Projective embedding of projective spaces

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Abstract

In this paper, embeddings $\phi : M \to P$ from a linear space $(M, \mathfrak{M})$ in a projective space $(P, \mathfrak{L})$ are studied. We give examples for $\dim M > \dim P$ and show under which conditions equality holds.
More precisely, we introduce properties (**G**) (for a line $L \in \mathfrak{L}$ and for a plane $E \subset M$ it holds that $|L \cap \phi(M)| \neq 1$) and (**E**) ($\phi(E) = \overline{\phi(E)} \cap \phi(M)$, whereby $\overline{\phi(E)}$ denotes the by $\phi(E)$ generated subspace of P). If (**G**) and (**E**) are satisfied then $\dim M = \dim P$. Moreover we give examples of embeddings of m-dimensional projective spaces in n-dimensional projective spaces with $m > n$ that map any $n+1$ independent points onto $n+1$ independent points. This implies that for a proper subspace T of M it holds $\phi(T) = \overline{\phi(T)} \cap \phi(M)$ if and only if $\dim T \leq n-1$, in particular (**E**) holds for $n \geq 3$. (cf. 4.1)

1 Introduction

An embedding $\phi : M \to P$ of a linear space $(M, \mathfrak{M})$ in a linear space $(P, \mathfrak{L})$ is an injective mapping that maps collinear points onto collinear points and noncollinear points onto noncollinear points. There are lots of papers concerning the embedding of linear spaces in projective spaces (cf. [3, Chap.6]). Important results are that every locally projective space $(M, \mathfrak{M})$ of $\dim M \geq 4$ (cf. [7, 10, 17, 19]) and every locally projective space $(M, \mathfrak{M})$ of $\dim M = 3$ satisfying the Bundle Theorem (cf. [8, 15]) is embeddable in a projective space $(P, \mathfrak{L})$. Due to the construction of the projective space the mentioned Embedding Theorems have the useful property that for every subspace T of $(M, \mathfrak{M})$ there exists exactly one subspace U of $(P, \mathfrak{L})$ with $\phi(T) = U \cap \phi(M)$. This property is equivalent to the two properties $(\mathbf{G}), (\mathbf{E})$;

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A linear space satisfying $(\mathbf{G}), (\mathbf{E})$ is called *locally complete* (cf. 2.4). For locally complete embeddings the dimension of M and P coincide (cf. 2.5). There are also projective embeddings of linear spaces which are not locally complete, but have the property that the dimension and order of M and P are equal (cf. [13, 14]). But there exist also embeddings which do not preserve the dimension. If $\phi(M)$ generates P, one obtaines $\dim M \geq \dim P$ (cf. 2.3), hence we have to consider only the case $\dim M > \dim P$. For example one can embed every linear space in a projective plane E by a free construction of E. (Then of course, E is not a Desarguesian plane.) Kalhoff constructed in [9] the embedding of any finite partial planes in a translation plane, and hence in a projective plane of Lenz class V.

In this paper we are interested in embeddings in Desarguesian projective planes and spaces. There are some papers which give a characterisation of embeddings of projective spaces in Desarguesian projective spaces. For a field K and the $(m+1)$-dimensional vector space (K^{m+1}, K) over K, let $\mathrm{PG}(m, K)$ denote the m-dimensional projective space over K with the 1-dimensional vector subspaces as points and the 2-dimensional vector subspaces as lines. M. Limbos [16] has shown for finite projective spaces that every embedding of $\mathrm{PG}(m, K)$ in $\mathrm{PG}(n, L)$ with $m > n$ is a product of the trivial embedding of $\mathrm{PG}(m, K)$ in $\mathrm{PG}(m, L)$ for a field extension L of K, and a projection of $\mathrm{PG}(m, L)$ in the subspace $\mathrm{PG}(n, L)$. In [16] a geometric construction of embeddings is given and the proof that every embedding can be obtained by this construction. H. Havlicek [6] and C.A. Faure, A. Froelicher [4, 5] give a similar characterisation for the infinite case, but without a construction. For an arbitrary field K an example of an embedding of $\mathrm{PG}(m, K)$ in $\mathrm{PG}(m-1, L)$ for a field extension L of K is given by A. Brezuleanu, D.-C. Rădulescu [1, (5.8)]. For a finite field K, J. Brown gives in [2] an analytic example of an embedding $\phi : \mathrm{PG}(m, K) \to \mathrm{PG}(2, L)$ for a field extension L of K. This examples does not satisfy $(\mathbf{E})$.

