Extension of vector-valued holomorphic and meromorphic functions

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Abstract

We present several results about the extension of vector-valued holomorphic or meromorphic functions from an open domain in $\mathbb{C}$ to a larger domain on which the function has a weakly holomorphic or meromorphic extension.

1 Introduction

The main problem which is considered in this article can be stated as follows: Let $\Omega_1 \subseteq \Omega_2$ be two non empty open connected subsets of $\mathbb{C}$, and let E be a complex Hausdorff locally convex space satisfying certain completeness assumptions. Which conditions on the space E ensure that every function $f : \Omega_1 \longrightarrow E$ such that $u \circ f$ admits a meromorphic extension to Ω_2 for each $u \in E'$ can be extended to Ω_2 as a meromorphic function with values in E? One of our main tools is the result proved by Bonet, Maestre and the author in [6]: *if E is locally complete and does not contain the countable product ω of copies of $\mathbb{C}$, then there is a canonical isomorphism between the space of meromorphic functions $M(\Omega, E)$ from a domain Ω in $\mathbb{C}$ to E and the ε-product of Schwartz $M(\Omega)\varepsilon E = L(E'_{co}, M(\Omega))$* when $M(\Omega)$ is endowed with the locally convex topology defined by Holdgrün in [18] and deeply studied by Grosse-Erdmann in [14].

Our main results give the following answers to the problem stated above. They constitute extensions of results due to Hai, Khue and Nga [17] and Grosse-Erdmann [13]: Suppose that E is locally complete and does not contain ω. The meromorphic

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extension $\hat{f} \in M(\Omega_2, E)$ exists if E'_β is suprabarrelled (Theorem 12), or if E is a barrelled complete Schwartz space (Theorem 16) or if E is a distinguished Fréchet space such that E''_β has a continuous norm (Theorem 17).

We study also the analogous problem for holomorphic functions, obtaining that whenever E is a locally complete locally convex space and $f : \Omega_1 \to E$ is a function such that $u \circ f$ admits a holomorphic extension to Ω_2 for each $u \in E'$ then also f can be holomorphically extended to Ω_2 (Theorem 3). The proofs use almost exclusively functional analytic techniques.

2 Notation and Preliminaries

Throughout this paper, E denotes a complex and Hausdorff locally convex space and Ω denotes a domain, i.e. a non-empty open and connected set, in $\mathbb{C}$. Our notation for locally convex spaces and functional analysis is standard. We refer to [21, 25, 27]. We recall the terminology which will be repeatedly used. In a topological vector space we denote by $\mathrm{cx}(A)$ and $\mathrm{acx}(A)$ the convex and the absolutely convex hull of A respectively. In a metric space we denote by $B(a, r)$, $D(a, r)$ and $S(a, r)$ the open ball, the closed ball and the sphere centered on a with radius r respectively. Given a subset A of a topological space we denote by $\overline{A}$ the closure of A and by ∂A its boundary. E_σ denotes E endowed with the weak topology $\sigma(E, E')$, E'_β denotes the strong dual of E, E'_{co} denotes the dual of E endowed with the topology of uniform convergence on absolutely convex compact sets of E and E'_μ denotes the dual endowed with the topology of uniform convergence in absolutely convex weakly compact sets, i.e. $E'_\mu = (E_\sigma)'_{co}$. A subspace S of E' is called *separating* if $S^\circ = \{0\}$, the polar taken in E. For two locally convex spaces E and F, we denote by $L(E, F)$ the space of continuous linear maps defined on E and with values in F. A locally convex space E is said to be Montel if it is barrelled and each bounded set in E is relatively compact. The space of holomorphic functions $H(\Omega)$ is an example of a Fréchet-Montel space. If E is a Montel space, $E'_{co} = E'_\beta$ holds. For E and F locally convex spaces, the space $L_e(F'_{co}, E)$, that is, the space $L(F'_{co}, E)$ endowed with the topology of the uniform convergence on the equicontinuous subsets of F', is called ε-product of Schwartz and denoted by $E\varepsilon F$. We remark that, in this paper, we will not use the topology defined in the space $E\varepsilon F$. Actually, we are only interested in which vectors belong to an ε-product. The ε-product of Schwartz has the following property [24, 43.3.(3)]:

$$E\varepsilon F = L_e(F'_{co}, E) \simeq L_e(E'_{co}, F) = F\varepsilon E.$$

Let I be an index set, the product of locally convex spaces each one of them isomorphic to E is denoted by E^I, and their direct sum is denoted by $E^{(I)}$. $\mathbb{C}^{\mathbb{N}}$ is denoted by ω and $\mathbb{C}^{(\mathbb{N})}$ by φ. We refer to [29] for elementary properties of holomorphic and meromorphic functions. The space of E-valued functions holomorphic on Ω is denoted by $H(\Omega, E)$. For equivalent definitions of vector-valued holomorphic and meromorphic functions we refer to [9, 13].

Let E be a locally convex space. A disc in E is a subset which is bounded and absolutely convex. Given a disc B, we denote by E_B the linear span of B endowed with the norm topology $\|\cdot\|_B$, where $\|x\|_B = \inf\{\lambda \in \mathbb{R}^+ : x \in \lambda B\}$. If E_B is a

Banach space B is called a Banach disc. Recall that a locally complete space is a locally convex space in which every closed disc is a Banach disc.

A sequence $(x_n)_n$ in E is said to be *locally convergent* if there is a disc B in E such that the sequence converges to x in E_B. Given a subset A of E, a point x is a *local limit point* of A if there is a sequence in A locally convergent to x. A is called *locally closed* if every local limit point of A belongs to A. Every locally complete subspace of E is locally closed and a locally closed subspace of a locally complete space is locally complete [27, Proposition 5.1.20]. In this paper we deal with locally complete locally convex spaces, and for this kind of spaces a function is holomorphic if and only if it is weakly holomorphic [7, Lemma 3.1.1]. The spaces in which this happens were called *differentially stable* by Nachbin [26].

