Division problems for Fourier ultra-hyperfunctions

Michael Langenbruch

Dedicated to Professor J. Schmets on the occasion of his 65th birthday

Abstract

We characterize the surjective convolution operators T_μ on the space $(P_{**})'$ of Fourier ultra-hyperfunctions by means of a slowly decreasing condition for the Fourier transform $\widehat{\mu}$ and then study the existence of continuous linear right inverses for T_μ.

1 Introduction

The subject of this paper are convolution operators on the space $(P_{**})'$ of Fourier ultra-hyperfunctions defined as the dual space of the space

$$P_{**} := P_{**}(\mathbb{C}^d) := \{f \in \mathcal{H}(\mathbb{C}^d) \mid \forall k : \|f\|_k := \sup_{|\Im(z)| \le k} |f(z)| e^{k|z|} < \infty\}$$

of entire test functions (see [12]). Notice the analogy to the definition of standard Fourier hyperfunctions (see [4, 5]) and of Schwartz' tempered distributions. $(P_{**})'$ is a space of entire rather than of real analytic functionals which has some interesting features that suggest to study convolution operators in this space (e.g., the exponentials $f_\lambda(z) := \exp(\sum \lambda_j z_j)$ are contained in $(P_{**})'$ for any $\lambda \in \mathbb{C}^d$, hence the kernels of an ordinary differential equation coincide in $C^\infty(\mathbb{R}^d)$ and in $(P_{**})'$, which is not true for the standard Fourier hyperfunctions, see [8]). Though some of the proofs in the present paper are based on similar ideas as in the case of Fourier hyperfunctions (see [7]), the results are rather different and sometimes more natural for Fourier ultra-hyperfunctions.

References to the huge literature on division problems in various spaces are given in more detail in [7].

The paper is organized as follows: In the next section we show that $\mu \in (P_{**})'$ defines a convolution operator T_μ on $(P_{**})'$ iff the Fourier transform $\widehat{\mu}$ is defined by an entire function F such that for any k there is K such that

$$|F(z)| \leq C_1 e^{K|z|} \text{ if } |\Im(z)| \leq k.$$

For these μ we then show that T_μ is surjective on $(P_{**})'$ iff T_μ admits an elementary solution $\nu \in (P_{**})'$ iff there is $C > 0$ such that for any $t \in \mathbb{R}^d$ with $|t| \geq C$ there is $\zeta \in \mathbb{C}^d$ such that

$$|\zeta - t| \leq C \text{ and } |\widehat{\mu}(\zeta)| \geq e^{-C|\zeta|}.$$

We do not need here a specific condition on the connected components of $\widehat{\mu}^{-1}(0)$ unlike to the case of standard Fourier hyperfunctions (see [7]).

In section 3 we show that a surjective convolution operator T_μ on $P_{**}(\mathbb{C})'$ admits a continuous linear right inverse in $P_{**}(\mathbb{C})'_b$ iff there is k_0 such that

$$\widehat{\mu}(z) \neq 0 \text{ if } |\Im(z)| > k_0.$$

We also show that T_μ admits a continuous linear right inverse in $P_{**}(\mathbb{C}^d)'_b$ if $\widehat{\mu}$ satisfies a condition of hyperbolic type or of hypoelliptic type (see 3.1 and 3.3). Some examples are discussed at the end of sections 2 and 3.

2 Convolution operators

For $f \in P_{**}$ and $\mu \in (P_{**})'$ we define the Fourier transformation by

$$\widehat{f}(z) := \int f(x) e^{-i\langle x,z\rangle}\, dx \text{ and } \langle \widehat{\mu}, g\rangle := \langle \mu, \widehat{g}\rangle \text{ if } g \in P_{**},$$

where $\langle w, z\rangle := \sum_{j=1}^d w_j z_j$ for $z, w \in \mathbb{C}^d$.

The Fourier transform is a topological isomorphism in P_{**} and in $(P_{**})'_b$ (see [6, 3.6 and 5.5]).