In this paper we answer the question, if there exists an embedding $\phi : P \to P'$ of a Pappian projective space $(P, \mathfrak{L})$ in a Pappian projective space $(P', \mathfrak{L}')$ which does not preserve dimension, but satisfy property $(\mathbf{E})$. We show the corresponding statements for higher dimensions. We show that for $\dim P' = n$ there are embeddings which map any $n+1$ independent points of P onto $n+1$ independent points of P'. It follows that the image of an $(n-1)$-dimensional subspace T of P generates an $(n-1)$-dimensional subspace $\overline{\phi(T)}$ of P' with $\phi(T) = \overline{\phi(T)} \cap \phi(P)$. We remark that there exist also embeddings of projective spaces in projective planes satisfying property $(\mathbf{G})$.

2 Locally Complete Embeddings

A *linear space* $(P, \mathfrak{L}, \mathrm{I})$ will be defined as a set P of elements, called *points*, a distinct set $\mathfrak{L}$ of elements, called *lines*, and an incidence relation I such that any two distinct points are incident with exactly one line and every line is incident with at least two points. Usually one identifies a line $L \in \mathfrak{L}$ with the set of points incident with L, hence the lines of $(P, \mathfrak{L}, \mathrm{I}) = (P, \mathfrak{L})$ are subsets of P.

A *subspace* is a subset $U \subset P$ such that for all distinct points $x, y \in U$ the unique line incident with x, y is contained in U. Let $\mathfrak{U}$ denote the set of all subspaces. For every subset $X \subset P$ we define the following *closure operator*:

$$^{-} : \mathfrak{P}(P) \to \mathfrak{U} : \; X \mapsto \overline{X} := \bigcap_{\substack{U \in \mathfrak{U} \\ X \subset U}} U \tag{1}$$

The closure of X is a subspace containing X. For $U \in \mathfrak{U}$ we call $\dim U := \inf\{\mid X \mid -1 : X \subset U \text{ and } \overline{X} = U\}$ the *dimension* of U. A subspace of dimension two is a *plane*. A subset $X \subset P$ is *independent* if $x \notin \overline{X \setminus \{x\}}$ for every $x \in X$, and is a *basis* of a subspace U if X is independent and $\overline{X} = U$.

For two linear spaces $(M, \mathfrak{M})$ and $(P, \mathfrak{L})$, an injective mapping

$$\phi : M \to P, \; x \mapsto \phi(x) \tag{2}$$

is called an *embedding*, if ϕ maps collinear points onto collinear points and noncollinear points onto noncollinear points, i.e., $\{\phi(G) : G \in \mathfrak{M}\} = \{L \cap \phi(M) : L \in \mathfrak{L}$ and $|L \cap \phi(M)| \geq 2\}$. Hence $\big(\phi(M), \{\phi(G) : G \in \mathfrak{M}\}\big)$ is the restriction of $(P, \mathfrak{L})$ to $\phi(M)$. Clearly:

Lemma 2.1 *If ϕ is an embedding of $(M, \mathfrak{M})$ in $(P, \mathfrak{L})$, and ψ is an embedding of $(P, \mathfrak{L})$ in $(P', \mathfrak{L}')$, then $\psi \circ \phi$ is an embedding of $(M, \mathfrak{M})$ in $(P', \mathfrak{L}')$.*

Let $Y \mapsto \overline{Y}$ denote the closure of $(P, \mathfrak{L})$ and $X \mapsto \langle X \rangle$ the closure of $(M, \mathfrak{M})$. By [12, (1.1)]:

Lemma 2.2 *If ϕ is an embedding of $(M, \mathfrak{M})$ in $(P, \mathfrak{L})$, and U a subspace of $(P, \mathfrak{L})$ and $X \subset M$, then:*

1. *$\phi^{-1}(U \cap \phi(M))$ is a subspace of M.*
2. *$\phi(\langle X \rangle) \subset \overline{\phi(X)}$ and $\overline{\phi(\langle X \rangle)} = \overline{\phi(X)}$.*
3. *If $\phi(X)$ is independent in P, then X is independent in M.*

Lemma 2.3 *If $\phi : M \to P$ is an embedding of a linear space $(M, \mathfrak{M})$ in a linear space $(P, \mathfrak{L})$ satisfying $\overline{\phi(M)} = P$, then $\dim M \geq \dim P$.*