Lemma 1. *Let Ω be a domain in $\mathbb{C}$, let E be a locally complete locally convex space and let F be a locally closed subspace of E. If $f \in H(\Omega, E)$ and there exists a non-empty open subset V of Ω with $f(V) \subset F$, then $f \in H(\Omega, F)$.*

Proof. It is enough to prove that given $a \in \partial V \cap \Omega$ there exists $r > 0$ such that $f(B(a,r)) \subset F$. Let $r > 0$ such that $D(a,r) \subset \Omega$. We define the set

$$B_1 := \left\{ \frac{f(z) - f(t)}{|z-t|} : z, t \in D(a,r),\ z \neq t \right\},$$

which is seen to be bounded as in the proof of [6, Proposition 2] (see also [4]). Since $u \circ f$ is continuous on $D(a,r)$ for each $u \in E'$, the set $f(D(a,r))$ is (weakly) bounded in E. We set

$$B := \overline{\mathrm{acx}}\{f(D(a,r)) \cup B_1\}.$$

B is a Banach disc since E is locally complete. Moreover, we have that the restriction of f to $B(a,r)$ is continuous considering in the image the topology inherited from E_B, since

$$\|f(z) - f(t)\|_B \leq |z-t|.$$

Thus, if we take a sequence $(z_n)_n \subset B(a,r) \cap V$ which converges to a, we have that $(f(z_n))_n \subset F$ converges to $f(a)$ in E_B. We apply that F is locally closed to get $f(a) \in F$. Since $f \in H(V,F)$ (F is locally complete and then differentially stable), the $n-th$ derivatives $f^{(n)} \in H(V,F)$. Thus, the same argument shows that, for $n \in \mathbb{N}$, $f^{(n)}(a) \in F$. The restriction of the functionals of E' to E_B form a separating subspace of E_B' since the topology of E_B is finer than the topology of E and, by the assumptions, $u \circ f \in H(B(a,r))$ for every $u \in E'$. We can apply [13, Theorem 5.2] (cf. [15, Theorem 1]) to conclude that $f : B(a,r) \to E_B$ is holomorphic. Hence $f^{(n)}(a) \in F \cap E_B$ and, for every $z \in B(a,r)$,

$$f(z) = \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!} f^{(n)}(a)$$

holds in E_B. Since F is locally closed, $f(z) \in F$. ∎

Let Ω be a domain in $\mathbb{C}$. A function f defined on Ω with values in a locally convex space E is called *meromorphic* if there exists a subset D discrete in Ω such that $f \in H(\Omega \setminus D, E)$ and for each $\alpha \in D$ there exists $k \in \mathbb{N}$ such that $(z-\alpha)^k f(z)$

admits holomorphic extension in α. We write $f \in M(\Omega, E)$ ($M(\Omega)$ if $E = \mathbb{C}$). A function f defined on an open non-empty set $\Omega \subseteq \mathbb{C}$ with values in a locally convex space E is called *weakly meromorphic* if there exists a set D discrete in Ω such that f is holomorphic in $\Omega \setminus D$ and $u \circ f$ is a meromorphic function in Ω with poles contained in D for each $u \in E'$. We denote by $WM(\Omega, E)$ the space of weakly meromorphic functions defined on Ω with values in E. A function $f : \Omega \to E$ is called *very weakly meromorphic* if $u \circ f$ is meromorphic for every $u \in E$. The space of very weakly meromorphic functions defined on Ω with values in E is denoted by $Mer^{\omega}(\Omega, E)$ (cf. [13]). It was proved in [6] that for a locally complete space E, $WM(\Omega, E) = M(\Omega, E)$ if and only if E does not contain ω. For a space like this, [6, Proposition 6] shows that the mapping $T : M(\Omega, E) \longrightarrow L(E'_{co}, M(\Omega)) = M(\Omega)\varepsilon E$, $T(f)(u) = u \circ f$ is an isomorphism, if one identifies (as we do) meromorphic functions which coincide except on a discrete set. $M(\Omega)$ is endowed with the complete locally convex topology studied in [14] by Grosse-Erdmann. This topology is generated by the seminorms

$$\|f\|_{K,b} = \sup_{z \in K} |(f - \sum_{\alpha \in K} h^{\alpha}(f))(z)| + \sum_{\alpha \in K} \sum_{n=1}^{\infty} b_{\alpha}^{n} |(a_{\alpha}^{-n}(f))|,$$

where K runs over the compact subsets of Ω, $b = (b_{\alpha}^{n})_{\alpha \in K, n \in \mathbb{N}}$, $b_{\alpha}^{n} \geq 0$ for every $\alpha \in K$ and for every $n \in \mathbb{N}$, and $h^{\alpha}(f) = \sum_{j=1}^{\infty} a_{\alpha}^{-j}(f)(z-\alpha)^{-j}$ is the principal part of f at α, where $a_{\alpha}^{-j}(f) = 0$ except for a finite number.

Remark 2. As mentioned above we identify meromorphic functions which coincide except on a discrete set. With this identification, the locally convex space $M(\Omega)$ is Hausdorff. As a consequence of the principle of isolated zeros of holomorphic functions, two meromorphic functions on Ω which coincide in a set D which has an accumulation point in Ω only can differ in a discrete subset of Ω, and then both represent the same vector in the locally convex space $M(\Omega)$.

Grosse Erdmann [13, Theorem 2.6] showed that if E is locally complete and E'_{β} is Baire, then $M(\Omega, E) = Mer^{\omega}(\Omega, E)$. We conjecture that this holds for every locally complete space E which does not contain ω. Partial positive results can be found in Section 4.

3 Holomorphic extension

In this introductory section we deal with E-valued functions f defined on subsets $A \subseteq \Omega$ and such that $u \circ f$ admits a holomorphic extension to Ω for each $u \in S \subseteq E'$, obtaining results on holomorphic extension of f. For literature concerning this problem we refer to [1, 2, 3, 11, 12, 13, 15, 19, 20].

From [6, Proposition 2] it follows that if E is a locally complete locally convex space then for each holomorphic function $f : \Omega \longrightarrow E$ and for each compact subset K of Ω the subset $\overline{\mathrm{acx}} f(K)$ is compact in E. Therefore one can easily obtain that the canonical identification $H(\Omega, E) \simeq H(\Omega)\varepsilon E$ is valid for locally complete spaces E. That is, a linear map $T : E' \longrightarrow H(\Omega)$ belongs to $H(\Omega)\varepsilon E$ if and only if there exists $f \in H(\Omega, E)$ such that $T(u) = u \circ f$ for each $u \in E'$ (see [21, Theorem 16.7.4] where it is done for complete spaces).

Theorem 3. *Let E be locally complete locally convex space, and let $\Omega_1 \subseteq \Omega_2$ be two complex domains. If $f : \Omega_1 \longrightarrow E$ is a function such that $u \circ f$ admits a holomorphic extension to Ω_2 for each $u \in E'$, then f can be holomorphically extended to Ω_2.*

Proof. First we assume E to be a distinguished space, i.e. with barrelled strong dual, and we observe that if $\Omega_1 \subseteq \Omega_2$ are two complex domains and $f : \Omega_1 \longrightarrow E$ is a function such that $u \circ f$ admits a holomorphic extension $\widehat{u \circ f}$ to Ω_2, then the linear mapping $T : E'_\beta \longrightarrow H(\Omega_2)$, $u \mapsto \widehat{u \circ f}$, has closed graph, and it is continuous as a consequence of Pták's closed graph theorem (see [27, Theorem 7.1.12]). Since $H(\Omega_2)$ is a Montel space we have that $T^t \in L(H(\Omega_2)_{co}, E''_\beta)$. Thus, T is also continuous if we endow E'' with the topology $\sigma(E'', E')$, topology which is locally complete by [27, Corollary 5.1.35]. The symmetry of the ε-product of Schwartz [24, 43.3.(3)] yields that $T^{tt} \in H(\Omega_2)\varepsilon E''$, E'' endowed with the (locally complete) weak star topology. Hence there exists a holomorphic function $g : \Omega \longrightarrow (E'', \sigma(E'', E'))$ such that $T^{tt}(u) = u \circ g$ for each $u \in E'$. But for each $z \in \Omega_1$, if we denote by ∂_z the evaluation functional, we have $u \circ g(z) = \partial_z(T^{tt}(u)) = u(T^t(\partial_z)) = \partial_z(T(u)) = u \circ f(z)$. This yields that g extends f. Lemma 1 implies that $g(\Omega_2) \subseteq E$. Thus, $g : \Omega_2 \longrightarrow (E, \sigma(E, E'))$ is holomorphic and the result follows from the differential stability of the locally complete space E.

