Completeness of certain function spaces

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Abstract

We give an example of a complete locally convex m-topology on the algebra of infinite differentiable functions on $[0,1]$ which is strictly coarser than the natural Fréchet-topology but finer than the topology of pointwise convergence. A similar construction works on the algebra of continuous functions on $[0,1]$. Using this examples we can separate different notions of diffotopy and homotopy.

1 Introduction

Our notation concerning locally convex spaces is standard, we refer e.g. to [2] and [3]. Let $\mathscr{K}$ be a family of compact subsets of $[0,1]$ which is closed with respect to finite unions. We introduce locally convex topologies $\theta_{\mathscr{K}}$ and $\tau_{\mathscr{K}}$ on $\mathscr{C}([0,1])$ and $\mathscr{C}^\infty([0,1])$, respectively. Namely, $\theta_{\mathscr{K}}$ is defined by the family of seminorms

$$p_K(f) := \sup_K |f|, \ K \in \mathscr{K},$$

and $\tau_{\mathscr{K}}$ is defined by the family of seminorms

$$p_{n,K} := \sup_{0\le\nu\le n} p_K(f^{(\nu)}), \ K \in \mathscr{K}, n \in \mathbb{N}_0.$$

To force the topologies to be finer than the topology of pointwise convergence, we assume, in addition, that $\cup\mathscr{K} = [0,1]$. Equipped with the pointwise multiplication, $(\mathscr{C}([0,1]), \theta_{\mathscr{K}})$ and $(\mathscr{C}^\infty([0,1]), \tau_{\mathscr{K}})$ are locally m-convex algebras, i.e. they admit a fundamental system of submultiplicative seminorms. $\mathscr{K} := \{[0,1]\}$ leads to the natural topologies. We write $\mathscr{K}_1 \le \mathscr{K}_2$ if for every $K_1 \in \mathscr{K}_1$ there exists $K_2 \in \mathscr{K}_2$ with $K_1 \subseteq K_2$. The following proposition is easy to prove.

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Proposition 1 The following statements are equivalent:

1. $\mathscr{K}_1 \leq \mathscr{K}_2$,
2. $\theta_{\mathscr{K}_1} \subseteq \theta_{\mathscr{K}_2}$,
3. $\tau_{\mathscr{K}_1} \subseteq \tau_{\mathscr{K}_2}$.

In particular, if $[0,1] \notin \mathscr{K}$ then $\theta_{\mathscr{K}}$ and $\tau_{\mathscr{K}}$ are strictly coarser than the natural topologies on $\mathscr{C}([0,1])$ and $\mathscr{C}^\infty([0,1])$, respectively.

2 Completeness

Let us denote $\mathscr{K}_{seq}$ the system of all compact subsets of $[0,1]$ having only finitely many accumulation points and let us denote $\mathscr{K}_{seq}^0$ the system of all compact subsets K of $[0,1]$ such that there is $\varepsilon > 0$ with $[0,\varepsilon] \cap K \in \mathscr{K}_{seq}$.

Theorem 2

1. If $\mathscr{K}_{seq} \leq \mathscr{K}$ then $(\mathscr{C}([0,1]), \theta_{\mathscr{K}})$ is complete.
2. If $\mathscr{K}_{seq}^0 \leq \mathscr{K}$ then $(\mathscr{C}^\infty([0,1]), \tau_{\mathscr{K}})$ is complete.

Proof. (1) Let $\theta := \theta_{\mathscr{K}}$ and let Φ be a Cauchy filter in $(\mathscr{C}([0,1]), \theta)$. Since $\cup\mathscr{K} = [0,1]$ this filter converges pointwise to a function f and for any $K \in \mathscr{K}$ its restriction to K converges in the Banach space $\mathscr{C}(K)$ to a function f_K with $f|_K = f_K$. In particular, $f|_K$ is continuous. Since $\mathscr{K}_{seq} \leq \mathscr{K}$ the function f is sequentially continuous, hence continuous. Therefore Φ converges in all the spaces $(\mathscr{C}([0,1]), p_K)$, $K \in \mathscr{K}$, to f and this means precisely that it is θ-convergent to f.
(2) Set $\tau := \tau_{\mathscr{K}}$ and let Φ be a Cauchy filter in $(\mathscr{C}^\infty([0,1]), \tau)$. Then $D : (\mathscr{C}^\infty([0,1]), \tau) \to (\mathscr{C}([0,1]), \theta)^{\mathbb{N}_0}$, $f \mapsto (f^{(n)})_{n\in\mathbb{N}_0}$ is an isomorphism onto its range. Using (1) we obtain that $D(\Phi)$ converges to some $F = (f_n)_{n\in\mathbb{N}_0}$. It remains to show that the continuous functions f_n are differentiable and $f_n' = f_{n+1}$, $n \in \mathbb{N}_0$. To this end, we use that $\mathscr{K}_{seq}^0 \leq \mathscr{K}$. Since $[\varepsilon, 1] \in \mathscr{K}$ for each $\varepsilon \in (0,1)$, we see that f_n is differentiable on $(0,1]$ and its derivative is $f_{n+1}|_{(0,1]}$. Since f_{n+1} is continuous this shows that f_n is also differentiable at 0 and that $f_n'(0) = f_{n+1}(0)$. ∎