Proof. Let $X \subset M$ be a subset generating M, i.e. $\langle X \rangle = M$. Then $P = \overline{\phi(M)} = \overline{\phi(\langle X \rangle)} = \overline{\phi(X)}$ by 2.2. Therefore $\phi(X)$ is a generating set of P with $|X| = |\phi(X)|$, hence $\dim P \leq \dim M$. ∎

We call an embedding ϕ of $(M, \mathfrak{M})$ in $(P, \mathfrak{L})$ *locally complete*, if for every nonempty subspace T of M, there is exactly one subspace U of P with $\phi(T) = U \cap \phi(M)$.

By [12, (1.5)] we have:

Lemma 2.4 *For an embedding ϕ of $(M, \mathfrak{M})$ in $(P, \mathfrak{L})$ the following statements are equivalent:*

1. *ϕ is locally complete.*

2. *For every subspace T of $(M, \mathfrak{M})$ and for every subspace U of $(P, \mathfrak{L})$ with $\phi(M) \cap U \neq \emptyset$ we have*

$$U = \overline{U \cap \phi(M)} \qquad \text{and} \qquad \phi(T) = \overline{\phi(T)} \cap \phi(M)$$

3. *The following properties* $(\mathbf{G}), (\mathbf{E})$ *are satisfied.*

   (G) *For every line $L \in \mathfrak{L}$, $|L \cap \phi(M)| \neq 1$*

   (E) *For every plane E of M, $\phi(E) = \overline{\phi(E)} \cap \phi(M)$*

A linear space $(P, \mathfrak{L})$ satisfies the *exchange condition* if

$$\text{for } S \subset P \text{ and } x, y \in P \text{ with } x \in \overline{S \cup \{y\}} \setminus \overline{S} \text{ it follows that } y \in \overline{S \cup \{x\}}. \tag{3}$$

Lemma 2.5 *If ϕ is a locally complete embedding of a linear space $(M, \mathfrak{M})$ in a linear space $(P, \mathfrak{L})$ satisfying the exchange condition, then* $\dim M = \dim P$.

Proof. Since ϕ is locally complete, $P = \overline{P \cap \phi(M)} = \overline{\phi(M)}$, hence, by Lemma 2.3, $\dim P \leq \dim M$. Now let $x \in \phi(M)$. Since $(P, \mathfrak{L})$ is an exchange space, there is a basis C of P containing x (cf. [11, §8]. By Lemma 2.4, $(\mathbf{G})$ holds. Moreover for every $y \in C \setminus \{x\}$, there exists a $y' \in \left(\overline{x, y} \cap \phi(M)\right) \setminus \{x\}$. Hence we obtain a basis $C' \subset \phi(M)$ of P with $|C| = |C'|$. Let $T := \langle \phi^{-1}(C') \rangle$ denote the subspace of M generated by $\phi^{-1}(C')$, i.e. $C' \subset \phi(T)$ and $P = \overline{C'} = \overline{\phi(T)}$. We get $\phi(T) = \overline{\phi(T)} \cap \phi(M) = P \cap \phi(M) = \phi(M)$, hence $M = T$ is generated by $\phi^{-1}(C')$ and $\dim M \leq \dim P$. ∎

The Lemma 2.5 applies in particular, if $(P, \mathfrak{L})$ is a projective space.

Theorem 2.6 *Let $(P, \mathfrak{L}), (M, \mathfrak{M})$ be linear spaces satisfying the exchange condition and* $\dim M > \dim P$. *If $\phi : M \to P$ is an embedding satisfying* $(\mathbf{G})$, *then there exist subspaces $M' \subset M, P' \subset P$ with* $\dim M' > \dim P' = 2$ *such that $\phi|_{M'} : M' \to P'$ is an embedding satisfying* $(\mathbf{G})$.

