lie on the curve

$$x = \tilde{x}(\xi_i(\alpha), \alpha), \quad \alpha \in (\alpha_0 - \rho_2, \alpha_0].$$

Similarly, we can prove that if $C_{111} > 0$ (resp. $C_{111} < 0$) then there is postcritical bifurcation (resp. subcritical bifurcation) in the solution set of (26) at the point

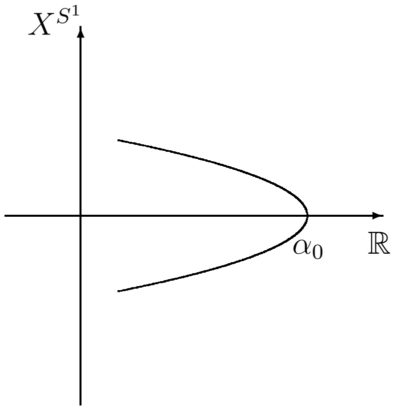


Figure 2: The scheme of subcritical bifurcation

$(0, \beta_0)$. It is sufficient to make a finite-dimensional reduction with $G : X^{S^1} \times \mathbb{R} \times \mathbb{R}_+ \to Y^{S^1}$ defined by

$$G(x, \xi, \beta) = F^{S^1}(x, \alpha_0, \beta) + (\xi - (x, e)_H)\, e$$

and check that $C_{12} := -(E^{S^1})'''_{xx\beta}(0, p_0)ee < 0$ (see (29)).

Summarizing, we have just proved the following result.

Theorem 4.6. *If* $C_{111} \neq 0$ *then* $\Gamma_{2,\alpha} \cap \widehat{\Gamma}_{2,\beta} = \{(0, p_0)\}$. *Another words, at* $(0, p_0)$ *at least four* C^∞ *branches of nontrivial radial solutions of* (2) *meet, causing the plate to choose between different forms of equilibrium.*

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