A note on quasinormable weighted Fréchet spaces of holomorphic functions

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Abstract

A characterization in terms of the weights of quasinormable weighted Fréchet spaces of holomorphic functions on the disc is presented, when the weights are radial and grow logarithmically.

1 Introduction and Notation

This article studies quasinormable weighted Fréchet spaces of holomorphic functions $HW(\mathbb{D})$ resp. $HW_0(\mathbb{D})$. For an increasing sequence $W = (w_n)_{n\in\mathbb{N}}$ of strictly positive continuous functions (weights) on the open unit disc $\mathbb{D}$ of $\mathbb{C}$ we consider the projective limit of the Banach spaces $Hw_n(\mathbb{D}) := \{f \in H(\mathbb{D}); \|f\|_n := \sup_{z\in\mathbb{D}} w_n(z)|f(z)| < \infty\}$ resp. $H(w_n)_0(\mathbb{D}) := \{f \in H(\mathbb{D}); w_n f$ vanishes at ∞ on $\mathbb{D}\}$, $n \in \mathbb{N}$.
Under rather general assumptions we obtained in [19] a necessary condition for quasinormability in terms of the sequence of weights (which are considered as growth conditions in the sense of [6]) and their associated growth conditions. A method of Bonet-Ěnglis-Taskinen (see [9]) is used to see that, under some restrictions on the weights, the necessary condition is also sufficient. This result complements those we obtained in [19] for radial weights satisfying a polynomial growth condition.
The class of quasinormable Fréchet spaces was studied by Grothendieck in [10] as a class "containing the most usual Fréchet function spaces" (see [10] p. 107). In the case of Köthe echelon spaces quasinormability was studied by Bierstedt-Meise-Summers [8], Meise-Vogt [13], Valdivia [16], [17] and Vogt [18]. For weighted Fréchet spaces of continuous functions Bierstedt-Meise [7] and Bastin-Ernst [2] obtained a characterization in terms of the weights. In fact, we get a result similar to the one of

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Bastin-Ernst. In contrast to the case of continuous functions the so called *associated growth conditions* mentioned by Anderson and Duncan in [1] and studied thoroughly by Bierstedt-Bonet-Taskinen in [6] are needed to get the characterization. The sufficiency is obtained here under the assumption that the systems of weights satisfy the condition (LOG) of Bonet-Ěnglis-Taskinen (see [9]), see the details below.

Our notation on locally convex spaces is standard; see for example Jarchow [11], Köthe [12], Meise-Vogt [14] and Pérez Carreras-Bonet [15]. For a locally convex space E, E' is the topological dual and E'_b the strong dual. If E is a locally convex space, $\mathcal{U}_0(E)$ and $\mathcal{B}(E)$ stand for the families of all absolutely convex 0-neighborhoods and absolutely convex bounded sets in E, respectively.
A locally convex space E is called *quasinormable* if

$$\forall U \in \mathcal{U}_0(E)\ \exists V \in \mathcal{U}_0(E)\ \forall \lambda > 0\ \exists B \in \mathcal{B}(E):\ V \subset B + \lambda U.$$

Each normed space is quasinormable. By [14, Lemma 26.14] a Fréchet space E with a 0-neighborhood base $(U_n)_{n\in\mathbb{N}}$ is quasinormable if and only if

$$\forall n \in \mathbb{N}\ \exists m > n\ \forall k \geq n\ \forall \varepsilon > 0\ \exists \delta > 0:\ U_m \subset \delta U_k + \varepsilon U_n.$$

In the sequel $\mathbb{D}$ denotes the open unit disc of $\mathbb{C}$. The space $H(\mathbb{D})$ of all holomorphic functions on $\mathbb{D}$ is endowed with the topology *co* of uniform convergence on the compact subsets of $\mathbb{D}$. Let $W = (w_n)_{n\in\mathbb{N}}$ be an increasing sequence of strictly positive continuous functions on $\mathbb{D}$. For every $n \in \mathbb{N}$ the spaces

$$\begin{aligned} Hw_n(\mathbb{D}) &:= \{f \in H(\mathbb{D});\ \|f\|_n := \sup_{z\in\mathbb{D}} w_n(z)|f(z)| < \infty\} \text{ and} \\ H(w_n)_0(\mathbb{D}) &:= \{f \in H(\mathbb{D}); w_n f \text{ vanishes at } \infty \text{ on } \mathbb{D}\} \end{aligned}$$

endowed with the norm $\|.\|_n$ are Banach spaces. The weighted Fréchet spaces of holomorphic functions are defined by

$$HW(\mathbb{D}) := \mathrm{proj}_n Hw_n(\mathbb{D}) \text{ and } HW_0(\mathbb{D}) := \mathrm{proj}_n H(w_n)_0(\mathbb{D}).$$