Proof. By Lemma 2.5 $(\mathbf{E})$ is not satisfied, since $\dim M > \dim P$. Hence there exists a plane $E \subset M$ with $\phi(E) \neq \left(\overline{\phi(E)} \cap \phi(M)\right)$. Therefore $M' := \phi^{-1}(\overline{\phi(E)} \cap \phi(M))$ is a subspace with $E \subset M'$ and $E \neq M'$, i.e. $\dim M' > 2$. Since E is a plane, also $P' := \overline{\phi(E)} = \overline{\phi(M')}$ is a plane, and the restriction of ϕ to M' is an embedding. For a line $L \subset P'$ we have $L \cap \phi(M) = L \cap \phi(M')$. Hence if $x \in L \cap \phi(M')$ we have $G := \phi^{-1}(L \cap \phi(M')) \in \mathfrak{M}$, since ϕ satisfies $(\mathbf{G})$. Because $G \subset M'$, also $\phi|_{M'}$ satisfies $(\mathbf{G})$. ∎

Theorem 2.7 *Let $(P, \mathfrak{L}) = \mathrm{PG}(m, K)$ and $(P', \mathfrak{L}') = \mathrm{PG}(n, L)$ be projective spaces and $\phi : P \to P'$ an embedding, then K is isomorphic to a subfield of L.*

Proof. Let E be a plane of P. Then $\phi(E) \simeq \mathrm{PG}(2, K)$ is a subplane of the Desarguesian projective plane $\overline{\phi(E)} \simeq \mathrm{PG}(2, L)$, hence K is isomorphic to a subfield of L (cf. [18, (8.2)], [6, (3.6.1)]). ∎

3 A mapping of a vector space in a vector space over a field extension

In this section let $n, s \in \mathbb{N}$ be integers with $n \geq 2$, let K be a commutative field, and $L = K(t)$ an extension field of K with a transcendental or algebraic element t of degree at least $2^{s(n+1)}$ over K. We consider the two left vector spaces (K^{n+s+1}, K) and (L^{n+1}, L). For $i \in 0, 1, \ldots, n$ let $\mathfrak{x}_i \in K^{n+s+1}$, more precisely

$$\mathfrak{x}_i = (x_{i,0}, x_{i,1}, \ldots, x_{i,n+s}) \tag{4}$$

with elements $x_{i,k} \in K$. We denote the rows of the matrix

$$\mathbf{X} := \begin{pmatrix} \mathfrak{x}_0 \\ \vdots \\ \mathfrak{x}_n \end{pmatrix} = \begin{pmatrix} x_{0,0} & \ldots & x_{0,n+s} \\ \vdots & \vdots & \vdots \\ x_{n,0} & \ldots & x_{n,n+s} \end{pmatrix} = \left(\mathfrak{a}_0^T,\ \ldots\ , \mathfrak{a}_{n+s}^T\right), \tag{5}$$

$$\text{where} \quad \mathfrak{a}_k^T = \begin{pmatrix} x_{0,k} \\ \vdots \\ x_{n,k} \end{pmatrix} \quad \text{for} \quad k = 0, 1, \ldots, n+s. \tag{6}$$

Since the column rank and the row rank of $\mathbf{X}$ are equal, we have:

Lemma 3.1 *The following statements are equivalent:*

1. *The vectors $\mathfrak{x}_0, \mathfrak{x}_1, \ldots, \mathfrak{x}_n$ are linearly independent in (K^{n+s+1}, K).*
2. *The matrix $\mathbf{X} = (\mathfrak{a}_0^T, \mathfrak{a}_1^T, \ldots, \mathfrak{a}_{n+s}^T)$ has rank $n+1$.*
3. *There exist distinct integers $i_0, i_1, \ldots, i_n \in \{0, 1, \ldots, n+s\}$ such that $\mathfrak{a}_{i_0}, \mathfrak{a}_{i_1}, \ldots, \mathfrak{a}_{i_n}$ are linearly independent in (K^{n+1}, K).*

Now we consider arbitrary vectors $\mathfrak{a}_0, \mathfrak{a}_1, \ldots, \mathfrak{a}_{n+s} \in K^{n+1} \subset L^{n+1}$ and define

$$\mathfrak{b}_i^T := \mathfrak{a}_i^T + \sum_{j_i=1}^{s} t^{2^{(j_i-1)(n+1)+i}} \mathfrak{a}_{n+j_i}^T \in L^{n+1} \qquad \text{for} \qquad i = 0, 1, \ldots, n. \tag{7}$$

For example, for $s = 2$ we obtain: $\quad \mathfrak{b}_i^T := \mathfrak{a}_i^T + t^{2^i}\mathfrak{a}_{n+1}^T + t^{2^{(n+1)+i}}\mathfrak{a}_{n+2}^T.$