To conclude, we observe that every locally complete space E is a subspace of a suitable product Y of Banach spaces [25, Remark 24.5 (a)]. Then Y is distinguished and $f : \Omega_1 \longrightarrow Y$ admits weak holomorphic extensions to Ω_2. By the above argument there exists $\widehat{f} \in H(\Omega_2, Y)$ extending f and $f(\Omega_1) \subset E$. Lemma 1 yields the conclusion. ∎

Corollary 4. *Let Ω_1 and Ω_2 two domains in $\mathbb{C}$ with $\Omega_1 \subseteq \Omega_2$ and let E be a barrelled space. If $f : \Omega_1 \to E'$ is a function such that $u \circ f$ admits a holomorphic extension to Ω_2 for each $u \in E$, then we can get a function $g \in H(\Omega_2, E'_\beta)$ extending f.*

Proof. Observe that since E is barrelled E'_σ (and then E'_β) is quasicomplete. Therefore Theorem 3 yields that f has a holomorphic extension $g : \Omega_2 \longrightarrow (E', \sigma(E', E))$. Hence $g : \Omega_2 \longrightarrow E'_\beta$ is a locally bounded function such that $u \circ f$ is holomorphic for each $u \in E \subseteq E''$. Hence the result is a direct consequence of the Grosse-Erdmann's criterion [15, Theorem 1] ([13, Theorem 5]). ∎

Remark 5. In [3, Corollary 3], Theorem 3 is obtained for sequentially complete spaces. Thus, an alternative proof of Theorem 3 can be obtained by applying [3, Corollary 3] and Lemma 1 to $f : \Omega_1 \to \widehat{E}$, where $\widehat{E}$ is the completion of E. Requiring E to be sequentially complete but removing the hypothesis that E is Hausdorff, Corollary 4 is also obtained in [3, Corollary 1]. In the setting of Banach spaces, the strongest result of holomorphic extension deduced from weak holomorphic extensions seems to be [1, Theorem 3.5].

The next two stated results are inspired by a theorem due to Grosse-Erdmann. We need the following definition to formulate them.

Let Ω be a complex domain. A subset M of Ω is said to be a set which *determines the locally uniform convergence in* $H(\Omega)$ (cf. [15]), if the seminorms

$$p_K(f) = \sup_{z \in K \cap M} |f(z)| \qquad (K \subset \Omega \,\text{compact}, f \in H(\Omega))$$

define the usual topology in $H(\Omega)$.

Our next two theorems show that Theorem 2 in [15], valid for Fréchet spaces, even for B_r-complete spaces [15, Remark 2 (a)], is also valid for semireflexive spaces and even for locally complete spaces, if stronger assumptions on S are supposed in the later case. Recall that a locally convex space is called semireflexive if $E = E''$ as vector spaces. If the topological equality holds too, then E is called reflexive. A space E is semireflexive if and only if each bounded set B in E is relatively $\sigma(E, E')$-compact (cf. [25, Proposition 23.18]) and, consequently $E'_\beta = E'_\mu$ holds. Since every absolutely convex $\sigma(E, E')$-compact set is a Banach disc, every semireflexive space E is locally complete.

Theorem 6. *Let E be a semireflexive locally convex space, let Ω be a domain in $\mathbb{C}$, let $M \subset \Omega$ be a set which determines the locally uniform convergence in $H(\Omega)$ and let S be a separating subspace of E'. If $f : M \to E$ is a function such that:*

(i) $u \circ f$ has a holomorphic extension to Ω for each $u \in S$,

(ii) $f(K \cap M)$ is bounded in E for all compact subsets K of Ω,

then f has a (unique) holomorphic extension to Ω.

Proof. If $u \in S$, we denote by $\widehat{u \circ f}$ the holomorphic extension of $u \circ f$ to Ω. For every compact subset $K \in \Omega$ we have

$$p_K(\widehat{u \circ f}) = \sup_{z \in K \cap M} |\widehat{u \circ f}(z)| = \sup_{e \in f(K \cap M)} |u(e)| \leq \sup_{e \in \overline{\mathrm{acx}} f(K \cap M)} |u(e)|. \tag{1}$$

The weak compactness of $\overline{\mathrm{acx}} f(K \cap M)$ together with the fact that the topology of $H(\Omega)$ is generated by the seminorms p_K imply that, if we consider in S the topology inherited from E'_μ, the map $T : S \to H(\Omega), u \mapsto \widehat{u \circ f}$, which is linear since M determines the locally uniform convergence in $H(\Omega)$, is continuous. As S is separating, S is dense in E'_μ. Consequently, since $H(\Omega)$ is complete, T admits a (unique) continuous extension $\widehat{T} : E'_\mu \to H(\Omega)$. But $E'_\mu = (E_\sigma)'_{co}$, and the property of being locally complete depends only on the dual pair. Therefore we get a holomorphic function $g : \Omega \to E_\sigma$ such that $\widehat{T}(u) = u \circ g$ holds for every $u \in E'$. This yields that, for each $z \in M$ and for each $u \in S$, the equality $u \circ f(z) = u \circ g(z)$ holds. Since S is separating $f(z) = g(z)$ for every $z \in M$. Thus, g extends f and $g \in H(\Omega, E_\sigma)$, that is, g is a weakly holomorphic function with values in the locally complete locally convex space E, and consequently g is holomorphic. ■

To obtain natural extensions of Theorem 6 for arbitrary locally complete spaces stronger assumptions on S are needed. Actually we require S to be a subspace of E' such that every $\sigma(E, S)$-bounded set is bounded in E. However condition (ii) in Theorem 6 is deduced from these assumptions by the next lemma, which provides a slight improvement of Proposition 2 in [6]. Recall that, for $n, m \in \mathbb{N}$, a function defined on an open subset of $\mathbb{R}^n$ and with values in $\mathbb{R}^m$ is called C^1 if it admits continuous partial derivatives of first order.