To define a convolution operator on $(P_{**})'$ we start with the usual convolution of functions: since

$$\widehat{f * g} = \widehat{f}\widehat{g} \text{ and } \widehat{fg} = (2\pi)^{-d} \widehat{f} * \widehat{g} \text{ for } f, g \in P_{**} \subset \mathcal{S} \tag{2.1}$$

we see that $f * g \in P_{**}$ if $f, g \in P_{**}$ and that the mapping

$$f* : P_{**} \to P_{**}, g \to f * g, \text{ is continuous.}$$

Therefore, we can define the convolution $S_\mu(f) \in (P_{**})'$ of a fixed $\mu \in (P_{**})'$ and $f \in P_{**}$ by the usual formula

$$\langle S_\mu(f), g\rangle := \langle \mu, \check{f} * g\rangle \text{ if } g \in P_{**}, \tag{2.2}$$

where $\check{f} := f(-\,\cdot\,)$ for $f \in P_{**}$.

Let $\nu \bullet f$ denote the product of $\nu \in (P_{**})'$ and a testfunction $f \in P_{**}$. Via Fourier transformation, $S_\mu(f)$ is the product of $\widehat{\mu}$ and $\widehat{f}$ since

$$\langle \widehat{\mu} \bullet \widehat{f}, g\rangle = \langle \mu, \widehat{\widehat{f}g}\rangle = \langle \mu, \check{f} * \widehat{g}\rangle = \langle \big(S_\mu(f)\big)\widehat{\ }, g\rangle \text{ if } g \in P_{**} \tag{2.3}$$

by (2.1) and Fourier inversion formula. We say that $\nu \in (P_{**})'$ is defined by an exponentially bounded measurable function F on $\mathbb{R}^d$ iff

$$\langle \nu, g\rangle = \int_{\mathbb{R}^d} F(x)g(x)\,dx \text{ if } g \in P_{**}.$$

If $S_\mu(f)$ is defined by a function $f_\mu \in P_{**}$ and if S_μ defines a continuous linear operator in P_{**} in this way, the convolution operator T_μ on $(P_{**})'$ is defined by duality, i.e.

$$T_\mu := (S_{\check{\mu}})^t : (P_{**})' \to (P_{**})'.$$

We now have the following characterization:

Proposition 2.1. *The following are equivalent for $\mu \in (P_{**})'$:*
a) For any $f \in P_{**}$, $S_\mu(f)$ is defined by some $f_\mu \in P_{**}$.
b) $S_\mu : P_{**} \to P_{**}$ is defined and continuous.
c) $\widehat{\mu}$ is defined by $F \in \mathcal{H}(\mathbb{C}^d)$ such that for any k there is K such that

$$|F(z)| \le C_1 e^{K|z|} \text{ on } W_k := \{z \in \mathbb{C}^d \mid |\Im(z)| < k\}. \tag{2.4}$$

We then have $S_\mu(f)(z) = (2\pi)^{-d}\widehat{(F\widehat{f}\,)}(-z)$ for any $f \in P_{**}$.

Proof. "*a) $\Rightarrow$ c)*" By (2.3), the Fourier inversion formula and a), $S(f) := \widehat{\mu} \bullet f = (2\pi)^{-d}\Big(S_\mu\Big(\widehat{\check{f}}\Big)\Big)\widehat{\ }$ is defined by some $f_\mu \in P_{**}$ for any $f \in P_{**}$. Since P_{**} is continuously embedded in $(P_{**})'$, S is a continuous operator in P_{**} by the closed graph theorem, that is, for any k there is K such that we have for the norms on P_{**}

$$\|\widehat{\mu} \bullet f\|_k \le C_k \|f\|_K \text{ if } f \in P_{**}. \tag{2.5}$$

For $t \ge 1$ let $g_t := \widehat{\mu} \bullet f_t$ for $f_t(z) := e^{-\langle z,z\rangle/t}$. $g_t \in P_{**}$ by a) since $f_t \in P_{**}$. By the definition of $\bullet$ we see that $g_4 = g_t f_4/f_t$ for any $t \ge 1$. For the entire function $F := g_4/f_4$ this implies by (2.5)

$$|F(z)| = |g_t(z)/f_t(z)| \le C_k \|f_t\|_K / |f_t(z)| \le C_k e^{2K^2} e^{tK^2/4 + |z|^2/t}$$

if $t \ge 1$ and $z \in W_k$. Taking the infimum with respect to $t \ge 1$ we get (2.4). Let h_j denote the Hermite polynomials. Then the Hermite functions are defined by $H_j := c_j h_j f_2$ and we thus get by the definition of $\bullet$

$$\int_{\mathbb{R}^d} F(x)H_j(x)dx = c_j \int_{\mathbb{R}^d} (\widehat{\mu} \bullet f_4)(x)(f_4 h_j)(x)dx = c_j \langle \widehat{\mu} \bullet f_4, h_j f_4\rangle$$

$$= c_j \langle \widehat{\mu}, h_j f_2\rangle = \langle \widehat{\mu}, H_j\rangle \text{ for any } j \in \mathbb{N}_0^d.$$

Since the Hermite functions are a basis in P_{**} by [6, 5.5], c) is proved.