For each $n \in \mathbb{N}$, let B_n, resp. $B_{n,0}$, be the closed unit ball of $Hw_n(\mathbb{D})$, resp. $H(w_n)_0(\mathbb{D})$, and $C_n := B_n \cap HW(\mathbb{D})$, resp. $C_{n,0} := B_{n,0} \cap HW_0(\mathbb{D})$. By $\overline{B_n}$, $\overline{B_{n,0}}$, $\overline{C_n}$, $\overline{C_{n,0}}$ we denote the *co*-closures of the corresponding sets. The sequence $\left(\frac{1}{n}C_n\right)_{n\in\mathbb{N}}$, resp. $\left(\frac{1}{n}C_{n,0}\right)_{n\in\mathbb{N}}$, constitutes a 0-neighborhood base of $HW(\mathbb{D})$, resp. $HW_0(\mathbb{D})$. Without loss of generality we may assume that $(C_n)_{n\in\mathbb{N}}$, resp. $(C_{n,0})_{n\in\mathbb{N}}$, is a 0-neighborhood base. Put

$$\overline{W} := \{\overline{w} : \mathbb{D} \to]0, \infty[;\ \overline{w} \text{ continuous on } \mathbb{D},\ w_n\overline{w} \text{ is bounded on } \mathbb{D}\ \forall n \in \mathbb{N}\},$$

and $C_{\overline{w}} := \{f \in HW(\mathbb{D}); |f| \leq \overline{w} \text{ on } \mathbb{D}\}$, resp. $C_{\overline{w},0} := C_{\overline{w}} \cap HW_0(\mathbb{D})$, $\overline{w} \in \overline{W}$. We write $\overline{C_{\overline{w}}}$ and $\overline{C_{\overline{w},0}}$ to refer to the *co*-closure. $(C_{\overline{w}})_{\overline{w}\in\overline{W}}$, resp. $(C_{\overline{w},0})_{\overline{w}\in\overline{W}}$, is a fundamental system of bounded subsets of $HW(\mathbb{D})$, resp. $HW_0(\mathbb{D})$.
Let v be a weight on $\mathbb{D}$. Its *associated growth condition* (see [6]) is defined by

$$\widetilde{v}(z) := \sup\{|g(z)|; g \in H(\mathbb{D}),\ |g| \leq v\},\ z \in \mathbb{D}.$$

A weight v on $\mathbb{D}$ is said to be *radial* if $v(z) = v(|z|)$ holds for every $z \in \mathbb{D}$.

We recall the following result. In [5] it is given in a more general setting, but we will restrict ourselves to the setting described in this paper.

Theorem 1 (Bierstedt-Bonet-Galbis [5]) *Let $W = (w_n)_{n\in\mathbb{N}}$ be an increasing sequence of non-negative continuous and radial functions on $\mathbb{D}$.*

(1) If $HW_0(\mathbb{D})$ contains the polynomials, then $\overline{B_{n,0}} = B_n$ and $\overline{C_n} = B_n$ hold for every $n \in \mathbb{N}$.

(2) If each w_n is strictly positive, then the $C_{\overline{w}}$ for $\overline{w} \in \overline{W}$ radial constitute a basis of bounded sets in $HW(\mathbb{D})$ and $C_{\overline{w}} = \overline{C_{\overline{w}} \cap HW_0(\mathbb{D})} = \overline{C_{\overline{w},0}}$ for every $\overline{w} \in \overline{W}$ radial.

(3) If each w_n approaches 0 monotonically as $r \to 1-$ the polynomials are dense in $HW_0(\mathbb{D})$.

2 Main Result

All the weights in this section are defined on the unit disc $\mathbb{D}$ of the complex plane. For every $n \in \mathbb{N}$ we denote $r_n := 1 - 2^{-2^n}$, $r_0 := 0$, and $I_n := [r_n, r_{n+1}]$. The following definition is inspired by a condition introduced by Bonet, Ĕnglis and Taskinen [9, Section 4].

Definition 2 *A sequence $W = (w_n)_{n\in\mathbb{N}}$ of weights on $\mathbb{D}$ satisfies the condition (LOG) if each weight in the sequence is radial and approaches monotonically 0 as $r \to 1-$ and there exist constants $0 < a < 1 < A$ such that*

(a) $Aw_k(r_{n+1}) \geq w_k(r_n)$ and

(b) $w_k(r_{n+1}) \leq aw_k(r_n)$.

for every k and n.

