A characterization of the family of secant or external lines of an ovoid of PG$(3, q)$

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Abstract

In this paper we characterize the family of secant lines of an ovoid of PG$(3, q)$ and the family of external lines to an ovoid of PG$(3, q)$.

1 Introduction

In the paper "*A characterization of the family of secant lines of an elliptic quadric in PG$(3, q)$, q odd*" [**2**] O. Ferri and G. Tallini characterize the family of secant lines of an ovoid of PG$(3, q)$, q odd. The same result is obtained for q even ($q > 2$) by M.J. de Resmini in the paper "*A characterization of the secants of an ovaloid in PG$(3, q)$, q even, $q > 2$* [**1**]. They got the following results.

Theorem 1.1 (Ferri-Tallini). *Let $\mathcal{F}$ be a family of lines of PG$(3, q)$, q odd, satisfying the following properties.*

I *Through every point of PG$(3, q)$ there are either q^2 or $\frac{q^2-q}{2}$ lines of $\mathcal{F}$.*

II *In every plane of PG$(3, q)$ there are either $\frac{q^2+q}{2}$ or zero lines of $\mathcal{F}$.*

III *Let p be a point on some line of $\mathcal{F}$. In every pencil with center p there are $\frac{q-1}{2}, \frac{q+1}{2}$ or q lines of $\mathcal{F}$.*

IV *Let π be a plane of PG$(3, q)$ containing at least one line of $\mathcal{F}$. Through every point of π there is at least one line of $\mathcal{F}$ contained in π.*

Then $\mathcal{F}$ is the family of secant lines to an elliptic quadric of PG$(3, q)$.

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Theorem 1.2 (de Resmini). *Let $\mathcal{F}$ be a family of lines of $PG(3,q)$, q even and $q>2$, satisfying the following properties.*

- **I'** *Through every point of $PG(3,q)$ there are either q^2 or n lines of $\mathcal{F}$, where $0<n<q^2$.*

- **II'** *Let p be a point on some line of $\mathcal{F}$. In every pencil with center p there are either $\frac{q}{2}$ or q lines of $\mathcal{F}$.*

Then $n=\frac{q^2-q}{2}$ and $\mathcal{F}$ is the family of secant lines to an ovoid of $PG(3,q)$.

In this paper we show that by using only properties I and II it is possible to prove both results. The theorem we will prove is the following.

Theorem 1.3. *A family $\mathcal{F}$ of lines of $PG(3,q)$, $q>2$, satisfying properties I and II is the family of secant lines to an ovoid of $PG(3,q)$.*

As an application to Theorem 1.3 we also get the following result.

Theorem 1.4. *Let $\mathcal{F}$ be a family of lines of $PG(3,q)$, $q>2$, satisfying the following properties.*

- **I*** *In every plane of $PG(3,q)$ there are either $\frac{q^2-q}{2}$ or q^2 lines of $\mathcal{F}$.*

- **II*** *Through every point of $PG(3,q)$ there are either $\frac{q^2+q}{2}$ or zero lines of $\mathcal{F}$.*

Then $\mathcal{F}$ is the family of external lines to an ovoid of $PG(3,q)$.

2 The characterization theorem

In this section $\mathcal{F}$ will be a family of lines of $\mathrm{PG}(3,q)$, $q>2$, satisfying Properties I and II. In order to simplify the exposition we will call ***black*** a point of $\mathrm{PG}(3,q)$ on q^2 lines of $\mathcal{F}$. Let Ω be the set of black points of $\mathrm{PG}(3,q)$. A plane containing no lines of $\mathcal{F}$ will be called a *tangent* plane, while a *secant* plane is a plane containing $\frac{q^2+q}{2}$ lines of $\mathcal{F}$. Next propositions will show that there are exactly q^2+1 black points, that Ω is an ovoid of $\mathrm{PG}(3,q)$ and that $\mathcal{F}$ is the family of secant lines to Ω.