"*c) $\Rightarrow$ b)*" By (2.3) and c) we know that $S_\mu(f) = (2\pi)^{-d}\widehat{(F\widehat{f}\,)}\check{\ }$ for any $f \in P_{**}$. This shows b) since the Fourier transformation and the multiplication with F are continuous operators in P_{**} by (2.4). ∎

Notice that (2.4) is not always satisfied: Easy counterexamples are provided by $\mu \in (P_{**})'$ such that $\widehat{\mu}$ is a hyperfunction with compact support. In the simplest case we can take $\mu \equiv 1$, i.e. $\widehat{\mu} = 2\pi\delta$. Also, elementary solutions $\nu \in (P_{**})'$ of surjective convolution operators T_μ on $(P_{**})'$ do not satisfy (2.4) if there is $z_0 \in \mathbb{C}^d$ such $\widehat{\mu}(z_0) = 0$ since the first assumption would imply that the kernel of T_μ is trivial contradicting the second assumption.

From now on we will always assume that μ satisfies (2.4). Therefore,

$$S_\mu : P_{**} \to P_{**} \text{ and } T_\mu := (S_{\check{\mu}})^t : (P_{**})'_b \to (P_{**})'_b$$

are defined, linear and continuous, and $\widehat{\mu}$ is an entire function.

Recall that $\nu \in (P_{**})'$ is an elementary solution for T_μ if $T_\mu(\nu) = \delta$. Surjective convolution operators on $(P_{**})'$ can now be characterized as follows:

Theorem 2.2. *Let $\mu \in (P_{**})'$ satisfy (2.4). The following are equivalent:*

a) The convolution operator $T_\mu : (P_{**})' \to (P_{**})'$ is surjective.

b) T_μ admits an elementary solution $\nu \in (P_{**})'$.

c) There is $C > 0$ such that for any $t \in \mathbb{R}^d$ with $|t| \geq C$ there is $\zeta \in \mathbb{C}^d$ such that

$$|\zeta - t| \leq C \text{ and } |\widehat{\mu}(\zeta)| \geq e^{-C|\zeta|}. \tag{2.6}$$

Proof. "$b) \Rightarrow c)$" Let $\nu \in (P_{**})'$ be an elementary solution for T_μ. Then $\widehat{\check{\nu}} \in (P_{**})'$ and thus there are j and C_1 such that

$$|\langle \widehat{\check{\nu}}, h\rangle| \leq C_1 \|h\|_j \text{ if } h \in (P_{**})'. \tag{2.7}$$

If (2.6) does not hold, for any $l \in \mathbb{N}$ there is $t_l \in \mathbb{R}^d$ with $|t_l| \geq 4l$ such that

$$|\widehat{\check{\mu}}(\zeta)| \leq e^{-l|\zeta|} \text{ if } |\zeta - t_l| \leq l. \tag{2.8}$$

Let $f_l(z) := \exp\Big(i\langle z, t_l\rangle - \langle z, z\rangle/(2c_l)\Big)$ for $c_l := |t_l|/l$. Then $f_l \in P_{**}$ and

$$\widehat{f_l}(z) := (2\pi c_l)^{d/2} \exp\Big(-\langle z - t_l, z - t_l\rangle c_l/2\Big) =: g_l(z)$$

$$1 = f_l(0) = |\langle T_\mu(\nu), f_l\rangle| = (2\pi)^{-d} |\langle \widehat{\check{\nu}}, \widehat{\check{\mu}} \widehat{f_l}\rangle| \leq C_1 \|\widehat{\check{\mu}} g_l\|_j \tag{2.9}$$

by 2.1 and (2.7). We will show that the right hand side of (2.9) tends to 0, a contradiction: let $|z - t_l| \leq l$. Since $|t_l| \geq 4l$, we get by (2.8)

$$\begin{aligned} |\widehat{\check{\mu}}(z) g_l(z)| &\leq C_2 c_l^{d/2} \exp\Big(-l|z| - |\Re(z - t_l)|^2 c_l/2 + |\Im(z)|^2 c_l/2\Big) \\ &\leq C_2 c_l^{d/2} e^{-l(|z| - |t_l|/2)} \leq C_3 e^{-l(|z| + |t_l|)/8}. \end{aligned} \tag{2.10}$$