Lemma 7. *Let E be a locally complete locally convex space, let S be a subspace of E' such that every $\sigma(E, S)$-bounded set is bounded in E, let K be a precompact set* *in* $\mathbb{R}^n$, $n \in \mathbb{N}$, *and let* $f : K \to E$ *be a function. If there exists an open set* $\Omega \subseteq \mathbb{R}^n$ *such that* $\overline{K} \subset \Omega$ *and* $u \circ f$ *admits* C^1 *extension to* Ω *for each* $u \in S$, *then the set* $\overline{acx} f(K)$ *is compact in* E.

Proof. Let K and Ω be as in the assumptions. We define

$$B_1 := \left\{ \frac{f(z) - f(t)}{\|z - t\|} : z, t \in K,\ z \neq t \right\}.$$

A similar method to the one used in the proof of [6, Proposition 2] shows that both B_1 and $f(K)$ are ($\sigma(E, S)$-) bounded sets. The details are left to the reader.

Now we define the (Banach) disc $B := \overline{\mathrm{acx}}(B_1 \cup f(K))$. The function $f : K \to E_B$ is uniformly continuous since $\|f(z) - f(t)\|_B \leq \|z - t\|$. Hence $f(K)$ is precompact in E_B. Since E_B is a Banach space, the set $\overline{\mathrm{acx}}^{E_B} f(K)$ is precompact and complete, i.e. compact, in E_B. This yields that $\overline{\mathrm{acx}}^{E_B} f(K)$ is compact in E, which completes the proof. ∎

Theorem 8. *Let* E *be a locally complete locally convex space, let* Ω *be domain in* $\mathbb{C}$, *let* $M \subset \Omega$ *a set which determines the locally uniform convergence topology in* $H(\Omega)$ *and let* S *be a subspace of* E' *such that every* $\sigma(E, S)$*-bounded set is bounded in* E. *If* $f : M \to E$ *is a function such that* $u \circ f$ *admits a holomorphic extension to* Ω *for each* $u \in S$, *then* f *admits a holomorphic extension to* S.

Proof. Applying Lemma 7, for every compact subset K of Ω, the set $\overline{\mathrm{acx}} f(K \cap M)$ is compact in E. The conclusion is obtained as in the proof of Theorem 6. ∎

Our final comments in this section are connected with possible extensions of Theorem 3. Given a locally complete space E, a domain $\Omega \subseteq \mathbb{C}$ and a holomorphic function $f : \Omega \to E$, we denote $S(f) := \{u \circ f : u \in E'\}$.

Proposition 9. *If* E *is a Fréchet space and* $f : \Omega \longrightarrow E$ *is a holomorphic function, then* $S(f)$ *is barrelled if and only if* $f(\Omega)$ *is contained in a finite dimensional subspace of* E.

Proof. Suppose that $f(\Omega)$ has finite dimensional range with basis $B = \{x_1, \ldots, x_n\}$. Then we can get a subset $U = \{u_1, \ldots, u_n\} \subset E'$ such that $u_i(x_j) = \delta_i^j$, where δ_i^j is the Dirac delta. Hence it follows that $\{u_1 \circ f, \ldots, u_n \circ f\}$ is a basis of $S(f)$.

Conversely, if $S(f)$ is barrelled, we have that $S(f)$ is the image of the continuous linear mapping $T : E'_{co} \longrightarrow H(\Omega)$, $u \mapsto u \circ f$. E'_{co} is B-complete since E is a Fréchet space; see [24, page 30 (5)]. Then $S(f)$ is isomorphic to a quotient of E'_{co} by the open mapping Theorem [27, Theorem 7.1.13]. Moreover, E'_{co} is a (gDF) space [27, Proposition 8.3.10] and this class of spaces is stable under the formation of separated quotients [27, Proposition 8.3.16]. Thus $S(f)$ is metrisable and nuclear as subspace of $H(\Omega)$ and has a fundamental sequence of bounded sets since it is (gDF). This implies that $S(f)$ is nuclear and normable, and then finite dimensional by the Dvoretzky-Rogers Theorem. If we suppose $f(\Omega)$ to be infinite dimensional then we can select a sequence $(z_n)_n$ such that $(f(z_n))_n$ is linearly independent. By the proof of [27, Theorem 2.1.3] we can get a sequence $(u_n)_n \subset E'$ such that $u_i(f(z_j)) = \delta_i^j$. Hence it follows that $(u_n \circ f)_n$ is linearly independent in $S(f)$, a contradiction. ∎

Proposition 10. *Let E be a (DF)-space and let f be an element of $H(\Omega, E)$. The space $S(f)$ is barrelled if and only if it is closed in $H(\Omega)$.*

Proof. E'_β is a Fréchet space. $S(f)$ is the range of the continuous linear mapping $T : E'_\beta \to H(\Omega), T(u) := u \circ f$. If $S(f)$ is barrelled, then T is open from E'_β onto $S(f)$ by the open mapping theorem. This implies that $S(f)$ is (isomorphic to a quotient of) a Fréchet space. ■

Proposition 11. *Every closed subspace F of $H(\Omega)$ can be written in the form $S(f)$ for certain $f \in H(\Omega, E)$ and E being a (DF)-space.*

Proof. Let F be a closed subspace of $H(\Omega)$. F is reflexive by [25, Proposition 23.26]. Let $f : \Omega \to F'_\beta$ be the map defined by $f(z) := \partial_z|F$. For each $g \in F$, $g \circ f = g \in H(\Omega)$. The differential stability of complete spaces shows that $f \in H(\Omega, F'_\beta)$ and we have $S(f) = F$. ■

4 Meromorphic extension

L. M. Hai, N. V. Khue, and N.T. Nga, in the main theorem of [17], have shown the following result.

> *Let Ω_1 and Ω_2 be two domains in $\mathbb{C}$ with $\Omega_1 \subseteq \Omega_2$ and let E be a Banach space. If $f : \Omega_1 \to E$ is a function such that $u \circ f$ admits a meromorphic extension to Ω_2 for every $u \in E'$, then f can be meromorphically extended to Ω_2.*

Actually, in [17] it is shown that the result is true assuming E to be only sequentially complete with Baire strong dual. Moreover, this theorem is valid for vector-valued functions of several variables. In this paper, our technique only allows us to deal with vector-valued functions of one variable. However, our method provides a generalization of the above theorem with weaker assumptions on E.

A locally convex space is said to be *suprabarrelled* if, given any increasing sequence $(E_n)_n$ of subspaces of E covering E, there exists p such that E_p is barrelled and dense in E [27, Definition 9.1.22]. Every Baire space is suprabarrelled [27, Observation 9.1.23]. Every space whose strong dual is suprabarrelled does not contain ω according to [5, Propositions 4 and 7].

