# Sectional category of fibrations of fibre $K(\mathbb{Q}, 2k)$.

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**Abstract**

We show that the sectional category of a non trivial fibration $p$ with fibre $K(\mathbb{Q}, 2k)$ has sectional category 1 although all $n$-fold fibre joins $p * \cdots * p$ are not trivial.

## 1 Introduction

We recall here some homotopic invariants related to the Lusternik-Schnirelmann category [8].

**Definition 1.** The category of a map $f : X \to Y$, denoted by $\mathrm{cat}(f)$, is the least integer $n$ such that $X$ can be covered by $n + 1$ open subsets $U_i$, for which the restriction of $f$ to each $U_i$ is null homotopic. The category of $X$, $\mathrm{cat}(X)$, is the category of the identity mapping on $X$.

We have the relation

$$\mathrm{cat}(f) \leq \min\{cat(X), cat(Y)\}. \tag{1}$$

The rational category of $X$, denoted by $cat_0(X)$, is defined by $cat_0(X) = cat(X_0)$. Here $X_0$ denotes the rationalization of $X$. For a mapping $f : X \to Y$, $cat_0(f)$ will denote $cat(f_0)$, where $f_0 : X_0 \to Y_0$ is the rationalization of $f$.

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Let $X$ be a simply connected CW-complex for which $H^i(X,\mathbb{Q})$ is a finite dimensional $\mathbb{Q}$-vector space, for each $i$. The Sullivan minimal model of $X$ is a free commutative cochain algebra $(\wedge Z, d)$ such that $dZ \subset \wedge^{\geq 2} Z$, with $Z^n \cong \mathrm{Hom}_{\mathbb{Q}}(\pi_n(X),\ \mathbb{Q})$ (see [11], [7]). Félix and Halperin showed that the rational category can be computed by the means of the Sullivan minimal model of $X$.

**Theorem 2.** *[5] If $(\wedge Z, d)$ is the Sullivan minimal model of $X$, then $cat_0(X)$ is the least integer $n$ such that $i$ has a retraction $\rho$ in the following diagram:*

$$\begin{tikzcd}
(\wedge Z, d) \arrow[d, "p"] \arrow[dr, "i"] & \\
(\wedge Z/\wedge^{>n} Z, \bar{d}) & \wedge Z \otimes \wedge T \arrow[l, "\simeq"'] \arrow[ul, dotted, bend right, "\rho"']
\end{tikzcd}$$

The rational Toomer invariant of $X$, written $e_0(X)$, is the largest integer $k$ such that some non trivial cohomology class is represented by a cocycle in $\wedge^{\geq k} Z$. It is always true that

$$e_0(X) \leq cat_0(X) \quad [13]. \tag{2}$$

**Definition 3.** Let $p : E \to B$ be a fibration. The sectional category of $p$, secat$(p)$, is the least integer $n$ such that $B$ can be covered by $(n+1)$ open subsets, over each of which $p$ has a section.

**Definition 4.** The genus of a fibration $X \to E \xrightarrow{p} B$ is the least integer $n$ such that $B$ can be covered by $(n+1)$ open subsets, over each of which $p$ is a trivial fibration, in the sense of fibre homotopy type [10, Chap.2, Sec.8].

It is straightforward that $\mathrm{secat}(p) \leq \mathrm{genus}(p)$ and equality holds when $p$ is a principal fibration.

Fibrations with fibre in the homotopy type of $X$ are obtained, up to fibre homotopy equivalence, as pull back of the universal fibration

$$X \to B\, aut^{\bullet} X \to B\, aut\, X \ [2],$$

where $aut\, X$ denotes the monoid of self-homotopy equivalences of $X$, $aut^{\bullet} X$ is the monoid of pointed self-homotopy equivalences of $X$, and $B$ is the Dold-Lashof functor from monoids to topological spaces [3].

Letting $\tilde{B}\, aut\, X \to B\, aut\, X$ be the universal covering, the induced fibration $X \to \tilde{B}\, aut^{\bullet} X \to \tilde{B}\, aut\, X$ is universal for fibrations with simply connected base spaces [4, Proposition 4.2]. Note that $\tilde{B}\, aut\, X$ is homeomorphic to $B\, aut_1(X)$, where $aut_1(X)$ denotes the path component of $aut\, X$ containing the identity.