Choose $J \geq j^2$ for j by (2.4). If $|z - t_l| \geq l$ and $z \in W_j$ we then get

$$|\widehat{\check{\mu}}(z) g_l(z)| \leq C_4 c_l^{d/2} e^{J|z| + (2|\Im(z)|^2 - |z - t_l|^2)c_l/2} \leq e^{-j|z| - |t_l|} \tag{2.11}$$

if l is large, since

$$|t_l| + (j + J)|z| + (2j^2 - |z - t_l|^2)c_l/2 \leq 2J|z - t_l| - |z - t_l|^2 c_l/2 + 3J|t_l|$$

$$\leq 2Jl - l|t_l|/2 + 3J|t_l| \leq -|t_l| \text{ for large } l.$$

The above claim follows from (2.10) and (2.11).

"$c) \Rightarrow a)$" P_{**} is a $(FS)-$space, hence reflexive. By Fourier transformation, Proposition 2.1 and the closed range theorem [11, 26.3] we thus get: T_μ is surjective in $(P_{**})'$ iff S_μ is injective with closed range in P_{**} iff $\widehat{\mu}P_{**}$ is closed in P_{**} iff for any $k \in \mathbb{N}$ there are $j \geq k$ and $C_1 \geq 1$ such that

$$\|f\|_k \leq C_1 \|\widehat{\mu} f\|_j \text{ if } f \in P_{**}. \tag{2.12}$$

We now recall the following fact (see [1, 3.1]): Let F, G and F/G be holomorphic on $\{z \in \mathbb{C}^d \mid |z| < R\}$.

$$|(F/G)(z)| \leq \sup_{|\eta|<R} |F(\eta)| \Big(\sup_{|\eta|<R} |G(\eta)| \Big)^{\frac{2|z|}{R-|z|}} |G(0)|^{\frac{-R-|z|}{R-|z|}} \text{ if } |z| < R. \tag{2.13}$$

Fix $k \in \mathbb{N}$ and let $w := t + iy \in W_k$. Choose $\zeta \in \mathbb{C}^d$ for t by (2.6) and apply (2.13) to $F(z) := \widehat{\mu}(\zeta + z) f(\zeta + z), f \in P_{**}, G(z) := \widehat{\mu}(\zeta + z), R := 2(C + k)$ and $|z| \leq R/2$. Since $|w - \zeta| \leq R/2$ we get

$$|f(w)|e^{k|w|} \leq C_1 \sup_{|\eta|<R} |\widehat{\mu}(\zeta + \eta) f(\zeta + \eta)| \sup_{|\eta|<R} e^{2J|\zeta+\eta|} e^{3C|\zeta|+k|w|} \leq C_2 \|\widehat{\mu} f\|_j$$

for $j := 2J + k + 3C$, if J is chosen for W_{C+R} by (2.4). This proves (2.12). ∎

T_μ is obviously defined for any $\mu \in \mathcal{H}(\mathbb{C}^d)'_b$, however T_μ need not be surjective (see [7, 3.2]). A simple example of a non surjective operator T_μ is provided by $\mu(x) := e^{-x^2/2}, x \in \mathbb{R}$, since $\widehat{\mu}(z) = (2\pi)^{1/2} e^{-x^2/2}$ does not satisfy (2.6). On the other hand, if $\mu(x) := e^{ix^2/2}, x \in \mathbb{R}$, then $\widehat{\mu}(z) = \pi^{1/2}(1+i)e^{-iz^2/2}$ (see [3, 7.6.1]) and T_μ is defined and surjective (and in fact bijective) since $|\widehat{\mu}(z)| = (2\pi)^{1/2} e^{\Re(z)\Im(z)}$ satisfies (2.4) and (2.6).