Theorem 3 *Let $W = (w_n)_{n\in\mathbb{N}}$ be an increasing sequence of strictly positive continuous radial functions on the unit disc $\mathbb{D}$ such that each w_n approaches monotonically 0 as $r \to 1-$ and such that (LOG) is satisfied. The following are equivalent:*

(a) $HW_0(\mathbb{D})$ is quasinormable.

(b) $HW(\mathbb{D})$ is quasinormable.

(c) For every $n \in \mathbb{N}$ there is $m > n$ such that for every $k \geq n$ and for every $\mu > 0$ we can find $\xi > 0$ such that

$$C_{m,0} \subset \xi C_{k,0} + \mu C_{n,0}.$$

(d) For every $n \in \mathbb{N}$ there is $m > n$ such that for every $k \geq n$ and for every $\varepsilon > 0$ we can find $\lambda > 0$ such that

$$\left(\frac{1}{w_m}\right)^{\sim} \leq \frac{\lambda}{w_k} + \frac{\varepsilon}{w_n} \text{ on } \mathbb{D}.$$

(e) *For every $n \in \mathbb{N}$ there is $m > n$ such that for every $\alpha > 0$ there is $\overline{w} \in \overline{W}$ with*

$$\left(\frac{1}{w_m}\right)^{\sim} \leq \overline{w} + \frac{\alpha}{w_n} \text{ on } \mathbb{D}.$$

Proof. Since $HW_0(\mathbb{D})$ is a Fréchet space, the equivalence of (a) and (c) follows from [14, Lemma 26.14]. The equivalence of (a) and (b) and the fact that (b) implies (e) are particular cases of results in a more general setting given in [19, Proposition 17 and 19]. It is easy to see that (d) follows from (e). It remains to show that (d) yields (3). Our proof was inspired by [9, Theorem 5].

Let $C > 0$ denote a constant such that $C > \sum_{k \in \mathbb{N}} a^k$, $0 < a < 1$ as in (b) of Definition 2, and $C > \frac{2A^t 2^{n-t}}{2^{2^n}}$ for all t and $n > t$, $1 < A$ as in (a) of Definition 2. We fix $n \in \mathbb{N}$ and select $m > n$ as in (d). For fixed $k \geq n$ and $\mu > 0$ we choose $\varepsilon = \frac{\mu}{4CA^2+2C}$. Then we select $\lambda > 0$ according to (d).
We fix $f \in C_{m,0}$. Hence $f \in HW_0(\mathbb{D})$ and

$$|f| \leq \left(\frac{1}{w_m}\right)^{\sim} \leq \frac{\lambda}{w_k} + \frac{\varepsilon}{w_n} \leq \max\left(\frac{2\lambda}{w_k}, \frac{2\varepsilon}{w_n}\right).$$

Put $u := \min(\frac{w_k}{2\lambda}, \frac{w_n}{2\varepsilon})$. Hence $f \in C_{u,0} := \{f \in HW_0(\mathbb{D}); \sup_{z \in \mathbb{D}} u(z)|f(z)| \leq 1\}$.
We write $u = \min(a_1 u_1, a_2 u_2)$ where $a_1 = \frac{1}{2\lambda}, a_2 = \frac{1}{2\varepsilon}$, $u_1 = w_k$ and $u_2 = w_n$.
u is a radial, continuous and non-increasing function. Each $g \in HW_0(\mathbb{D})$ can be approximated in $HW_0(\mathbb{D})$ by the functions $g_{r_n}(z) = g(r_n z)$ for large n; see e.g. [5]. Hence it suffices to show that f_{r_n} belongs to $\xi C_{k,0} + \mu C_{n,0}$ for each n big enough. Since the weight u is non-increasing, we get

$$\inf_{|z| \in I_n} u(z) = u(r_{n+1}) \geq u(r_{n+2}) = \inf_{|z| \in I_{n+1}} u(z) \geq A^{-2} u(r_n). \tag{1}$$

For every $n \in \mathbb{N}$ we can thus pick a $k(n) \in \{1, 2\}$ such that

$$u(r_n) = a_{k(n)} u_{k(n)}(r_n) = a_{k(n)} \sup_{|z| \in I_n} u_{k(n)}(z). \tag{2}$$

For $\nu \in \mathbb{N}$ let $N_1 = \{n \leq \nu;\ k(n) = 1\}$ and $N_2 = \{n \leq \nu;\ k(n) = 2\}$. Let us define, for all n the function $g_n(z) := f(r_{n+1}z) - f(r_n z)$ and $g_0(z) := f(0)$, and, for $i \in \{1, 2\}$,

$$h_i := \sum_{n \in N_i} g_n, \tag{3}$$

and $h_i := 0$ if $N_i = \emptyset$. Clearly $f_{r_\nu} = h_1 + h_2 + g_0$. The constant function g_0 belongs to $H(u_2)_0(\mathbb{D})$ and $|f(0)| \leq a_1^{-1} u_1^{-1}(0)$, hence $g_0 \in a_1^{-1} C_{k,0}$. Let us fix $i \in \{1, 2\}$. We pick $n \in N_i$, and estimate $|g_n(z)|$ for different z.