Lemma 3.2 $\det(\mathfrak{b}_0^T, \mathfrak{b}_1^T, \ldots, \mathfrak{b}_n^T) \neq 0$ *if and only if* $\operatorname{rank}(\mathfrak{a}_0^T, \mathfrak{a}_1^T, \ldots, \mathfrak{a}_{n+s}^T) = n+1.$

Proof. (i). First we introduce some notation to get a shorter representation. For $i \in \{0, \ldots, n\}$ and $j_i \in \{0, \ldots, s\}$ we define

$$\lambda_{i,j_i} := \begin{cases} 0 & if \quad j_i = 0 \\ 2^{(j_i-1)(n+1)+i} & if \quad j_i \neq 0 \end{cases} \quad \text{and } \mathfrak{a}_{i,j_i}^T := \begin{cases} \mathfrak{a}_i^T & if \quad j_i = 0 \\ \mathfrak{a}_{n+j_i}^T & if \quad j_i \neq 0 \end{cases},$$

$$\text{so} \qquad \mathfrak{b}_i^T := t^0\mathfrak{a}_i^T + \sum_{j_i=1}^{s} t^{2^{(j_i-1)(n+1)+i}} \mathfrak{a}_{n+j_i}^T = \sum_{j_i=0}^{s} t^{\lambda_{i,j_i}} \mathfrak{a}_{i,j_i}^T. \tag{8}$$

(ii).We recall that we can write every integer $k \in \{1, 2, \ldots, 2^{s(n+1)} - 1\}$ as a sum of elements of $\{2^r : r = 0, 1, \ldots, s(n+1)-1\} = \{2^{(j_i-1)(n+1)+i} : j_i = 1, 2, \ldots, s, \quad i = 0, 1, \ldots, n\} = \{\lambda_{i,j_i} : j_i = 1, 2, \ldots, s, \quad i = 0, 1, \ldots, n\}$ in a unique way. Hence we have for $j_i, k_i \in \{0, 1, \ldots, s\}$

$$\sum_{i=0}^{n} \lambda_{i,j_i} = \sum_{i=0}^{n} \lambda_{i,k_i} \quad \text{if and only if} \quad j_i = k_i \quad \text{for all} \quad i \in \{0, \ldots, n\} \tag{9}$$

$$\text{and} \prod_{i=0}^{n} t^{\lambda_{i,j_i}} = \prod_{i=0}^{n} t^{\lambda_{i,k_i}} \ \text{if and only if} \ j_i = k_i \quad \text{for all} \quad i \in \{0, \ldots, n\}. \tag{10}$$

(iii).By (i) we get

$$d := \det(\mathfrak{b}_0^T, \ldots, \mathfrak{b}_n^T) = \det\Big(\sum_{j_0=0}^{s} t^{\lambda_{0,j_0}} \mathfrak{a}_{0,j_0}^T, \ \ldots \ , \sum_{j_n=0}^{s} t^{\lambda_{n,j_n}} \mathfrak{a}_{n,j_n}^T \Big) =$$

$$= \sum_{j_0,\ldots,j_n=0}^{s} \Big(t^{\lambda_{0,j_0}} \cdot \ldots \cdot t^{\lambda_{n,j_n}}\Big) \det\Big(\mathfrak{a}_{0,j_0}^T, \ldots, \mathfrak{a}_{n,j_n}^T\Big) = \sum_{k \le m} t^k \det \mathbf{A}_k \tag{11}$$

with $k = \sum \lambda_{0,j_0}, + \cdots + \lambda_{n,j_n}, \ m = 2^{s(n+1)} - 1$ and $\mathbf{A}_k = (\mathfrak{a}_{i_0}^T, \mathfrak{a}_{i_1}^T, \ldots, \mathfrak{a}_{i_n}^T)$ with not necessarily distinct integers $i_j \in \{0, 1, \ldots, n+s\}$, $j = 0, 1, \ldots, n$.

(vi). Since $t \in L \setminus K$ has at least degree $m + 1 = 2^{s(n+1)}$ over K, we have $d = \sum_{k=0}^{m} t^k \det \mathbf{A}_k = 0$ if and only if $\det \mathbf{A}_k = 0$ for every $k \in \{0, 1, \ldots, 2^{s(n+1)} - 1\}$. This means in particular that for distinct elements $i_0, i_1, \ldots, i_n \in \{0, 1, \ldots, n+s\}$ the vectors $\mathfrak{a}_{i_0}^T, \mathfrak{a}_{i_1}^T, \ldots, \mathfrak{a}_{i_n}^T$ are linearly dependent. By Lemma 3.1 it follows that $\operatorname{rank}(\mathfrak{a}_0^T, \mathfrak{a}_1^T, \ldots, \mathfrak{a}_{n+s}^T) < n + 1$.