Theorem 12. *Let Ω_1 and Ω_2 be two domains in $\mathbb{C}$ with $\Omega_1 \subseteq \Omega_2$ and let E be a locally complete locally convex space with suprabarrelled strong dual. If $f : \Omega_1 \to E$ is a function such that $u \circ f$ admits a meromorphic extension to Ω_2 for every $u \in E'$, then f can be meromorphically extended to Ω_2*

Proof. Let f be as in the hypothesis of the theorem. For $u \in E'$, we denote by $\widehat{u \circ f}$ the meromorphic extension of $u \circ f$ to Ω_2; without loss of generality we can assume that $\widehat{u \circ f}$ does not have removable singularities on $\Omega_2 \setminus \Omega_1$. We also assume that $\widehat{u \circ f}$ takes the value 0 on its poles outside Ω_1. Given a domain $\Omega_1 \subseteq U \subseteq \Omega_2$, we call U *domain of meromorphy* of f in Ω_2 if either $U = \Omega_1$ or $\Omega_1 \subsetneq U$ and there exists a meromorphic extension f_U of f to U without removable singularities outside Ω_1 and such that, if we denote by P_U the discrete subset of $U \setminus \Omega_1$ in which f_U is not holomorphic, then $f_U(z) = 0$ for each $z \in P_U$. With these definitions it is clear that $u \circ f_U(z) = \widehat{u \circ f}(z)$ for each $u \in E'$ and for each $z \in U \setminus P_U$.

CLAIM. If U is a domain of meromorphy of f in Ω_2 then there exists a domain V of meromorphy of f in Ω_2 such that $U \cup (\partial U \cap \Omega_2) \subseteq V \subseteq \Omega_2$.

PROOF OF THE CLAIM. Notice that, according to our definition, Ω_1 is a domain of meromorphy of f to Ω_2 and we do not know a priori if f is meromorphic. We only can assume that f_U is an E-valued extension of f which is (weakly) holomorphic in a set $U \setminus (P_U \cup \overline{\Omega}_1)$ which could be empty. With these assumptions, we need to show that f is meromorphic in U and that f_U can be meromorphically extended to $\partial U \cap \Omega_2$. We fix $a \in \overline{U} \cap \Omega_2$ and we denote by A_n the subspace

$$\{u \in E' : (z-a)^n \widehat{u \circ f}(z) \text{ is holomorphic and bounded on } B(a, 1/n) \setminus \{a\}\}.$$

A_n is the subspace of E' formed by the functionals u for which $(z-a)^n \widehat{u \circ f}(z)$ is holomorphic on $B(a, 1/n)$ with a removable singularity at a. Then we can consider $(z-a)^n \widehat{u \circ f}(z)$ holomorphic on $B(a, 1/n)$ for every $u \in A_n$. By the hypothesis, we have

$$E' = \bigcup_{n=1}^{\infty} A_n.$$

We apply now that E'_β is suprabarrelled to get $n_0 \in \mathbb{N}$ such that A_{n_0} is barrelled and dense in E'_β. Let τ be the locally convex topology in $H(B(a, 1/n_0))$ defined by the pointwise convergence on $B(a, 1/n_0) \cap (U \setminus P_U)$. The principle of isolated zeros of holomorphic functions yields that τ is Hausdorff. The map

$$\begin{array}{rccl} T: & A_{n_0} & \to & (H(B(a, 1/n_0)), \tau) \\ & u & \mapsto & (z-a)^{n_0} \widehat{u \circ f}(z) \end{array}$$

is linear and continuous, if we consider on A_{n_0} the topology inherited from E'_β, since $\widehat{u \circ f}(z) = u \circ f_U(z)$ for $z \in U \setminus P_U$. Since τ is Hausdorff and weaker than the usual topology in $H(B(a, 1/n_0))$ we have that the map has closed graph in $A_{n_0} \times H(B(a, 1/n_0))$ if we endow the two spaces with their strong topologies. Therefore T is continuous as a consequence of Pták's Closed Graph Theorem. We apply now that A_{n_0} is $\beta(E', E)$-dense and that $H(B(a, 1/n_0))$ is complete to obtain a continuous linear extension of T to E'_β. We denote the extension by $\widehat{T}$. A similar argument to the one used in Theorem 3 yields

$$\widehat{T}^{tt} \in L(E', H(B(a, 1/n_0))), \tag{2}$$

E' endowed with the (locally complete) topology of uniform convergence on the absolutely convex $\sigma(E'', E')$-compact subsets of E''. This implies that there exists g defined on $B(a, 1/n_0)$ and with values in E'' which is $\sigma(E'', E')$-holomorphic such that $\widehat{T}^{tt}(u) = u \circ g$ for every $u \in E'$. Again as in the proof of Theorem 3, we can get $u \circ g(z) = u \circ f_U(z)$ for every $u \in A_{n_0}$ and for every $z \in B(a, 1/n_0) \cap U \setminus P_U$. Since A_{n_0} is $\beta(E', E)$-dense (i.e. separating in E''), we have

$$g(z) = (z-a)^{n_0} f_U(z) \in E$$

for each $z \in B(a, 1/n_0) \cap U \setminus P_U$. The assumption of local completeness in E yields that it is a locally closed subspace of $(E'', \sigma(E'', E'))$. Thus Lemma 1 shows that $g(z) \in E$ for every $z \in B(a, 1/n_0)$. Therefore g is holomorphic in E for the topology $\sigma(E, E')$, and then $g \in H(B(a, 1/n_0), E)$ since E is locally complete. Hence $h_a(z) = (1/(z-a)^{n_0})g(z)$ is a meromorphic function on $B(a, 1/n_0)$ with values in E which extends f_U. If $a \in \partial U \cap \Omega_2$ and $(1/(z-a)^{n_0})g(z)$ has a removable singularity at a, then we give to $h_a(a)$ the value which makes h_a a holomorphic function on $B(a, 1/n_0)$, assigning $h_a(a) = 0$ if a is a pole. If we write $V_a = B(a, 1/n_0)$, for every $a \in \overline{U} \cap \Omega_2$ we have found a meromorphic function h_a defined on V_a such that h_a restricted to $V_a \cap (U \setminus P_U)$ agrees with f_U (and then extends f), h_a does not have removable singularities outside Ω_1 and $h_a(a) = 0$ if a is a pole, a being the unique possible pole of h_a at V_a. If we define $V := \cup V_a$ and $f_V(z) = h_a(z)$ if $z \in V_a$, according to the principle of isolated zeros of holomorphic functions, f_V is well defined and meromorphic on V, and the claim is proved.