Remark 3 If $\mathscr{K} = \mathscr{K}_{seq}$ then $(\mathscr{C}^\infty([0,1]), \tau_{\mathscr{K}})$ is not complete since D (taken from the preceding proof) has in this case a dense range. Indeed, let $f_1, \ldots, f_n \in \mathscr{C}([0,1])$, and $K \in \mathscr{K}_{seq}$ be given. We may assume that K has only one accumation point, say x_0. We choose a polynomial p with $p^{(\nu)}(x_0) = f_\nu(x_0)$, $0 \leq \nu \leq n$. For every $\varepsilon > 0$ there is a neighbourhood of x_0 on which $|p^{(\nu)} - f_\nu| < \varepsilon$. Outside this neighbourhood there are only finitely many points of K, hence we find a smooth function g which coincide with p on a neighbourhood $U$of x_0 and satifies $g^{(\nu)} = f_\nu$ on $K \setminus U$.

Theorem 2 allows to construct an example separating two natural notions of diffotopy of homomorphisms between m-algebras (i.e. complete locally m-convex algebras), see also [1], 1.1.

Let A and B be m-algebras. Two continuous homomorphisms $\varphi, \psi : A \to B$ are called diffotopic if there is a continuous homomorphism $\alpha : A \to \mathscr{C}^\infty([0,1], B)$ with $\alpha(\cdot)(0) = \varphi$ and $\alpha(\cdot)(1) = \psi$. Here $\mathscr{C}^\infty([0,1], B)$ is identified with the complete π-tensor product $\mathscr{C}^\infty([0,1])\hat{\otimes}_\pi B$, where $\mathscr{C}^\infty([0,1])$ carries its natural Fréchet-topology. (Since $\mathscr{C}^\infty([0,1])$ is nuclear we can choose also the complete ε-tensor product)
Let us call φ and ψ pointwise diffotopic if there is a family of continuous homomorphisms $\alpha_t : A \to B$, $t \in [0,1]$ such that $\alpha_0 = \varphi$, $\alpha_1 = \psi$ and for any $a \in A$ the map $t \mapsto \alpha_t(a)$ from $[0,1]$ to B is smooth. Let $\mathscr{K} := \mathscr{K}^0_{seq}$ and $A := (\mathscr{C}^\infty[0,1], \tau_{\mathscr{K}})$, $B := \mathbb{C}$.
We show that the evaluations $\delta_0 : A \to \mathbb{C}$ and $\delta_1 : A \to \mathbb{C}$ are not diffotopic. Assume that there is a continuous homomorphism $\alpha : A \to \mathscr{C}^\infty([0,1])$ connecting δ_0 and δ_1. Then $f \mapsto \alpha(f)(x)$ is a continuous character on A for every $x \in [0,1]$, hence there is $g(x) \in [0,1]$ with $\alpha(f)(x) = f(g(x))$. So $\alpha(f) = f \circ g$ and α is a composition operator. Applying α to $f(x) = x$ we see that g is smooth. Since α connects δ_0 and δ_1 we obtain $g(0) = 0$ and $g(1) = 1$. The continuity of α ensures the existence of $K \in \mathscr{K}^0_{seq}$, $n \in \mathbb{N}$, and $C \geq 1$ such that

$$\sup_{x\in[0,1]} |f(g(x))| \leq C \sup_{0\leq\nu\leq n} \sup_{y\in K} |f^{(\nu)}(y)|$$

for every $f \in \mathscr{C}^\infty([0,1])$. But there is $x_0 \in [0,1]$ with $g(x_0) \notin K$. This contradicts the estimate above.
On the other hand, the homomorphism $\alpha : A \to \mathscr{C}^\infty([0,1])$, $f \mapsto f$ connects δ_0 and δ_1. If we equip $\mathscr{C}^\infty([0,1])$ with the topology of pointwise convergence, α becomes continuous. Hence δ_0 and δ_1 are pointwise diffotopic.

Remark 4 i) With the obvious modifications in the definitions (replace smooth by continuous and $\mathscr{C}^\infty([0,1])\hat{\otimes}_\pi B$ by $\mathscr{C}([0,1])\hat{\otimes}_\varepsilon B$) one can introduce also the concepts of homotopy and pointwise homotopy. Then δ_0 and δ_1 are not even homotopic.
ii) If A is ultrabornogical (e.g. if A is Fréchet) and $\mathscr{C}^\infty([0,1])\hat{\otimes}_\pi B$ has a web (e.g. if B is Fréchet) then the closed graph theorem implies that both notions of diffotopy coincide. An analogous result holds in case of homotopy.

References

[1] J. Cuntz, *Bivariante K-Theorie für lokalkonvexe Algebren und der bivariante Chern-Connes-Charakter*, Docum. Math. J. DMV 2 (1997), 139-182, http://www.mathematik.uni-bielefeld.de/documenta

[2] G. Köthe *Topological Vector Spaces II*, Springer-Verlag, New York-Berlin, 1979.

[3] R. Meise, D. Vogt, *Introduction to Functional Analysis*, Oxford University Press, New York, 1997.

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