Differential-delay equations are always surjective in $(P_{**})'$. In fact, we then have $\mu \in \operatorname{span}\{\partial^\alpha \delta_w \mid \alpha \in \mathbb{N}_0^d, w \in \mathbb{C}^d\}$ and $\widehat{\mu} \in \operatorname{span}\{z^\alpha \exp(\langle z, w \rangle) \mid \alpha \in \mathbb{N}_0^d, w \in \mathbb{C}^d\}$. Thus, let $\widehat{\mu} := \sum_{j=1}^k p_j e^{\langle \cdot, w_j \rangle}$ with distinct $w_j \in \mathbb{C}^d$ and polynomials p_j. Let $\deg p_j := m_j \leq m$ and $\max_{j \leq k} |\Re(w_j)| := r$. Then

$$g_t(z) := \widehat{\mu}(t + z) = \sum_{j=1}^k \sum_{l=0}^m \frac{p_j^{(l)}(t) e^{\langle t, w_j \rangle}}{l!} z^l e^{\langle z, w_j \rangle} \in \operatorname{span}_{j \leq k, |l| \leq m} \{z^l e^{\langle z, w_j \rangle}\}.$$

Since all norms on this space are equivalent, (2.6) follows:

$$\sup_{|z|\leq 1} |\widehat{\mu}(t + z)| \geq C_1 \sup_{j \leq k, |l| \leq m} |p_j^{(l)}(t) e^{\langle t, w_j \rangle}|/l! \geq C_2 e^{-r|t|}.$$

3 Right inverses

As a first class of convolution operators admitting a continuous linear right inverse we consider a condition of hyperbolic type:

Theorem 3.1. *Let $\mu \in (P_{**})'$ satisfy* (2.4) *and* (2.6)*. T_μ admits a continuous linear right inverse in $(P_{**})'$ if there is $N \in \mathbb{R}^d$ such that for any k there is k_0 such that*

$$\widehat{\mu}(z + i\tau N) \neq 0 \text{ if } z \in W_k \text{ and } |\tau| \geq k_0. \tag{3.1}$$

Proof. Let $|N| = 1$ (w.l.o.g.) and $M_{\widehat{\mu}}(f) := \widehat{\mu} f$ for $f \in P_{**}$. By Fourier transformation, it is sufficient to show that $M_{\widehat{\mu}}$ has a continuous linear left inverse in P_{**}.

a) For any k there is k_1 such that any $j \geq k_1$ there are $A, C_0 > 0$ such that

$$|\widehat{\mu}(w + i\tau N)| \geq C_0 e^{-A|w|} \text{ if } w \in W_k \text{ and } j \geq |\tau| \geq k_1. \tag{3.2}$$

When proving (3.2) we need the following minimum modulus theorem (see e.g. [9, 1.11]): Let $0 \neq g$ be holomorphic near $|z| \leq \varrho, z \in \mathbb{C}$. For any $0 < r < \varrho/4$ there are $H = H(r/\varrho) > 0$ and $r < \delta < \varrho/4$ such that

$$|g(\xi)| \geq |g(0)|^{1+H} / \sup_{|\eta|=\rho} |g(\eta)|^H \text{ if } |\xi| = \delta. \tag{3.3}$$

Fix k and choose k_1 for $2C+3k$ by (3.1) with C from (2.6). Let $k_1 \leq \tau \leq j$ (w.l.o.g.) and let $w \in W_k$. We first choose $\zeta \in \mathbb{C}^d$ for $t := \Re(w)$ by (2.6) and then apply (3.3) to $g(z) := \widehat{\mu}(\zeta + zN), r := \tau$ and $\rho := 4(1 + k/j)\tau$. Using also (2.4) we thus obtain $C_1, A_1 > 0$ (independent of w and τ) and $\tau < \delta < (1 + k/j)\tau \leq \tau + k$ such that

$$|\widehat{\mu}(\zeta + i\delta N)| \geq C_1 e^{-A_1|w|}.$$

(2.13) is now applied to $F \equiv 1, G(z) := \widehat{\mu}(\zeta + i\delta N + z), R := C + 3k$ and $|z| \leq C + 2k$ (notice, that $G(z) \neq 0$ for $|z| \leq R$ by (3.1) and the choice of k_1 since $\zeta + z \in W_{2C+3k}$). Since $|w + i\tau N - \zeta - i\delta N| \leq C + 2k$ we get by (2.4)

$$|\widehat{\mu}(w + i\tau N)| \geq C_2 e^{-A_2|w|}$$

for some constants $A_2, C_2 > 0$.

b) We may assume that $N = e_d := (0, \dots, 1)$ and write $z = (z', z_d) \in \mathbb{C}^{d-1} \times \mathbb{C}$. The left inverse for $M_{\widehat{\mu}}$ can now be given by means of an explicit formula which is a $\mathbb{C}^d-$variant of [7, (4.5)]: For $f \in P_{**}$ let

$$L(f)(z) := \frac{1}{2\pi i} \int_{|\Im(\tau)|=k_1} \frac{f(z', \tau) e^{-(\tau - z_d)^2}}{\widehat{\mu}(-z', -\tau)(\tau - z_d)} d\tau \text{ if } z \in W_k \tag{3.4}$$

where $k_1 > k$ is the constant from (3.2).

