The sets closest to ovoids in $Q^{-}(2n+1, q)$

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1 Introduction

An *ovoid* of a polar space is a set of points with the property that every maximal subspace contains exactly one point of it. The existence of ovoids in polar spaces was studied extensively, see for example [5, 2, 3, 4] and the overview in [1, Appendix VI]. Clearly, if a polar space contains an ovoid $\mathcal{O}$, then $|\mathcal{O}|$ is the minimum size of a set of points that meets every maximal subspace of that polar space.

By $Q^{-}(2n+1, q)$ we denote the elliptic quadric of $\mathrm{PG}(2n+1, q)$. An ovoid of $Q^{-}(2n+1, q)$ has $q^{n+1}+1$ points [1]. However, Thas [5] has shown that $Q^{-}(2n+1, q)$ has no ovoids for $n \geq 2$. We will improve this result by showing that $q^{n+1}+q^{n-1}$ is the minimum cardinality of a set of points that meets every maximal subspace of $Q^{-}(2n+1, q)$. More precisely, we prove the following theorem.

Theorem 1.1 *Let B be a set of points of $Q = Q^{-}(2n+1, q)$ such that every maximal subspace of Q has a point in B. Let $\perp$ be the related polarity of $\mathrm{PG}(2n+1, q)$. Then $|B| \geq q^{n+1}+q^{n-1}$ with equality if and only if $B = (U^{\perp} \setminus U) \cap Q$ for a subspace U of dimension $n-2$ with $U \subseteq Q$.*

If U is a subspace of Q of dimension $n-2$, then the set $B := (U^{\perp} \setminus U) \cap Q$ meets all maximal subspaces of Q. For, if S is a maximal subspace of Q, then $\dim(S \cap U^{\perp}) = 1 + \dim(S \cap U)$ and thus $S \cap U \neq \emptyset$.

Notice that the quotient space $U^{\perp}/U$ is a 3-space and that Q induces a $Q^{-}(3, q)$ on this 3-space (that is the set $\{\langle U, P\rangle \mid P \in (U^{\perp} \setminus U) \cap Q\}$ is a $Q^{-}(3, q)$ of $U^{\perp}/U$). Thus $B := (U^{\perp} \setminus U) \cap Q$ has cardinality $(q^2+1)q^{n-1}$.

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2 Proof of the theorem

In this section Q denotes the elliptic quadric $Q^-(2n+1,q)$, $n \geq 1$, and $\perp$ denotes the related polarity of $\mathrm{PG}(2n+1,q)$. A subspace of $\mathrm{PG}(2n+1,q)$ contained in Q will be called *singular*. In order to prove the theorem, we suppose that B is a minimal set of points of Q that meets all maximal singular subspaces and that $|B| \leq q^{n+1}+q^{n-1}$. We show in a series of lemmas that equality holds and that $B = (U^\perp \setminus U) \cap Q$ for a singular subspace U of dimension $n-2$. Put $\delta := |B| - q^{n+1}$, so

$$|B| = q^{n+1} + \delta \qquad \text{with} \qquad \delta \leq q^{n-1}.$$

Lemma 2.1 *Suppose that T is a maximal singular subspace that meets B in exactly one point P. Then $|P^\perp \cap B| \leq \delta$ and there exists a hyperplane S of T with $P \notin S$ such that every maximal singular subspace on S meets B in exactly one point.*

Proof. Consider a hyperplane S of T. Then Q induces an $Q^-(3,q)$ on $S^\perp/S$ and hence S lies in $|Q^-(3,q)| = q^2+1$ maximal singular subspaces. If $P \notin S$, then q^2 of these maximal singular subspaces do not contain P and hence $S^\perp$ contains q^2 points of B that do not lie in $P^\perp$. Since T has q^{n-1} hyperplanes S with $P \notin S$, and since distinct choices of S yield distinct points of B, it follows that $|B \setminus P^\perp| \geq q^{n-1} \cdot q^2 = q^{n+1}$. Hence $|P^\perp \cap B| \leq \delta$.

For some of the hyperplanes S of T with $P \notin S$, we must have that each maximal singular subspace on S contains a unique point in B, since otherwise we could improve the above bound to $|B \setminus P^\perp| \geq q^{n-1} \cdot (q^2+1)$, which is not possible, since $|B| \leq (q^2+1)q^{n-1}$ and $P \in B \cap P^\perp$. ∎

Lemma 2.2 *If $P \in B$, then $|P^\perp \cap B| \leq \delta$.*

Proof. By the minimality of B, some maximal singular subspace on P meets B only in P. Apply the previous lemma. ∎

By $Q(2n,q)$ we denote the parabolic quadric of $\mathrm{PG}(2n,q)$. Recall that $Q(2n,q)$ has $(1+q)(1+q^2)\dots(1+q^n)$ maximal subspaces and each point of $Q(2n,q)$ lies in $(1+q)(1+q^2)\dots(1+q^{n-1})$ of them ([1]). Hence, a set of points of $Q(2n,q)$ that meets all maximal singular subspaces of $Q(2n,q)$ has at least q^n+1 points with equality if it is an ovoid.