The genus is related to classifying spaces by the following

**Proposition 5.** *[8] If $X \to E \xrightarrow{p} B$ is a fibration, then*

$$\mathrm{genus}(p) = \mathrm{cat}(f), \tag{3}$$

*where $f : B \to B\, aut\, X$ is the classifying map of $p$.*

## 2 Fibrations with fibre a product of $n$ copies of $K(\mathbb{Q}, 2k)$.

Let $p : E \to B$ be a fibration with fibre a product of $n$ copies of $K(\mathbb{Q}, 2k)$. Then $p$ is represented by the KS-extension $A \to (A \otimes \wedge(y_1, y_2, \ldots, y_n), d)$, with $|y_i| = 2k$ and where $dy_i = \alpha_i$. The $\alpha_i$'s represent cohomology classes in $H^{2k+1}(A)$. A lower bound of the sectional category is given by the nilpotency index of the ideal generated by the $\alpha_i$'s [8]. Since the $\alpha_i$'s have odd degrees, this nilpotency index is $\leq n$. The following result provides an upper bound.

**Theorem 6.** *Let $X$ be a product of $n$ copies of $K(\mathbb{Q}, 2k)$ and $p$ a rational fibration with fibre $X$, then* $\mathrm{secat}(p) = genus(p) \leq n$.

*Proof.* We use a model of the classifying space $B\,aut_1(X)$, as described by Sullivan in [11]. A model of $B\,aut_1(X)$ is obtained as the Lie algebra of derivations of a Sullivan model of $X$. Since the Sullivan minimal model of $X$ is $(\wedge(x_1, \ldots, x_n), 0)$ where $|x_i| = 2k$, a Lie model of the classifying space is the abelian Lie algebra $\oplus_{i=1}^{n} \mathbb{Q}\alpha_i$, where all $\alpha_i$ have degree $2k$, and with zero differential. The classifying space $B\,aut_1(X)$ has therefore the rational homotopy type of a product of $n$ copies of $S^{2k+1}$. Applying Proposition 5 and the relation (1), we deduce that

$$\mathrm{genus}(p) \leq cat(S^{2k+1} \times \cdots \times S^{2k+1}) = n.$$

Using a model of the universal fibration as described in [12], a model of $B\,aut_1^{\bullet}(X)$ is given by $\otimes_{i=1}^{n}(\wedge(x_i, y_i), d)$, with $|x_i| = 2k$, $|y_i| = 2k+1, dx_i = y_i$. Therefore the total space is rationally contractible, hence the universal fibration is the path fibration. We conclude that every rational fibration $p$ with fibre $X$ is principal. This yields $\mathrm{genus}(p) = \mathrm{secat}(p)$. ∎

In particular we have the following

**Corollary 7.** *A non trivial rational fibration $p$ with fibre $K(\mathbb{Q}, 2k)$ verifies* $\mathrm{secat}(p) = \mathrm{genus}(p) = 1$.

## 3 Join and cojoin operations

If $F_1 \to E_1 \xrightarrow{p_1} B$ and $F_2 \to E_2 \xrightarrow{p_2} B$ are fibrations with the same base space, then the fibrewise join is the fibration $p_1 * p_2 : E_1 *_B E_2 \to B$, where elements of $E_1 *_B E_2$ are of the form $(t_1e_1, t_2e_2)$, $t_1 + t_2 = 1$, $p_1(e_1) = p_2(e_2)$, with the restriction that $t_ie_i$ is independent of $e_i$ if $t_i = 0$. Naturally $(p_1 * p_2)(t_1e_1, t_2e_2) = p_1(e_1) = p_2(e_2)$. Note that the fibre is the join $F_1 * F_2$. If $p$ is a fibration, then $p(n)$ will denote the fibrewise join of $n + 1$ copies of $p$. Schwarz proved the following

**Proposition 8.** *[8, 9] If $p : E \to B$ is a fibration, then the sectional category of $p$ is the least integer $n$ such that the $(n + 1)$-fold fibre join $p(n)$ admits a homotopic section.*

In the category of commutative differential graded algebras, we consider the subcategory of 1-connected objects, that is, each object $A$ verifies $A^0 = \mathbb{Q}$ and $A^1 = 0$. This assumption is sufficient to enable us to compute cojoins in that category [1], in which fibrations are surjective mappings while cofibrations are KS-extensions $A \rightarrowtail A \otimes \wedge V$ .