Indeed, for $f \in P_{**}$, $L(f)(z)$ is defined for any z by (3.2). $L(f)$ is welldefined by Cauchy's theorem and (3.2) again. It is also clear that $L(f)$ is an entire function and that $L(M_{\widehat{\mu}} f) = f$ by Cauchy's integral formula. Finally, $L(f) \in P_{**}$ by an easy estimate and $L : P_{**} \to P_{**}$ is continuous. ∎

Hyperbolic polynomials P satisfy (3.1). To see this, let P_m denote the principal part of P and let $\widetilde{Q}(x,t) := (\sum_\alpha |Q^{(\alpha)}(x,t)|^2 t^{2|\alpha|})^{1/2}$ for a polynomial Q. By [3, 12.4.6(iii)] we know that

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^d} \frac{(P_m^{(\alpha)})\widetilde{\ }(x,t)}{\widetilde{P_m}(x,t)} = 0 \text{ if } \alpha \neq 0 \text{ and } \lim_{t\to\infty} \sup_{x\in\mathbb{R}^d} \frac{(P-P_m)\widetilde{\ }(x,t)}{\widetilde{P_m}(x,t)} = 0 \tag{3.5}$$

if P is hyperbolic w.r.t. N. For $z = x+iy \in W_k$ and $t \geq k$ we thus get by Taylor expansion, [10, 3.3] and (3.5)

$$\begin{aligned}
&|P(z+itN)| \\
&\geq |P_m(x+itN)| - \sum_{\alpha\neq 0} |P_m^{(\alpha)}(x+itN)||y^\alpha| - \sum_\alpha |(P-P_m)^{(\alpha)}(x)||(y+itN)^\alpha| \\
&\geq C_1\widetilde{P_m}(x,t) - C_2\Big(\sum_{\alpha\neq 0} |P_m^{(\alpha)}(x+itN)| + (P-P_m)\widetilde{\ }(x,k+t)\Big) \\
&\geq C_1\widetilde{P_m}(x,t) - C_3\Big(\sum_{\alpha\neq 0} (P_m^{(\alpha)})\widetilde{\ }(x,t)| + (P-P_m)\widetilde{\ }(x,t)\Big) \geq C_1\widetilde{P_m}(x,t)/2 \neq 0
\end{aligned}$$

if t is large.

The condition (DN) of Vogt is fundamental for the existence of continuous linear right inverses. It is defined as follows (see e.g. [11, p. 359]): Let E be a Frechet space with fundamental system $(\|\ \|_k)_{k\in\mathbb{N}}$ of seminorms. E has (DN) iff there is p such that for each k there are n and C such that

$$\|x\|_k^2 \leq C\|x\|_p\|x\|_n \text{ for all } x \in E.$$

If T_μ is surjective on $(P_{**})'$, the sequence

$$0 \to \ker(T_\mu) \to (P_{**})' \xrightarrow{T_\mu} (P_{**})' \to 0$$

is exact. By Fourier transformation it is split iff the dual sequence

$$0 \to P_{**} \xrightarrow{M_{\widehat{\mu}}} P_{**} \to P_{**}/(\widehat{\mu}P_{**}) \to 0$$

splits (again, $M_{\widehat{\mu}}(f) := \widehat{\mu}f$ for $f \in P_{**}$). Since P_{**} is isomorphic to a power series space of infinite type by [6], the splitting theorem of Vogt (see [11, 30.1 and 29.2]) implies that

$$T_\mu \text{ has a right inverse in } (P_{**})'_b \text{ iff } (\ker(T_\mu))'_b \simeq P_{**}/(\widehat{\mu}P_{**}) \in (DN). \tag{3.6}$$

For operators T_μ in one variable we thus get

Theorem 3.2. *Let $d = 1$ and let $\mu \in (P_{**})'$ satisfy* (2.4) *and* (2.6). *Then T_μ admits a continuous linear right inverse in $P_{**}(\mathbb{C})'_b$ iff there is k_1 such that*

$$\widehat{\mu}(z) \neq 0 \textit{ if } |\Im(z)| \geq k_1. \tag{3.7}$$

Proof. (3.7) is sufficient by 3.1. If T_μ admits a continuous linear right inverse in $P_{**}(\mathbb{C})'_b$, $P_{**}/(\widehat{\mu}P_{**})$ has (DN) by (3.6), hence $P_{**}/(\widehat{\mu}P_{**})$ has a continuous norm, that is, a quotient seminorm $\|\ \|_k^{\sim}$ is a norm. Let $\widehat{\mu}(-w) = 0$. Then $g(z) := \widehat{\mu}(-z)\exp(-\langle z-w, z-w\rangle)/(z-w) \in P_{**}$ and $[g] \neq 0$ in $P_{**}/(\widehat{\mu}P_{**})$.