On the other hand, if $d \neq 0$, then there exist integers $i_0, i_1, \ldots, i_n \in \{0, 1, \ldots, n+s\}$ with $\det(\mathfrak{a}_{i_0}^T, \mathfrak{a}_{i_1}^T, \ldots, \mathfrak{a}_{i_n}^T) \neq 0$, hence $i_0, i_1, \ldots, i_n$ are distinct and $\mathfrak{a}_{i_0}^T, \mathfrak{a}_{i_1}^T, \ldots, \mathfrak{a}_{i_n}^T$ are linearly independent. By Lemma 3.1 we get $\operatorname{rank}(\mathfrak{a}_0^T, \mathfrak{a}_1^T, \ldots, \mathfrak{a}_{n+s}^T) = n + 1$. ■

Now we define the map

$$f : K^{n+s+1} \to L^{n+1}, \quad \mathfrak{x} = (x_0, \ldots, x_{n+s+1}) \mapsto \mathfrak{x}' = (x_0', \ldots, x_n')$$
$$\text{by} \quad x_i' = x_i + \sum_{j=1}^{s} t^{2^{(j-1)(n+1)+i}} x_{n+j} \quad , \quad \text{for} \quad i = 0, 1, \ldots, n. \tag{12}$$

Lemma 3.3 *1.* $f(K\mathfrak{x}) = Kf(\mathfrak{x}) \subset Lf(\mathfrak{x})$.

2. *The vectors* $\mathfrak{x}_0, \ldots, \mathfrak{x}_n$ *are linearly independent in* (K^{n+s+1}, K) *if and only if* $f(\mathfrak{x}_0), \ldots, f(\mathfrak{x}_n)$ *are linearly independent in* (L^{n+1}, L).

3. *In particular for three vectors* $\mathfrak{x}_0, \mathfrak{x}_1, \mathfrak{x}_2 \in K^{n+s+1}$, $\operatorname{rank}(\mathfrak{x}_0, \mathfrak{x}_1, \mathfrak{x}_2) = 3$ *if and only if* $\operatorname{rank}\Big(f(\mathfrak{x}_0), f(\mathfrak{x}_1), f(\mathfrak{x}_2)\Big) = 3$.

Proof. 1. By definition $f(\lambda\mathfrak{x}) = \lambda f(\mathfrak{x})$ for $\lambda \in$ K. Clearly $Kf(\mathfrak{x}) \subset Lf(\mathfrak{x})$.

2. For $\mathfrak{x}_0, \ldots, \mathfrak{x}_n \in K^{n+s+1}$ with $\mathfrak{x}_i = (x_{i,0}, x_{i,1}, \ldots, x_{i,n+s})$, we consider the matrix

$$\mathbf{X}' := \begin{pmatrix} f(\mathfrak{x}_0) \\ \vdots \\ f(\mathfrak{x}_n) \end{pmatrix}$$
$$= \begin{pmatrix} x_{0,0} + \sum_{j=1}^{s} t^{2^{(j-1)(n+1)}} x_{0,n+j} & \ldots & x_{0,n} + \sum_{j=1}^{s} t^{2^{(j-1)(n+1)+n}} x_{0,n+j} \\ \vdots & \vdots & \vdots \\ x_{n,0} + \sum_{j=1}^{s} t^{2^{(j-1)(n+1)}} x_{n,n+j} & \ldots & x_{n,n} + \sum_{j=1}^{s} t^{2^{(j-1)(n+1)+n}} x_{n,n+j} \end{pmatrix}$$
$$= \left(\mathfrak{a}_0^T + \sum_{j=1}^{s} t^{2^{(j-1)(n+1)}} \mathfrak{a}_{n+j}^T, \ \ldots \ , \mathfrak{a}_n^T + \sum_{j=1}^{s} t^{2^{(j-1)(n+1)+n}} \mathfrak{a}_{n+j}^T \right)$$
$$= \left(\mathfrak{b}_0^T, \ \ldots \ , \mathfrak{b}_n^T \right).$$