We complete the proof assuming the claim. We define M as the set formed by the pairs (V, f_V), such that V is a proper domain of meromorphy of f to Ω_2, i.e. $\Omega_1 \subset V \subseteq \Omega_2$ and f_V is a meromorphic extension of f to V. M is not empty by the claim applied to $U = \Omega_1$. We define in M the order relation $(V, f_V) \leq (U, f_U)$ if $V \subseteq U$ and $f_U|_V = f_V$. Let $(V_i, f_{V_i})_{i \in I}$ be a completely ordered chain in M. $V := \cup_{i \in I} V_i$ is a domain and $f_V(z) := f_{V_i}(z)$ if $z \in V_i$ is well defined and meromorphic. This yields that (V, f_V) is an upper bound of the chain. We apply Zorn's Lemma to get a maximal element (W, f_W) of M. If we suppose that W is strictly included in Ω_2, then we apply the claim to $U = W$ obtaining a contradiction with the maximality of W.

■

Remark 13. The claim stated in the proof of Theorem 12 might seem unnecessary. Actually, after proving that f can be extended throughout its boundary, it seems to be possible to obtain the conclusion by a simple repetition of the argument. But, for E locally complete without extra assumptions, it could happen that a function $f : \Omega_1 \longrightarrow E$ satisfies that $u \circ f$ admits a meromorphic extension to Ω_2 for each $u \in E'$ and that there exists a domain $\Omega_1 \subseteq V \subseteq \Omega_2$ and a meromorphic function $g : V \longrightarrow E$ extending f such that there exists $u \in E'$ for which $u \circ g$ does not admit a meromorphic extension to Ω_2, and thus the hypothesis on (g, V) differ from those on (f, Ω_1). To clarify this, for $n \in \mathbb{N}$ we take $f_n : \mathbb{C} \longrightarrow \mathbb{C}$ meromorphic with one unique pole at $1 - 1/(n+1)$ in which it takes the value 0 and without removable singularities, we set $D := \{1 - 1/n : n \geq 2\}$, and we define $h_n : \mathbb{C} \longrightarrow \mathbb{C}$, by $h_n(z) = f_n(z)$ if $z \in \mathbb{C} \setminus D$ and $h_n(z) = 0$ for $z \in D$. We set $\Omega_1 = B(0, 1/2)$ and $\Omega_2 = \mathbb{C}$. Clearly, $f : B(0, 1/2) \longrightarrow \omega$, $z \mapsto (f_n(z))_{n=1}^{\infty}$, is a function which can be weakly meromorphically extended to $\mathbb{C}$. If we define $g : B(0, 1) \longrightarrow \omega$ by $g(z) = (h_n(z))_n$, we have that g is an extension of f to $B(0, 1)$ which is easily checked to be meromorphic with their set of poles contained in D by [13, Theorem 6.5], g does not have removable singularities because each $\alpha \in D$ is a pole of one coordinate and then it is a pole of g, g takes the value 0 at each pole and, for each $u \in \varphi$, if we get the weak extensions $\widehat{u \circ f}$ without removable singularities and taking the value 0 at its poles, which are contained in D, then $u \circ g(z) = \widehat{u \circ f}(z)$ for every $z \in B(0, 1) \setminus D$ since the two functions are holomorphic on $B(0, 1) \setminus D$ and they agree in $B(0,1/2)$. However, for each coordinate vector $u_n \in \varphi$, the function $u_n \circ g = h_n$ is not continuous on almost all $\alpha \in D$ because each $f_n \in M(\mathbb{C})$ only has a finite number of zeros in $B(0,1)$ and h_n vanishes on D. Since D is not discrete in $\mathbb{C}$, we conclude that h_n does not have a meromorphic extension to $\mathbb{C}$. This is why we had to show that this situation can not happen in spaces with suprabarrelled strong dual

Corollary 14. *Let Ω_1 and Ω_2 be two domains in $\mathbb{C}$ with $\Omega_1 \subseteq \Omega_2$ and let E be a suprabarrelled space. If $f : \Omega_1 \to E'$ is a function such that $u \circ f$ admits a meromorphic extension to Ω_2 for each $u \in E$, then there is $\hat{f} \in M(\Omega_2, E'_\beta)$ extending f.*

Proof. Since E is barrelled $(E', \sigma(E', E))$ is locally complete by [27, Corollary 5.1.35]. We can apply Theorem 12 to obtain a meromorphic function $\hat{f} : \Omega_2 \to (E', \sigma(E', E))$ extending f. This yields that there exists a discrete set D in Ω_2 such that $\hat{f}$ is continuous and then locally bounded in $\Omega_2 \setminus D$. Again the barrelledness of E implies that every $\sigma(E', E)$-bounded set is $\beta(E', E)$-bounded and consequently $\hat{f} : \Omega_2 \setminus D \longrightarrow E'_\beta$ is locally bounded. Moreover, for each $u \in E$, $u \circ \hat{f}$ is a meromorphic function which has all its poles in D. Moreover the order of these poles is bounded by its order in $\hat{f}$. Hence we can apply [13, Theorem 6.5] ([15, Theorem 4]) to obtain $\hat{f} \in M(\Omega_2, E'_\beta)$. ∎

To obtain more results in the same direction, we make a distinction in the notation for poles and removable singularities in very weakly meromorphic functions. Given $f \in Mer^\omega(\Omega, E)$ we denote by $P(f)$ the subset of Ω formed by the points which are poles of $u \circ f$ for some $u \in E'$ and we denote by $A(f)$ the subset of $\Omega \setminus P(f)$ formed by the points which are removable singularities of $u \circ f$ for some $u \in E'$. Notice that there exist very weakly meromorphic functions with only removable singularities which are not weakly meromorphic. Indeed, if we take a sequence (z_n) with some accumulation point in $\mathbb{C}$ and a sequence of functions $f_n : \mathbb{C} \to \mathbb{C}$ holomorphic with a removable singularity at z_n, the function $f : \mathbb{C} \to \omega$, $z \mapsto (f_n(z))_n$ verifies that $f \in Mer^\omega(\mathbb{C}, \omega) \setminus WM(\mathbb{C}, \omega)$.

Lemma 15. *Let Ω be a complex domain, let E be a locally complete locally convex space which does not contain ω and let $f \in Mer^\omega(\Omega, E)$. If $P(f)$ is discrete in Ω, then $f \in M(\Omega, E)$.*

Proof. By [6, Theorem 5], we only have to show that $f \in WM(\Omega, E)$, and for this we have to see that $A(f)$ is discrete in Ω. According to the definitions, for every $z \in \Omega \setminus (P(f) \cup A(f))$ and for every $u \in E'$, $u \circ f$ is holomorphic in z. As, by hypothesis, $P(f)$ is discrete in Ω, if we show that $A(f)$ is discrete in Ω, we will have that, for the discrete subset $D := P(f) \cup A(f)$ of Ω, $u \circ f \in H(\Omega \setminus D) \cap M(\Omega)$ holds for each $u \in E'$, which permits to conclude.