1. Assume first $|z| \geq r_{n-1}$. Then

$$|r_n z| \geq (1 - 2^{-2^n})(1 - 2^{-2^{n-1}}) \geq (1 - 2 \cdot 2^{-2^{n-1}}) \in I_{n-2}$$

and similarly for $|r_{n+1}z|$; hence

$$r_{n-2} \leq |r_n z| \leq |r_{n+1} z| \leq r_{n+1}. \tag{4}$$

Since $f \in C_{u,0}$ we have for these z, by (1)

$$|g_n(z)| \le |f(r_n z)| + |f(r_{n+1}z)| \le 2 \sup_{r_{n-2}\le r\le r_{n+1}} u(r)^{-1} = 2u(r_{n+1})^{-1}. \tag{5}$$

Now (5) can still be estimated using (2) by

$$2A^2 u(r_n)^{-1} = 2A^2 a_i^{-1} u_i(r_n)^{-1}. \tag{6}$$

2. Assume $2 \le t \le n$ and $|z| \in I_{n-t}$. We have

$$\begin{aligned} |g_n(z)| &= |f(r_n z) - f(r_{n+1}z)| \le \sup_{\xi\in I_{n-t}\cup I_{n-t-1}} |f'(\xi)||r_{n+1}-r_n| \\ &\le \sup_{\xi\in I_{n-t}\cup I_{n-t-1}} |f'(\xi)| 2^{-2^n}. \end{aligned} \tag{7}$$

We estimate $|f'(\xi)|$ using the Cauchy formula

$$|f'(\xi)| \le \int_{|\eta|=r_n} \frac{|f(\eta)|}{|\eta-\xi|^2}\, d\eta \le u(r_n)^{-1} 2^{2^{n-t+1}} \tag{8}$$

since $|\eta-\xi| \ge 2^{-2^{n-t+1}} - 2^{-2^n} \ge 2^{-1}\cdot 2^{-2^{n-t+1}}$. We use $2^n - 2^{n-t+1} \ge 2^{n-1}$ and from (7) and (8) we obtain

$$|g_n(z)| \le 2^{-2^{n-1}} \cdot u(r_n)^{-1} \le 2^{-2^{n-1}} \cdot a_{k(n)}^{-1} u_{k(n)}(r_n)^{-1}. \tag{9}$$

Here we used (2). Moreover, using (a) of Definition 2 t times, we can continue the estimate by

$$\le 2^{-2^{n-1}} \cdot A^t a_{k(n)}^{-1} u_{k(n)}(z)^{-1}. \tag{10}$$

Since $n > t$ we have $2^{-2^n} A^t \le C2^{-(n-t)}$ (for all n and t), hence (10) is bounded by

$$C2^{-(n-t)} a_{k(n)}^{-1} u_{k(n)}(z)^{-1} = C2^{-(n-t)} a_i^{-1} u_i(z)^{-1}. \tag{11}$$

So altogether

$$|g_n(z)| \le C2^{-(n-t)} a_i^{-1} u_i(z)^{-1}. \tag{12}$$

To complete the proof, let now $z \in \mathbb{D}$; we want to show that

$$|h_i(z)| \le (2CA^2 + C) a_i^{-1}. \tag{13}$$

Let $t \in \mathbb{N}$ be such that $|z| \in I_t$, then

$$|h_i(z)| \le \sum_{n\in N_i, n\le t+1} |g_n(z)| + \sum_{n\in N_i, n>t+1} |g_n(z)| =: G_i(z) + H_i(z). \tag{14}$$

(a) Consider $G_i(z)$. In this case (6) of 1. implies

$$G_i(z) \le \sum_{n\in N_i, n\le t+1} 2A^2 a_i^{-1} u_i(r_n)^{-1}.$$

By using (b) of Definition 2 $(t-n)$ times, this is bounded by a constant times

$$\sum_{n\le t+1} 2A^2 a_i^{-1} u_i(r_t)^{-1} a^{-t+n} \le 2CA^2 a_i^{-1} u_i(z)^{-1}. \tag{15}$$

Remember that $a < 1$ and $C > \sum_{k\in\mathbb{N}} a^k$.