Hence by Lemma 3.2 we have $\det(\mathbf{X}') = \det(\mathfrak{b}_0^T, \mathfrak{b}_1^T, \ldots, \mathfrak{b}_n^T) \neq 0$ iff $\operatorname{rank}(\mathfrak{a}_0^T, \mathfrak{a}_1^T, \ldots, \mathfrak{a}_{n+s}^T) = n+1$, i.e. by Lemma 3.1, iff $\mathfrak{x}_0, \ldots, \mathfrak{x}_n$ are linearly independent.
Since $n \geq 2$, 3. is a consequence of 2. ∎

4 Embeddings satisfying (E)

Using the map f introduced in the preceding section, we now construct projective embeddings.
Let $(P, \mathfrak{L})$ be a Pappian projective space with $\dim P = n + s$ for $n, s \in \mathbb{N}$ with $n \geq 2$. Then we can represent $(P, \mathfrak{L}) = \mathrm{PG}(n+s, K)$ by an $(n+s+1)$-dimensional vector space (K^{n+s+1}, K) over a commutative field K. Let us denote by $(P', \mathfrak{L}') := \mathrm{PG}(n, L)$ the n-dimensional projective space with the underlying vector space (L^{n+1}, L) where L is the field extension of K introduced in the preceding section. We recall that three points $a = K\mathfrak{a}, b = K\mathfrak{b}, c = K\mathfrak{c}$ are noncollinear if and only if $\operatorname{rank}(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) = 3$ for vectors $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in K^{n+s+1}$.

Theorem 4.1 *1. For every $n, s \in \mathbb{N}$ with $n \geq 2$ and every Pappian projective space $(P, \mathfrak{L})$ of dimension $n+s$, there exists an embedding $\phi : P \to P'$ in an n-dimensional projective Pappian space $(P', \mathfrak{L}')$ such that any $n+1$ points $x_0, \ldots, x_n \in P$ are independent in $(P, \mathfrak{L})$ if and only if $\phi(x_0), \ldots, \phi(x_n)$ are independent in $(P', \mathfrak{L}')$.*

2. For a proper subspace T of P it holds that $\phi(T) = \overline{\phi(T)} \cap \phi(P)$ if and only if $\dim T \leq n-1$

*3. For $n \geq 3$, ϕ satisfies (**E**).*

Proof. 1. Using the map f of Lemma 3.3, we define

$$\phi : P \to P', \ x = K\mathfrak{x} \mapsto \phi(x) := Lf(\mathfrak{x}) \tag{13}$$

By Lemma 3.3(1), ϕ is well defined, and by Lemma 3.3(3), ϕ maps collinear points onto collinear point and noncollinear points onto noncollinear points, hence ϕ is an embedding. Since $x_0 = K\mathfrak{x}_0, \dots, x_n = K\mathfrak{x}_n$ are independent iff $\mathfrak{x}_0, \dots, \mathfrak{x}_n$ are linearly independent, and $\phi(x_0) = Lf(\mathfrak{x}_0), \dots, \phi(x_n) = Lf(\mathfrak{x}_n)$ are independent iff $f(\mathfrak{x}_0), \dots, f(\mathfrak{x}_0)$ are linearly independent, one obtain by Lemma 3.3(2) that $x_0, \dots, x_n \in P$ are independent iff $\phi(x_0), \dots, \phi(x_n)$ are independent.

2. For $r \leq n-1$, let T be an r-dimensional subspace of $(P, \mathfrak{L})$ with a basis $a_0, \dots, a_r$. Assume that $\phi(T) \neq \overline{\phi(T)} \cap \phi(P)$. Then there exists a point $b \in P$ with $\phi(b) \in \left(\overline{\phi(T)} \setminus \phi(T)\right)$, i.e. $b \notin T$ and $a_0, \dots, a_r, b$ are independent in $(P, \mathfrak{L})$. Since $\phi(b) \in \overline{\phi(T)} = \overline{\phi(a_0), \dots, \phi(a_r)}$ (cf. Lemma 2.2(2)), it follows that $\phi(a_0), \dots, \phi(a_r), \phi(b)$ are dependent in $(P'\mathfrak{L}')$.
Since $r+2 \leq n+1$, by 1., the points $\phi(a_0), \dots, \phi(a_r), \phi(b)$ are independent since $a_0, \dots, a_r, b$ are independent, a contradiction to the assumption $\phi(T) \neq \overline{\phi(T)} \cap \phi(P)$. Hence $\phi(T) = \overline{\phi(T)} \cap \phi(P)$ for $\dim T \leq n-1$. For every proper subspace T of P with $\dim T \geq n$, there are $n+1$ independent points $a_0, \dots, a_n \in T$. By 1. $\phi(a_0), \dots \phi(a_n)$ are independent in P', hence $P' = \overline{\phi(a_0), \dots, \phi(a_n)} \subset \overline{\phi(T)}$ and $\phi(T) \neq \phi(P) = \overline{\phi(T)} \cap \phi(P) = P' \cup \phi(P)$, since T is a proper subspace of P.

