Note on Hilbert-Schmidt composition operators on weighted Hardy spaces

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Abstract

We show that if C_φ is a Hilbert-Schmidt composition operator on an appropriately weighted Hardy space, then there exists a capacity, associated to the weight sequence of the space, so that the set on which the radial limit of φ is unimodular has capacity zero. This extends recent results by Gallardo-Gutiérrez and González.

Let $\mathbb{D}$ be the open unit disk in the complex plane and suppose that $(X, \|\cdot\|)$ is a Hilbert space of analytic functions on $\mathbb{D}$. Following [3] we say that X is a weighted Hardy space if the set $\{z^j : j = 0, 1, 2, \dots\}$ of monomials is a complete orthogonal system. We put $\beta_j = \|z^j\|$. Then $\beta := \{\beta_j\}$ is called the weight sequence and X is denoted by $H^2(\beta)$.

Many classical function spaces are weighted Hardy spaces. For example, the standard Hardy space H^2, the α-Dirichlet space $\mathcal{D}_\alpha$, $0 \leq \alpha < 1$, of all analytic functions whose first derivative is square integrable with respect to the measure $(1-|z|^2)^\alpha dA(z)$, and the Bergman space A^2 of all square integrable analytic functions are particular instances of $H^2(\beta)$ with $\beta_j \equiv 1$, $\beta_j \sim (1+j)^{(1-\alpha)/2}$ and $\beta_j = (1+j)^{-1/2}$ respectively.

Now suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ is an analytic self-map of the unit disk and consider the corresponding composition operator C_φ acting on $H^2(\beta)$, i.e.

$$C_\varphi(f) = f \circ \varphi, \ f \in H^2(\beta).$$

We are interested in the behavior of those φ which induce Hilbert-Schmidt composition operators.

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Recall that an operator T on a Hilbert space is called Hilbert-Schmidt if

$$\sum_j \|Te_j\|^2 < \infty \tag{1}$$

for some orthonormal basis $\{e_j\}$. It can be shown that the above quantity is independent of the choice of the basis. We denote the sum in (1) by $\|T\|_{HS}^2$. So, in the case of a composition operator on $H^2(\beta)$ we have

$$\|C_\varphi\|_{HS}^2 = \sum_j \frac{\|\varphi^j\|^2}{\beta_j^2}.$$

Hilbert-Schmidt composition operators on H^2, $\mathcal{D}_\alpha$ and A^2 have been studied in [2] and [7]. Gallardo-Gutiérrez and González [4], [5] recently found an interesting Fatou-type necessary condition in order for φ to induce a Hilbert-Schmidt composition operator on $\mathcal{D}_\alpha$. Namely, if C_φ is Hilbert-Schmidt on $\mathcal{D}_0$ then the radial limit

$$\varphi_*(t) := \lim_{r\to 1^-} \varphi(re^{it})$$

can have modulus 1 only on a set of logarithmic capacity zero. Similarly, if C_φ is Hilbert-Schmidt on $\mathcal{D}_\alpha$, $0 < \alpha < 1$ then the set $\{|\varphi_*| = 1\}$ has zero Riesz α-capacity.

The argument in [4], [5] is based on the characterization:

$$C_\varphi \text{ is Hilbert-Schmidt on } \mathcal{D}_\alpha \iff \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2+\alpha}}(1-|z|^2)^\alpha dA(z) < \infty,$$

and on the minimization of the energy integral of an appropriate function. In a general weighted Hardy space, concrete integral characterizations like the one above are unavailable, and therefore, the techniques in [4], [5] do not seem to apply.

The purpose of this note is to extend the results of Gallardo-Gutiérrez and González to a certain class of weighted Hardy spaces, using a very simple general argument. We shall show that if $H^2(\beta)$ is a "small" space which is "not too small", and C_φ is Hilbert-Schmidt on $H^2(\beta)$, then there is a natural capacity, associated to the weight sequence β, so that the set $\{|\varphi_*| = 1\}$ has capacity zero. Here "small" means that $\{\beta_j^{-1}\}$ is, essentially, a sequence of Fourier coefficients, whereas "not too small" means that

$$\sum_j \beta_j^{-2} = \infty.$$

So, we will work with "mildly weighted" Hardy spaces.

In order to make the above into a precise statement we introduce some notation and terminology.

For non-negative x and y, $x \lesssim y$ means $x \le Cy$ for some constant $C > 0$, not necessarily the same at each occurrence. $x \sim y$ means ($x \lesssim y$ & $y \lesssim x$).

As usual, we identify the unit circle $\mathbb{T}$ with $[-\pi, \pi)$.