(b) Consider $H_i(z)$. Then 2. (12) implies

$$H_i(z) \leq \sum_{n \in N_i, n > t+1} C2^{-(n-t)} a_i^{-1} u_i(z)^{-1} \leq C a_i^{-1} u_i(z)^{-1}. \tag{16}$$

We obtain

$$\begin{aligned} f_{r_\nu} &= g_0 + h_1 + h_2 \in (1 + 2CA^2 + C) a_1^{-1} C_{k,0} + (2CA^2 + C) a_2^{-1} C_{n,0} \\ &= (2 + 4CA^2 + 2C)\lambda C_{k,0} + (4CA^2 + 2C)\varepsilon C_{n,0} \\ &= \xi C_{k,0} + \mu C_{n,0}, \end{aligned}$$

where $\xi := (2 + 4CA^2 + 2C)\lambda$. ■

A characterization of weighted Fréchet spaces of holomorphic functions having Stefan Heinrich's density condition similar to the one above can be given with a similar, technically more complicated approach. A Fréchet space has the density condition if every bounded set of its strong dual is metrizable. This condition was studied thoroughly by Bierstedt-Bonet (see [3, 4]).

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References

[1] Anderson, J.M., Duncan, J., Duals of Banach spaces of entire functions, *Glasgow Math. J.* **32** (1990) 215-220.

[2] Bastin, F., Ernst, B., A criterion for $CV(X)$ to be quasinormable, *Results Math.* **14**, no. 3-4, (1988) 223-230.

[3] Bierstedt, K.D., Bonet, J., Stefan Heinrich's density condition and the characterization of the distinguished Köthe echelon spaces, *Math. Nachr.* **135** (1988) 149-180.

[4] Bierstedt, K.D., Bonet, J., Some aspects of the modern theory of Fréchet spaces, *RACSAM Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math.* **97**, no. 2, (2003) 159-188.

[5] Bierstedt, K.D., Bonet, J., Galbis, A., Weighted spaces of holomorphic functions on balanced domains, *Michigan Math. J.* **40**, no. 2, (1993) 271-297.

[6] Bierstedt, K.D., Bonet, J., Taskinen, J., Associated weights and spaces of holomorphic functions, *Studia Math.* **127**, no. 2, (1998) 137-168.

[7] Bierstedt, K.D., Meise, R., Distinguished echelon spaces and the projective description of weighted inductive limits of type $V_dC(X)$, pp. 169-226 in *Aspects of Mathematics and its Applications*, North-Holland Math. Library **34**, Amsterdam, 1986.

[8] Bierstedt, K.D., Meise, R., Summers, W.H., Köthe sets and Köthe sequence spaces, pp. 27-91 in *Functional Analysis, Holomorphy and Approximation Theory (Rio de Janeiro 1980)*, North-Holland Math. Stud. **71**, Amsterdam, 1982.

[9] Bonet,J., Ěnglis, M., Taskinen, J., Weighted L^∞-estimates for Bergman projections, *Studia Math.* **171**, no. 1, (2005) 67-92.

[10] Grothendieck, A., Sur les espaces (F) et (DF), *Summa Brasil Math.* **3** (1954) 57-123.

[11] Jarchow, H., *Locally Convex Spaces*, Math. Leitfäden, B.G. Teubner, Stuttgart, 1981.

[12] Köthe, G., *Topological Vector Spaces I*, Grundlehren der math. Wiss. **159**, Springer-Verlag, New York-Berlin, 1969.

[13] Meise, R., Vogt, D., A characterization of quasinormable Fréchet spaces, *Math. Nachr.* **122** (1985) 141-150.

[14] Meise, R., Vogt, D., *Introduction to Functional Analysis*, Oxford Graduate Texts in Math. **2**, The Clarendon Press, Oxford University Press, New York, 1997.

[15] Pérez-Carreras, P., Bonet, J., *Barrelled Locally Convex Spaces*, North-Holland Math. Stud. **131**, Amsterdam, 1987.

[16] Valdivia, M., On quasinormable echelon spaces, *Proc. Edinbourgh Math. Soc.(2)* **24** (1981) 73-80.

[17] Valdivia, M., *Topics in locally convex spaces*, North-Holland Math. Stud. **67**, Amsterdam, 1982.

[18] Vogt, D., Some results on continuous linear maps between Fréchet spaces, pp. 349-381 in *Functional analysis: Surveys and Recent Results III*, North-Holland Math. Stud. **90**, Amsterdam, 1984.

[19] Wolf, E. (2005). Weighted Fréchet spaces of holomorphic functions, *Studia Math.* **174** (2006), no. 3, 255-275.

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