$$\begin{tikzcd}
 & & B *^A C \arrow[drr, "p'"] \arrow[dd, "i''"] & & \\
A \arrow[urr, "(g,\sigma i)"] \arrow[rrrr, "g"] \arrow[dr, "f"] \arrow[dd, tail, "i"'] & & & & C \arrow[dd, tail, "i'"] \\
 & B & E \arrow[drr, "p"] & & \\
D \arrow[ur, "\tau"] \arrow[urr, "\sigma"] \arrow[rrrr, "g'"'] & & & & C'
\end{tikzcd}$$

Figure 1: Cojoin operation

Consider two maps $f : A \to B$ and $g : A \to C$ between commutative differential graded algebras (see Figure 1). Factorize $f = \tau \circ i$, where $i$ is a cofibration and $\tau$ a weak equivalence, form then the push out of $i$ and $g$. Now factorize $g' = p \circ \sigma$ where $p$ is a fibration and $\sigma$ a weak equivalence. The pullback of $p$ and $i'$, $B *^A C$, is called the cojoin of the maps $f$ and $g$. If $A$ is the zero object of the cojoin category, that is, $A^0 = \mathbb{Q}$ and $A^+ = 0$, then $B *^A C$ is simply written $B * C$ and is called the cojoin of $B$ and $C$.

We will use the cojoin process to prove the following

**Theorem 9.** *Let $K(\mathbb{Q}, 2k) \to E \xrightarrow{p} B$ be a non trivial fibration between rational spaces. The fibrations $p(n)$ verify the following properties:*

1. *$p(1) = p * p$ admits a section,*
2. *For all $n \geq 1$, $p(n)$ is not trivial.*

*Proof.* First of all, note that a fibration $p$ with fibre $K(\mathbb{Q}, 2k)$ is trivial if and only if $\mathrm{genus}(p) = \mathrm{secat}(p) = 0$.

Let $p$ be a non trivial fibration with fibre $K(\mathbb{Q}, 2k)$. Consider the KS-extension

$$(A, d_A) \xrightarrow{\imath} (A \otimes \wedge x, d) \longrightarrow (\wedge x, 0)$$

modelling the fibration $p$. The element $\alpha = dx \in A$ represents a non-trivial cohomology class in $H^{2k+1}(A, d_A)$, otherwise the fibration is trivial. Such a fibration does not admit a section. A model of $p*p$ is the cojoin $\imath * \imath$ where $\imath : (A, d_A) \rightarrowtail (A \otimes \wedge x, d)$ .

Now consider the push out

$$\begin{tikzcd}
A \arrow[r, tail, "\imath"] \arrow[d, "\jmath"'] & A \otimes \wedge x \arrow[d, "\bar{\jmath}"'] \\
A \otimes \wedge y \arrow[r, tail, "\bar{\imath}"] & A \otimes \wedge(x, y),
\end{tikzcd}$$

where $A \otimes \wedge y$ is canonically isomorphic to $A \otimes \wedge x$.

Factorize $\bar{\jmath} : A \otimes \wedge x \longrightarrow A \otimes \wedge(x, y)$ as

$$A \otimes \wedge x \xrightarrow{\simeq} (A \otimes \wedge(x, y, t), \tilde{d}) \xrightarrow{\pi} A \otimes \wedge(x, y),$$

where $\tilde{d}x = \alpha$, $\tilde{d}y = \alpha + t$ and $\tilde{d}t = 0$. The mapping $\pi$ is such that $\pi|_{A\otimes\wedge x} = \bar{\jmath}$, $\pi(y) = y$ and $\pi(t) = 0$. The total space of the fibre join $p * p$ is the pullback

$$
\begin{tikzcd}
(\mathcal{A}_1, D_1) \arrow[r] \arrow[d, "\pi'"] & A \otimes \wedge(x, y, t) \arrow[d, "\pi"] \\
A \otimes \wedge y \arrow[r, "\bar{\imath}"] & A \otimes \wedge(x, y)
\end{tikzcd}
$$

The natural inclusion mapping $\iota(1) : A \to \mathcal{A}_1$ is a model of the fibre join fibration $p * p$.