Proposition 2.1. *On every line of $\mathcal{F}$ there are exactly two black points.*

Proof : Let ℓ be a line of $\mathcal{F}$. Let a be the number of black points on ℓ and let μ_ℓ be the number of lines of $\mathcal{F}$, different from ℓ, meeting ℓ. Since every plane through ℓ is a secant plane and as in those planes there are the lines of $\mathcal{F}$ meeting ℓ we get

$$\mu_\ell=(q+1)(\frac{q^2+q}{2}-1)=a(q^2-1)+(q+1-a)(\frac{q^2-q}{2}-1). \qquad (1)$$

Hence $a=2$ and the assertion follows. ■

From the previous proposition, counting $|\Omega|$ by considering the lines of $\mathcal{F}$ through a black point p we have,

$$|\Omega| \geq q^2 + 1. \tag{2}$$

We can now prove the following

Proposition 2.2. *A line containing two black points is a line of* $\mathcal{F}$.

Proof : Let p and p' be two different black points and let ℓ be the line pp'. Suppose, by way of contradiction, that ℓ is not in $\mathcal{F}$. Denote again by a the number of black points on ℓ and by μ_ℓ the number of lines of $\mathcal{F}$ meeting ℓ. Let π be a secant plane through ℓ. Denote by ρ_π the number of lines of $\mathcal{F}$ contained in π and containing the point p. Since $\ell \notin \mathcal{F}$ then we have $\rho_\pi \leq q$. If $\pi_1, \ldots, \pi_m$ are the secant planes through ℓ, then we have

$$q^2 = \sum_i \rho_{\pi_i} \leq mq$$

and hence there are at least q secant planes through ℓ.

If $m = q$, then

$$\mu_\ell = q\frac{q^2+q}{2} = aq^2 + (q+1-a)\frac{q^2-q}{2}$$

and hence $a = 1$, while $a \geq 2$.

Hence $m = q + 1$ and

$$\mu_\ell = (q+1)\frac{q^2+q}{2} = aq^2 + (q+1-a)\frac{q^2-q}{2}.$$

Therefore $a = 2$. If follows that Ω is a cap, hence, since $q > 2$, it $|\Omega| \leq q^2 + 1$ [**3**]. Count $|\Omega|$ by considering all lines through p. We obtain $|\Omega| \geq q^2 + 2$ and this is a contradiction. It follows that $\ell \in \mathcal{F}$ and hence the assertion. ■

From propositions 2.1 and 2.2 it follows that the set Ω of black points is a cap and hence $|\Omega| \leq q^2 + 1$ [**3**]. From Equation (2) $|\Omega| = q^2 + 1$ and hence Ω is an ovoid, and propositions 2.1 e 2.2 show that $\mathcal{F}$ is the family of secant lines to Ω.

3 Applications: Theorem 1.4

It is well known that the points of the dual space $\mathrm{PG}^*(3, q)$ of $\mathrm{PG}(3, q)$ are the planes of $\mathrm{PG}(3, q)$ and the lines are the pencils of planes with axis a line of $\mathrm{PG}(3, q)$. By identifying a pencil of planes with axis the line t, with the line t itself, the planes and the lines of $\mathrm{PG}(3, q)$ can be seen as the points and the lines of $\mathrm{PG}^*(3, q)$.

With such an identification a "point" π is on a line t if the point π contains t. If Ω is an ovoid of $\mathrm{PG}(3, q)$, the $q^2 + 1$ tangent planes to Ω are, in the dual space, the points of an ovoid Ω^* and the secant lines to Ω are the external lines to Ω^*, while the external lines to Ω are the secant lines to Ω^*. Moreover if Ω' is an ovoid of $\mathrm{PG}^*(3, q)$ there is an ovoid Ω of $\mathrm{PG}(3, q)$ such that $\Omega^* = \Omega'$. We can now prove Theorem 1.4. Let $\mathcal{F}$ be a family of lines of $\mathrm{PG}(3, q)$ satisfying properties I^* and II^*. In the dual space the family $\mathcal{F}$ satisfies properties I and II and hence by Theorem 1.3 it is the family of secant lines to an ovoid Ω' of $\mathrm{PG}^*(3, q)$. Let Ω be the ovoid of $\mathrm{PG}(3, q)$ such that $\Omega^* = \Omega'$. Then it follows that $\mathcal{F}$ is the family of external lines to Ω.

References

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