3. By 2., (**E**) is satisfied for $n \geq 3$. ■

Corollary 4.2 *For every $n, s \in \mathbb{N}$ with $n \geq 2$ and every finite projective space $(P, \mathfrak{L})$ of dimension $n+s$, there exists an embedding $\phi : P \to P'$ in an n-dimensional finite projective Desarguesian space $(P', \mathfrak{L}')$ such that any $n+1$ points $x_0, \dots, x_n \in P$ are independent in $(P, \mathfrak{L})$ if and only if $\phi(x_0), \dots, \phi(x_n)$ are independent in $(P', \mathfrak{L}')$. For a proper subspace T of P it holds that $\phi(T) = \overline{\phi(T)} \cap \phi(P)$ if and only if* $\dim T \leq n-1$, *and for $n \geq 3$, ϕ satisfies* (**E**).

Proof. If P is finite, then ord$P$ is finite and $(P, \mathfrak{L}) = \mathrm{PG}(n+s, K)$ for a commutative field K. There exists a finite field extension $L = K(t)$ of finite degree t at least $2^{s(n+1)}$, hence L, and therefore also P' are finite and the assertion follows with 4.1. ■

If we set n = 2 we obtain:

Corollary 4.3 *Every Pappian projective space is embeddable in a Pappian projective plane.*

Proof. For a Pappian projective space $(P, \mathfrak{L})$ of finite dimension, Corollary 4.3 is a direct consequence of Theorem 4.1 with $n = 2$. For $\dim P = \infty$ we modify the construction of the last section, by taking a transcendental element t_b for every element b of a basis B of P. Then for $T = \{t_b : b \in B\}$ and $L := K(T)$ we get the result analogous to the proofs of Lemma 3.1 to 3.3. ■

Let $(M, \mathfrak{M})$ be a linear space. Two lines $G, L \in \mathfrak{L}$ are called *parallel* if $G = L$, or if G, L are contained in a common plane and $G \cap L = \emptyset$. For $x \in M \setminus L$ let

$$\pi(x, L) := |\{G \in \mathfrak{M} : x \in G \text{ and } G, L \text{ parallel }\}| \tag{14}$$

denote the number of all parallel lines of L passing x. For $m \in \mathbb{N}$, $(M, \mathfrak{M})$ is called an $[0, m]$-*space*, if for each non-incident point-line pair (x, L) we have that $\pi(x, L) \in [0, m] = \{0, 1, \dots, m\}$. Let $\pi(L) := max\{\pi(y, L) : y \in M \setminus L\}$. If $|L| + \pi(L) - 1 \geq 3m+1$ and $\dim M \geq 3$, then by [14, Theorem (2.10)], ord$M := |L| + \pi(L) - 1$ is constant for every line $L \in \mathfrak{M}$. If ord$M \geq 3m+2$ and $\dim M \geq 3$, then by [14, Embedding Theorem (4.5)], $(M, \mathfrak{M})$ is embeddable in a projective space $(P, \mathfrak{L})$ with $\dim M = \dim P$ and ord$M = $ ord$P$. Hence:

Corollary 4.4 *Every finite* $[0,m]$*-space* $(M,\mathfrak{M})$ *with* $\dim M \geq 3$ *and* $\mathrm{ord}M \geq 3m+2$ *is embeddable in a finite Pappian projective plane.*

Proof. Since M is finite, also $\mathrm{ord}M = \mathrm{ord}P$ and $\dim M = \dim P$ is finite and $(M,\mathfrak{M})$ is embeddable in a finite projective space $(P,\mathfrak{L})$. Now by 4.2 for $n=2$, $(P,\mathfrak{L})$ is embeddable in a finite Pappian plane $(P',\mathfrak{L}')$ and by 2.1 the assertion follows. ∎

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