Let $z_0 \in A(f)$. We define the increasing sequence of subspaces of E'

$$E_n := \{u \in E' : u \circ f \text{ is holomorphic on } B(z_0, 1/n) \setminus \{z_0\}\},\ n \in \mathbb{N}.$$

Since $f \in Mer^\omega(\Omega, E)$, for each $u \in E'$, the set formed by the poles and removable singularities of $u \circ f$ is discrete in Ω. Therefore, we can write

$$E' = \bigcup_{n \in \mathbb{N}} E_n.$$

Now, since E does not contain ω, we apply [5, Proposition 4 and 7] to obtain $n_0 \in \mathbb{N}$ such that E_{n_0} is $\sigma(E', E)$-dense. Since $P(f)$ is discrete, we can choose n_0 large enough to verify (a) $B(z_0, 2/n_0) \subset \Omega$ and (b) $B(z_0, 2/n_0) \bigcap P(f) = \emptyset$. Condition (b) implies that, for each $u \in E'$, the restriction of $u \circ f$ to $B(z_0, 2/n_0)$ has only removable singularities. Moreover, since $u \circ f$ is meromorphic in Ω for each $u \in E'$, the set of removable singularities of $u \circ f$ in the closed ball $D(z_0, 1/n_0)$ is finite for every $u \in E'$. Thus, we have that, for every $u \in E'$, the function $u \circ f$, restricted to $D(z_0, 1/n_0)$ is continuous except on a finite subset. Hence

$$\sup_{z \in B(z_0, 1/n_0)} |u(f(z))| < \infty$$

for each $u \in E'$. Consequently, $f(B(z_0, 1/n_0))$ is bounded in E and the restriction of f to $B(z_0, 1/n_0) \setminus \{z_0\}$ is a locally bounded function such that $u \circ f$ is holomorphic for each $u \in E_{n_0}$. We obtain now that f is holomorphic on $B(z_0, 1/n_0) \setminus \{z_0\}$ as a consequence of [13, Theorem 5.2], concluding then that $A(f)$ is discrete. This completes the proof. ■

A locally convex space is said to be a Schwartz space if for each absolutely convex 0-neighbourhood U in E there exists a 0-neighbourhood V so that for each $\varepsilon > 0$, points $x_1, \ldots, x_n \in V$ exist such that $V \subset \bigcup_{i=1}^{n}(x_i + \varepsilon U)$. Given a subspace E of a locally convex space G, we can always identify algebraically E' with the quotient space G'/E°. A complete Schwartz Hausdorff locally convex space E has the following property [21, pages 179 and 201]: *For each Hausdorff locally convex space G which contain E as a subspace, the quotient topology induced by G'_β in $E' = G'/E^\circ$ coincides with the strong topology $\beta(E', E)$.*

Theorem 16. *Let E be a barrelled complete Schwartz space which does not contain ω. If $\Omega_1 \subseteq \Omega_2$ are domains in $\mathbb{C}$, and $f : \Omega_1 \to E$ is a function with the property that $u \circ f$ admits a meromorphic extension to Ω_2 for every $u \in E'$, then f admits a meromorphic extension to Ω_2.*

Proof. We denote by $\widehat{u \circ f}$ the meromorphic extension of $u \circ f$.

We consider E as a subspace of the product of a family of Banach spaces $(E_i)_{i \in I}$ (cf. [25, Remark 24.5 (a)]). Therefore, we can write

$$\begin{array}{rccl} f: & \Omega_1 & \to & \prod_{i \in I} E_i \\ & z & \mapsto & (f_i(z))_{i \in I}. \end{array}$$

Since each f_i is a meromorphic function which takes its values in a Banach space and $u \circ f$ can be meromorphically extended to Ω_2 for each $u \in E_i'$, we can get a meromorphic extension $\hat{f}_i : \Omega_2 \to E_i$. We apply [6, Proposition 6] (or Theorem 12) to conclude that the map

$$\begin{array}{rccl} T_{\hat{f}_i}: & E_i' & \to & M(\Omega_2) \\ & u & \mapsto & u \circ \hat{f}_i \end{array}$$

is continuous if we consider in E_i' the strong topology $\beta(E_i', E_i)$, since this topology is finer than the topology of the space $(E_i')_{co}$. Therefore, the linear map

$$\begin{array}{rccl} T_f: & \bigoplus_{i \in I} E_i' & \to & M(\Omega_2) \\ & (u_i)_{i \in I} & \mapsto & \sum_{i \in I} u_i \circ \hat{f}_i \end{array}$$

is continuous. Since f takes its values in E, we use Remark 2 to obtain $E^\circ \subset KerT_f$. Therefore the map

$$\begin{array}{rccl} \widehat{T}_f: & \bigoplus_{i\in I} E_i^{'}/E^\circ & \rightarrow & M(\Omega_2) \\ & [(u_i)_{i\in I}] & \mapsto & \sum_{i\in I} u_i \circ \hat{f}_i \end{array}$$

is continuous. As E is a complete Schwartz space, we have that $\widehat{T}_f$ is a continuous linear map defined on E'_β with values in $M(\Omega_2)$. Moreover, for each $u \in E'$, $\widehat{u \circ f}$ and $\widehat{T}_f(u)$ coincide in Ω_1 with $u \circ f$. Again Remark 2 yields $\widehat{T}_f(u) = \widehat{u \circ f}$ for every $u \in E'$ in the locally convex space $M(\Omega_2)$. We apply that E is a Montel space [25, Remark 24.24], to conclude $\widehat{T}_f \in L(E'_{co}, M(\Omega_2)) = M(\Omega_2)\varepsilon E$. As E does not contain ω, we can apply [6, Proposition 6] to obtain a meromorphic function $g : \Omega_2 \rightarrow E$, such that $\widehat{T}_f(u) = u \circ g$ for each $u \in E'$. Therefore, for $u \in E'$, we have $u \circ g = \widehat{u \circ f}$ in the topological vector space $M(\Omega_2)$. Thereby, again Remark 2 implies that, for each $u \in E'$, there exists a subset D_u discrete in Ω_2 such that $u \circ g(z) = \widehat{u \circ f}(z)$ for each $z \in \Omega_2 \setminus D_u$. We define

$$h(z) := \begin{cases} f(z) & \text{if} \quad z \in \Omega_1 \\ g(z) & \text{if} \quad z \in \Omega_2 \setminus \Omega_1, \end{cases}$$

$h \in Mer^\omega(\Omega_2, E)$ and $u \circ g = u \circ h$ in the topological vector space $M(\Omega_2)$ since $u \circ g(z) = u \circ h(z)$ for each $u \in E^{'}$ and for each $z \in \Omega_2 \setminus D_u$. As $g \in M(\Omega_2, E)$, we have that $P(h) = P(g)$ is a discrete set in Ω_2. Lemma 15 implies $h \in M(\Omega_2, E)$. ∎

Notice that theorem 16 is valid for every Fréchet-Schwartz space with a continuous norm (recall that a Fréchet space has a continuous norm if and only if it does not contain ω). However, we have a better result for Fréchet spaces. Recall that a Fréchet space E is distinguished if and only if E'_β is ultrabornological [25, Proposition 25.12].