We now notice that for any k there is k_2 such that

$$P_{**} \text{ is dense in } \mathcal{H}_{k_2} := \{f \in \mathcal{H}(W_{k_2}) \mid \|f\|_{k_2} < \infty\} \text{ w.r.t. } \|\ \|_{k+K} \tag{3.8}$$

where K is chosen for k by (2.4). Indeed, the proof of [6, 3.4] shows that there is k_2 such that the Hermite expansion of $f \in \mathcal{H}_{k_2}$ converges to f with respect to $\|\ \|_{k+K}$.

If $|\Im(w)| > k_2$ then $h(z) := \exp(-\langle z-w, z-w\rangle)/(z-w) \in \mathcal{H}_{k_2}$ and we may choose $h_n \in P_{**}$ by (3.8) such that $\|h-h_n\|_{k+K} \to 0$, and therefore

$$0 \neq \|[g]\|_k^{\sim} = \|[\widehat{\mu}(h-h_n)]\|_k^{\sim} \leq \|\widehat{\mu}(h-h_n)\|_k \leq C_1\|h-h_n\|_{k+K} \to 0,$$

a contradiction. ∎

A right inverse also exists for operators of hypoelliptic type (see 3.3 below). This is based on the following observation: Let F be an entire function such that there is $N \in \mathbb{C}^d$ such that for any k there is K such that

$$|\langle z, \overline{N}\rangle| \leq K \text{ if } F(z) = 0 \text{ and } |\Pi(z)| \leq k, \tag{3.9}$$

where Π is the orthogonal projection onto $N^\perp$. Then

$$\mathcal{H}(\mathbb{C}^d)/(F\mathcal{H}(\mathbb{C}^d)) \text{ has } (DN). \tag{3.10}$$

Indeed, we may assume that $N = e_d := (0,\ldots,1)$. A left inverse for the multiplication operator M_F on $\mathcal{H}(\mathbb{C}^d)$ is then provided by

$$L(f)(z) := \frac{1}{2\pi i}\int_{|\tau|=K+1} \frac{f(z',\tau)}{F(z',\tau)(\tau - z_d)}d\tau \text{ if } |z| \leq k$$

for $K \geq k$ from (3.9). Hence, $F\mathcal{H}(\mathbb{C}^d)$ is a complemented (closed) subspace of $\mathcal{H}(\mathbb{C}^d)$ and the sequence

$$0 \to \mathcal{H}(\mathbb{C}^d) \xrightarrow{M_F} \mathcal{H}(\mathbb{C}^d) \to \mathcal{H}(\mathbb{C}^d)/(F\mathcal{H}(\mathbb{C}^d)) \to 0$$

is split. Hence, $\mathcal{H}(\mathbb{C}^d)/(F\mathcal{H}(\mathbb{C}^d))$ is isomorphic to a subspace of $\mathcal{H}(\mathbb{C}^d)$, and (3.10) follows from [11, 29.2] since $\mathcal{H}(\mathbb{C}^d)$ has (DN).

(3.9) is satisfied for $N = e_d$ if $F(z) := \sum_{j=0}^k F_j(z')z_d^j$ and $F_j \in \mathcal{H}(\mathbb{C}^{d-1})$.

Theorem 3.3. *Let $\widehat{\mu}$ satisfy* (2.4), (2.6) *and* (3.9). *T_μ admits a continuous linear right inverse in $(P_{**})'_b$ if*

$$|\Im(z)| \to \infty \text{ if } \widehat{\mu}(z) = 0 \text{ and } |\Re(z)| \to \infty. \tag{3.11}$$