Lemma 2.3 *If R is a point with $R \notin Q$, then $|R^\perp \cap B| \leq q^n + \delta$.*

Proof. Put $H := R^\perp$. Then $Q \cap H$ is an $Q(2n,q)$ and thus H contains maximal singular subspaces. Therefore $B \cap H \neq \emptyset$.

Let P be a point of $B \cap H$. Then Q induces on $(P^\perp \cap H)/P$ an $Q(2n-2,q)$ and $B' = \{\langle P, X\rangle \mid X \in P^\perp \cap H \cap B, X \neq P\}$ is a set of points of this $Q(2n-2,q)$ with $|B'| < |B \cap P^\perp| \leq \delta \leq q^{n-1}$. Since a set of points of $Q(2n-2,q)$ that meets all maximal subspaces of $Q(2n-2,q)$ has at least $1+q^{n-1}$ points, it follows that the $Q(2n-2,q)$ has a maximal singular subspace that contains no 'point' of B'. This shows that P lies on a maximal singular subspace T of H that meets B only in P.

Consider a hyperplane S of T with $P \notin S$. Then S lies in q^2+1 maximal singular subspaces and $q+1$ of these are contained in H. Since $S \cap B = \emptyset$, each of the q^2-q

maximal singular subspaces on S that is not contained in H meets B in a point that is not in H. This gives q^2-q points of B in $S^\perp \setminus H$. For distinct hyperplanes S of T, the sets $(S^\perp \cap Q)\setminus H$ are disjoint, since $T^\perp \cap Q = T \subseteq H$. Since there are q^{n-1} hyperplanes S of H with $P \notin S$, it follows that $|B\setminus H| \ge q^{n-1}(q^2-q)$. Hence $|H\cap B| \le |B| - q^{n-1}(q^2-q) = q^n+\delta$. ■

Lemma 2.4 *(a)* $|B| = q^{n+1}+q^{n-1}$.

(b) If $P\in B$, *then* $|P^\perp \cap B| = q^{n-1}$.

(c) If P *and* R *are distinct points of* B *and if* $\langle P,R\rangle$ *is a secant line of* Q, *then* $P^\perp \cap R^\perp \cap B = \emptyset$.

Proof. Consider a point $P\in B$. Then $|P^\perp\cap B|\le\delta$ and hence there exists a point $R\in B$ with $R\notin P^\perp$. Then $l := \langle P,R\rangle$ is a secant line of Q and the $(2n-1)$-space $l^\perp = P^\perp\cap R^\perp$ lies in $q+1$ hyperplanes, which are $X^\perp$ with $X\in l$. If $X\in l\cap Q$, that is $X=P$ or $X=R$, then $|X^\perp\cap B|\le\delta$ by Lemma 2.2. Otherwise $X\in l\setminus Q$ and then $|X^\perp\cap B|\le q^{n-1}+\delta$ by Lemma 2.3. Since the $q+1$ hyperplanes $X^\perp$ with $X\in l$ cover the whole space, it follows that

$$|B| \le 2\delta + (q-1)(q^n+\delta).$$

with equality only if $l^\perp\cap B=\emptyset$ and $|P^\perp\cap B| = |R^\perp\cap B| = \delta$.

Since $|B| = q^{n+1}+\delta$, we obtain $q^n\le\delta q$. Hence $\delta = q^{n-1}$ and we obtain equality. This proves all parts. ■

Lemma 2.5 $B = (U^\perp\setminus U)\cap Q$ *for a singular subspace* U *of dimension* $n-2$.

Proof. Consider $P\in B$. Then $|P^\perp\cap B| = q^{n-1}$ by Lemma 2.4 (a). If X and Y are distinct points of $P^\perp\cap B$, then P is a point of B in $X^\perp\cap Y^\perp$ and therefore Lemma 2.4 (c) shows that $\langle X,Y\rangle$ is a singular line. Hence, $P^\perp\cap B$ spans a singular subspace S. As $|P^\perp\cap B| = q^{n-1}$, we have $\dim(S)\ge n-1$, so S is a maximal singular subspace.

Consider any point $R\in B$ that is not in $P^\perp$. Then $P^\perp\cap R^\perp\cap B=\emptyset$ by Lemma 2.4 (c). Hence, the hyperplane $R^\perp\cap S$ of S contains no point of B. As $|P^\perp\cap B| = q^{n-1}$ and $P^\perp\cap B\subseteq S$, this gives $P^\perp\cap B = S\setminus(S\cap R^\perp)$.

Since this holds for all points R of $B\setminus P^\perp$, it follows that S has a hyperplane U with $U = R^\perp\cap S$ for all $R\in B\setminus P^\perp$. Hence $U\subseteq X^\perp$ for all points $X\in B$. As $|B| = (q^2+1)q^{n-1}$ and $U\cap B=\emptyset$, this implies that $B = (U^\perp\setminus U)\cap Q$. ■

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