Theorem 17. *Let E be a distinguished Fréchet space such that E''_β has a continuous norm. If $\Omega_1 \subseteq \Omega_2$ are domains in $\mathbb{C}$ and $f : \Omega_1 \rightarrow E$ satisfies that $u \circ f$ admits a meromorphic extension to Ω_2 for each $u \in E'$, then f admits a meromorphic extension to Ω_2.*

Proof. We can choose a sequence of Banach spaces $(E_n)_n$ such that E is a subspace of $\prod_{n\in\mathbb{N}} E_n$ [25, Remark 24.5 (a)]. We write

$$\begin{array}{rccl} f: & \Omega_1 & \rightarrow & \prod_{n\in\mathbb{N}} E_n \\ & z & \mapsto & (f_n(z))_{n\in\mathbb{N}}. \end{array}$$

As in the proof of Theorem 16, for each n, we get $\hat{f}_n \in M(\Omega_2, E_n)$ such that $\hat{f}_n$ restricted to Ω_1 coincides with f_n. We fix $u \in E^{'}$. By the Hahn-Banach Theorem, there exists $(u_n)_n \in \bigoplus_n E_n^{'}$ such that, for every $(e_n)_n \in E \subset \prod_n E_n$, $u(e) = \sum_n u_n(e_n)$. Therefore, $u \circ f = \sum u_n \circ f_n$, and, again as a consequence of Remark 2 we have that $\widehat{u \circ f} = \sum_n u_n \circ \hat{f}_n$ in the locally convex space $M(\Omega_2)$. We define now the subspace of $M(\Omega_2)$

$$F := \operatorname{span}\{\widehat{u \circ f} : u \in E'\}.$$

Let P_n be the discrete subset of Ω_2 formed by the poles of the meromorphic functions $\hat{f}_n$. If we define $P := \bigcup_n P_n$ we have that P is countable and the set of the poles of the functions which are in F is contained in P. We recall the projective description of the topology of the space of meromorphic functions given in [14]. For every exhaustion $(O_n)_{n=1}^{\infty}$ in Ω_2, i.e. each O_n is a relatively compact subdomain of Ω_2 such that $\overline{O}_n \subset O_{n+1}$ and $\Omega = \bigcup_{n=1}^{\infty} O_n$, $M(\Omega_2)$ is a closed subspace of

$$\prod_{n\in\mathbb{N}} H(O_n) \times \mathbb{C}^{(O_n\times\mathbb{N})}.$$

If the principal part of f at α is $h^{\alpha}(f) = \sum_{n=1}^{k} a_{\alpha}^{n}(f)(z-\alpha)^{-j}$, k being an element of $\mathbb{N}$, then the projection of f over each $\mathbb{C}^{(O_n\times\mathbb{N})}$ is defined by $P_{\alpha}^{n}(f) = a_{\alpha}^{n}(f)$ for every $(\alpha, n) \in (O_n, \mathbb{N})$, and the projection of f over each $H(O_n)$ is obtained as the difference between f and the sum of its principal parts in O_n. Then F can be considered as a subspace of

$$\prod_{n\in\mathbb{N}} H(O_n) \times \mathbb{C}^{((O_n\cap P)\times\mathbb{N})}. \tag{3}$$

This product is a webbed space according to the definition given in [25, page 287] (cf. [25, Lemma 24.28, Corollary 24.29]). Therefore the closure of F in $M(\Omega_2)$ is webbed because it is closed in the webbed space (3). We define

$$\begin{array}{rccc} T: & E'_{\beta} & \to & M(\Omega_2) \\ & u & \mapsto & \widehat{u\circ f}. \end{array}$$

We have $T(E') \subset F$. Moreover T is continuous if we consider in F the Hausdorff locally convex topology of pointwise convergence on $\Omega_1 \setminus P$. As this topology is weaker than the topology inherited from $M(\Omega_2)$, T is a linear map with closed graph and it takes values in the webbed space $\overline{F}$. Since E'_{β} is ultrabornological by hypothesis, we can apply De Wilde's Closed Graph theorem [25, Theorem 24.31] to obtain that T is continuous. We apply that $M(\Omega_2)$ is a Montel space [14, Theorem 3] and the symmetry of the ε-product of Schwartz to obtain $T^{tt} \in L(E'''_{co}, M(\Omega_2)) = M(\Omega_2)\varepsilon E''_{\beta}$. By hypothesis, E''_{β} has a continuous norm. It follows from [6, Proposition 6] that there exists a meromorphic function $g : \Omega_2 \to E''_{\beta}$, such that $T^{tt}(u) = u \circ g$ for each $u \in E'''$. Therefore, for every $u \in E'$ and for every $v \in M(\Omega_2)'$ we have

$$v(u\circ g) = v(T^{tt}(u)) = u(T^{t}(v)) = v(T(u)) = v(\widehat{u\circ f}).$$

Hence, $u \circ g = \widehat{u\circ f}$ in the topological vector space $M(\Omega_2)$. Thereby, for each $u \in E^{'}$ there exists a subset D_u discrete in Ω_2 such that $u\circ g(z) = \widehat{u\circ f}(z)$ for each $z \in \Omega_2 \setminus D_u$. If we proceed as in the proof of Theorem 16, we can apply Lemma 15 together with the hypothesis that E''_{β} has continuous norm to get $h \in M(\Omega_2, E''_{\beta})$ extending f. Since $h(\Omega_1) \subset E$ we can apply Lemma 1 to conclude that $h(\Omega_2) \subset E$ except on a discrete set. This yields $h \in M(\Omega_2, E)$. ∎

Remark 18. (a) Clearly, every Fréchet space whose bidual has a continuous norm has a continuous norm itself. Examples showing that the converse is not generally true can be found in [8, 31].

(b) For every complex domain Ω, applying Theorems 12, 16 and 17 to $\Omega_1 = \Omega_2 = \Omega$ we obtain that if E is a locally complete space with Baire strong dual or E is a complete barrelled Schwartz space which does not contain ω or E is a distinguished Fréchet space whose bidual has a continuous norm, then $Mer^{\omega}(\Omega, E) = M(\Omega, E)$ holds.

(c) The product of a DFS and a FS space with a continuous norm satisfies the assumptions of Theorem 16 but not those of Theorems 12 and 17

(d) In Theorem 17 we can not apply the argument of Theorem 3 to avoid the assumption that E is distinguished, because infinite products of Banach spaces contain ω as subspace.

All the counterexamples that we have found for functions which admit weak meromorphic extension but not a meromorphic extension are with range space ω (see Remark 4 and [6]). We conjecture that all the results stated in this section can be extended to all the locally complete locally convex spaces which do not contain subspaces isomorphic to ω.

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