Proof. By (3.6) and (3.10), it is sufficient to show that the canonical mapping

$$S : P_{**}/(\widehat{\mu}P_{**}) \to \mathcal{H}(\mathbb{C}^d)/(\widehat{\mu}\mathcal{H}(\mathbb{C}^d))$$

is a topological isomorphism. To prove this we first notice that S is clearly welldefined. S is injective by the proof of "$c) \Rightarrow a)$" in 2.2 (use (2.6) and (2.13)). Let

$$V_{\widehat{\mu}} := \{z \in \mathbb{C}^d \mid \widehat{\mu}(-z) = 0\}.$$

The surjectivity of S is seen as follows: choose $\varphi \in C^\infty(\mathbb{C}^d)$ such that $\varphi(z) = 1$ if $\operatorname{dist}(z, V_{\widehat{\mu}}) \leq 1$ and $\varphi(z) = 0$ if $\operatorname{dist}(z, V_{\widehat{\mu}}) \geq 2$ and such that $|\varphi|$ and $\|\operatorname{grad}\varphi\|$ are bounded on $\mathbb{C}^d$. We must show that for any $f \in \mathcal{H}(\mathbb{C}^d)$ there are $f_1 \in P_{**}$ and $f_2 \in \mathcal{H}(\mathbb{C}^d)$ such that $f = f_1 + \widehat{\widehat{\mu}} f_2$. For this, we will find

$$g \in \mathcal{L} := \{g \in L^2_{loc}(\mathbb{C}^d) \mid \forall k : |f|_k^2 := \int_{W_k} |f(z)|^2 e^{2k|z|} dz < \infty\}$$

solving

$$\overline{\partial} g = \overline{\partial}(\varphi f / \widehat{\widehat{\mu}}). \tag{3.12}$$

Then $f_1 := \varphi f - \widehat{\widehat{\mu}} g$ and $f_2 := (1-\varphi) f / \widehat{\widehat{\mu}} + g$ will prove the claim (use the arguments from below). To solve (3.12) we notice that

$$F_k(z) := \overline{\partial}\Big(f(z)\varphi(z) e^{\langle z, z\rangle} / \mu(-z)\Big), z \in W_k$$

is bounded and has bounded support by (3.11). Hence $F_k \in L^2(W_k)$ and by [2, 4.4.2] there is G_k such that $\overline{\partial} G_k = F_k$ on W_k and $G_k/(1 + |\cdot|^2) \in L^2(W_k)$. Therefore, $g_k := G_k \exp(-\langle z, z\rangle)$ satisfies (3.12) on W_k and $|g_k|_k$ is finite. For $j \geq k$, $g_{jk} := (g_j - g_k) \mid_{W_k}$ is holomorphic on W_k and $g_{jk} \in \mathcal{L}_k$. We therefore can switch from L^2-norms to $\sup -$norms for g_{jk}, that is, $g_{jk} \mid_{W_{k-1}} \in \mathcal{H}_{k-1}$. By (3.8), for any k there is k_2 such that $\mathcal{L} \cap \ker(\overline{\partial}) = P_{**}$ is dense in $\mathcal{H}_{k_2}$ w.r.t. $\| \ \|_k$. The classical Mittag-Leffler argument therefore shows that (3.12) can be solved with $g \in \mathcal{L}$. ■

Any hypoelliptic partial differential operator with constant coefficients admits a continuous linear right inverse on $(P_{**})'$ by 3.3.

An interesting example for 3.2 is given by $\mu := (\delta_i - \delta_{-i})/2 \in P_{**}(\mathbb{C})'$. Then $T_\mu = (\tau_i - \tau_{-i})/2$, where $\tau_{\pm i}$ is the shift by $\pm i$. $\widehat{\mu}(z) = \sinh(z)$ satisfies (2.6), but not (3.7). T_μ is surjective but does not admit a right inverse in $P_{**}(\mathbb{C})'_b$. The kernel of T_μ (i.e. the $2i-$periodic elements in $P_{**}(\mathbb{C})'$) is $\operatorname{span}\{e^{j\pi z} \mid j \in \mathbb{Z}\} \simeq \varphi$, where φ is the space of all finite sequences (see [8]).

On the other hand, if $\mu := (\delta_{-1} - \delta_1)/(2i)$, then $T_\mu = (\tau_{-1} - \tau_1)/(2i)$ and $\widehat{\mu}(z) = \sin(z)$ satisfies (2.6) and (3.7). T_μ admits a right inverse, that is, the space of $2-$periodic elements is complemented in $P_{**}(\mathbb{C})'_b$ (see [8] for more details).

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Department of Mathematics, University of Oldenburg,
D-26111 Oldenburg, GERMANY
E-mail: langenbruch@mathematik.uni-oldenburg.de