For $f \in L^1(\mathbb{T})$ its Fourier coefficients are given by

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt, \ n \in \mathbb{Z}.$$

If Φ is a kernel on $\mathbb{T}$, that is, a non-negative, radially decreasing, integrable function, then we define the Φ-capacity of a set $E \subset \mathbb{T}$ by

$$\mathrm{Cap}^{\Phi}(E) = \inf\{\|f\|_2^2 : f \in L^2(\mathbb{T}),\ f \geq 0,\ \Phi * f \geq 1 \text{ on } E\}.$$

Finally, if $\beta = \{\beta_j\}$ is a positive sequence and Φ is a kernel on $\mathbb{T}$, then we say that (β, Φ) is an admissible pair if

$$\Phi \in L^1(\mathbb{T}) \setminus L^2(\mathbb{T}), \quad \text{and} \quad \widehat{\Phi}(j) \sim \beta_{|j|}^{-1},\ j \in \mathbb{Z}.$$

Now, to motivate the statement of our result, let us look more closely at the relation between the capacities and the spaces considered in [4] and [5].

$\mathcal{D}_0$ is $H^2(\beta)$ with $\beta_j \sim (1+j)^{1/2}$. So, (β, Φ) is an admissible pair with $\Phi(t) = |t|^{-1/2}$, and the logarithmic capacity is induced by the kernel Φ.

Similarly, $\mathcal{D}_\alpha$ $0 < \alpha < 1$ is $H^2(\beta)$ with $\beta_j \sim (1+j)^{(1-\alpha)/2}$, (β, Φ) is an admissible pair with $\Phi(t) = |t|^{-(1+\alpha)/2}$ and the Riesz α-capacity is induced by Φ as before.

These observations naturally suggest the following generalization of the results in [4] and [5].

Theorem. *Let $H^2(\beta)$ be a weighted Hardy space such that (β, Φ) is an admissible pair for some Φ. If C_φ is Hilbert-Schmidt on $H^2(\beta)$ then, outside a* Cap^{Φ}*- null set, the radial limit φ_* exists and*

$$\mathrm{Cap}^{\Phi}(\{|\varphi_*| = 1\}) = 0.$$

To prove the theorem, first we observe that functions in $H^2(\beta)$ have radial limits Cap^{Φ}-almost everywhere. Indeed, for any $f \in H^2(\beta)$ we have

$$f(re^{it}) = \sum_{n=0}^{\infty} a_n \beta_n^{-1} r^n e^{int},$$

where $\{a_n\} \in l^2$. Since $\widehat{\Phi}(n) \sim \beta_{|n|}^{-1}$, there exists a function g in the usual Hardy space $H^2(\mathbb{T})$ with $\|g\|_2 \sim \|f\|$ such that

$$f(re^{it}) = \sum_{n=-\infty}^{\infty} \widehat{\Phi}(n)\widehat{g}(n) r^{|n|} e^{int} = P_r * \Phi * g(t),$$

where P_r is the Poisson kernel. This means that f belongs to a Dirichlet-type space in the sense of [6]. Therefore, the radial limit f_* exists outside a Cap^{Φ}- null set and in fact $f_* = \Phi * g$, Cap^{Φ}- almost everywhere. In particular, $\varphi_*^j = \Phi * g_j$ for some g_j as above.

Now for $0 < \lambda < 1$

$$\begin{aligned}
\mathrm{Cap}^{\Phi}(\{|\varphi_*| \geq \lambda\}) \sum_{j=1}^{\infty} \lambda^{2j} \beta_j^{-2} &\lesssim \sum_{j=1}^{\infty} j \beta_j^{-2} \int_0^{\lambda} \mathrm{Cap}^{\Phi}(\{|\varphi_*| \geq s\}) s^{2j-1} ds \\
&\leq \sum_{j=1}^{\infty} \beta_j^{-2} \int_0^1 \mathrm{Cap}^{\Phi}(\{|\varphi_*^j| \geq s\}) s ds \\
&\leq \sum_{j=1}^{\infty} \beta_j^{-2} \int_0^1 \mathrm{Cap}^{\Phi}(\{\Phi * |g_j| \geq s\}) s ds.
\end{aligned}$$

By the capacitary strong type inequality ([1, p. 189, Theorem 7.1.1]) we have

$$\int_0^1 \operatorname{Cap}^\Phi(\{\Phi * |g_j| \geq s\}) s ds \lesssim \|g_j\|_2^2.$$

Therefore

$$\operatorname{Cap}^\Phi(\{|\varphi_*| \geq \lambda\}) \sum_{j=1}^\infty \lambda^{2j} \beta_j^{-2} \lesssim \sum_{j=0}^\infty \|g_j\|_2^2 \beta_j^{-2} \sim \|C_\varphi\|_{HS}^2. \tag{2}$$

Since $\Phi \notin L^2(\mathbb{T})$, we see that

$$\sum_{j=1}^\infty \beta_j^{-2} = \infty,$$

so letting $\lambda \to 1^-$ we obtain

$$\operatorname{Cap}^\Phi(\{|\varphi_*| = 1\}) = 0.$$

Note that (2) actually gives an estimate for the rate of convergence of the capacitary size of the sublevel set $\{|\varphi_*| \geq \lambda\}$ as $\lambda \to 1^-$.

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