Note that

$$\mathcal{A}_1 = \Big\{(u, v) \in [A \otimes \wedge y] \bigoplus [A \otimes \wedge(x, y, t)] : \bar{\imath}(u) = \pi(v)\Big\}.$$

One can verify that the algebra $\mathcal{A}_1$ is isomorphic to $A \otimes (\wedge y \underset{\sim}{\oplus} t. \wedge (x, y))$, of which the underlying vector space is $A \otimes (\wedge y \oplus t. \wedge (x, y))$, but $y^m.tx^n y^r = tx^n y^{m+r}$. Moreover $D_1 y = \alpha + t$, $D_1 t = 0$, $D_1 y^n = ny^{n-1}\alpha + ny^{n-1}t$ and for $r \geq 1$ or $s \geq 1$, $D_1(x^r y^s t) = r\alpha x^{r-1} y^s t + sx(\alpha + t)y^{s-1}t = r\alpha x^{r-1}y^s t + sx\alpha y^{s-1}t$. The cohomology of the fibre $(\mathbb{Q} \otimes_A \mathcal{A}_1, \bar{D}_1)$ is isomorphic to $t. \wedge^+ x \otimes \wedge y$. The projection map is surjective onto $[tx]$ because $[tx + \alpha y]$ maps to $[tx]$, but there is no cohomology class in $\mathcal{A}_1$ that maps to $[tx^2]$. Suppose in fact that there exists such a class $[u]$. We write $u = tx^2 + \delta y^2 + \rho tx + \sigma ty + \mu t + \nu y + \theta$, with $|\delta| = 2k + 1$, $|\rho| = |\sigma| = 2k$, $|\mu| = 4k$, $|\nu| = 4k + 1$ and $|\theta| = 6k + 1$. The equation $D_1 u = 0$ implies $2\alpha = -d_A(\rho)$ which is in contradiction with our assumption on $\alpha$. This shows that the fibration is not trivial.

Furthermore one can define a retraction $\rho : \mathcal{A}_1 \to A$ as follows:

$$\rho|_A = id_A, \;\; \rho(y) = 0, \; \rho(t) = -\alpha \text{ and } \rho(x^r y^s t) = 0 \text{ for } r > 0 \text{ or } s > 0.$$

It is easily checked that $\rho$ commutes with the differentials. Hence the fibration $p * p$ has sectional category 1 as expected (see Corollary 7).

To show that $p(n)$ is not trivial, we have to repeat the above cojoin process. Computations yield

$$(\mathcal{A}_n, D_n) = (A \otimes (\wedge y_n \underset{\sim}{\oplus} t_n.(V_{n-1} \otimes \wedge y_n)), D_n),$$

where $|y_n| = 2k, |t_n| = 2k + 1$. The algebras $V_i, i \geq 1$ are defined inductively by the formula

$$V_1 = \wedge y_1 \underset{\sim}{\oplus} t_1. \wedge (y_0, y_1), \; V_i = \wedge y_i \underset{\sim}{\oplus} t_i.(V_{i-1} \otimes \wedge(y_i)),$$

where $|y_i| = 2k$ and $|t_i| = 2k + 1$.

The differential verifies $D_n(y_0) = \alpha$ and $D_n(y_p) = \alpha + t_p$ for $p = 1, \ldots, n$. The same argument as in the case $n = 1$ works. The element $t_n \ldots t_1 y_0^2$ represents a nonzero cohomology class in the quotient that can not lift into a cocycle in $\mathcal{A}_n$. Therefore the fibration is not trivial. ∎

The following example shows that Theorem 6 does not hold if the fibre is a product of distinct Eilenberg-MacLane spaces.

*Example* 10. Consider the space $X = K(\mathbb{Q}, 2) \times K(\mathbb{Q}, 4)$. The minimal Sullivan model of $B\, aut_1(X)$ is $(\wedge(x_3, y_3, x_5), d)$, with $dx_3 = dy_3 = 0,\ dx_5 = x_3y_3$. Here subscripts indicate degrees. Applying Theorem 2 in conjunction with the inequality (2), we deduce that $\mathrm{cat}(B\, aut_1(X)) = 3$ since the nilpotency index of $(\wedge(x_3, y_3, x_5), d)$ is three and $x_3y_3z_5$ represents a nonzero cohomology class. Therefore the genus of the universal fibration is 3.

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