# Existence of integrals for finite dimensional quasi-Hopf algebras

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## 1 Introduction

If $A$ is a finite dimensional Hopf algebra and $\int \subseteq A$ is the space of integrals in $A$, it is well known that $dim(\int) = 1$. The proof given in [12] actually shows the existence and uniqueness of integrals in $A^*$ and it relies on the structure of Hopf modules over $A$, namely one has to prove that $A^*$ is a right $A$-Hopf module and then the result follows from the fundamental theorem for Hopf modules (see [12] for details).

It is very natural to ask if the result remains true if $A$ is not a Hopf algebra, but a quasi-Hopf algebra (this question arose in [9], where the following version of Maschke's theorem for quasi-Hopf algebras was proved: $A$ is semisimple if and only if $\varepsilon(\int) \neq 0$). The answer is positive for some particular quasi-Hopf algebras, for instance for Dijkgraaf-Pasquier-Roche's quasi-Hopf algebras $D^\omega(G)$ (where $G$ is a finite group and $\omega$ is a normalized 3-cocycle on $G$) and for their generalizations $D^\omega(H)$ introduced in [1] (where $H$ is a finite dimensional cocommutative Hopf algebra and $\omega : H \otimes H \otimes H \to k$ is a normalized 3-cocycle in Sweedler's cohomology). But if one tries to generalize the proof given in [12] to quasi-Hopf algebras some problems occur, for example it is not clear which could be the appropriate definition for a Hopf module over a quasi-Hopf algebra.

The existence and uniqueness of integrals for finite dimensional Hopf algebras have been reproved in [11], [8] by avoiding the use of Hopf modules. In this note we shall prove the *existence* of integrals for finite dimensional quasi-Hopf algebras, by generalizing the short and direct proof given by A. Van Daele in [11] for the Hopf algebra case. It seems that the method in [11] does not yield a proof for the *uniqueness* property.

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## 2 The existence of integrals

Throughout, $k$ will be a fixed field and all algebras, linear spaces etc. will be over $k$; unadorned $\otimes$ means $\otimes_k$.

**Definition 2.1.** *(see [3], [6]) Let $A$ be a $k-$algebra, $\Delta : A \to A \otimes A$, $\varepsilon : A \to k$ two algebra homomorphisms. $A$ is called a quasi-bialgebra if there exists an invertible element $\Phi \in A \otimes A \otimes A$ such that, for all elements $a \in A$, we have:*

(2.1) $(I \otimes \Delta)(\Delta(a)) = \Phi((\Delta \otimes I)(\Delta(a))\Phi^{-1}$,

(2.2) $(\varepsilon \otimes I)(\Delta(a)) = a$ *and* $(I \otimes \varepsilon)(\Delta(a)) = a$,

(2.3) $(I \otimes I \otimes \Delta)(\Phi)(\Delta \otimes I \otimes I)(\Phi) = (1 \otimes \Phi)(I \otimes \Delta \otimes I)(\Phi)(\Phi \otimes 1)$,

(2.4) $(I \otimes \varepsilon \otimes I)(\Phi) = 1 \otimes 1$,

*where $I = id_A$. The map $\Delta$ is called the coproduct or the comultiplication and $\varepsilon$ the counit.*

*$A$ is called a quasi-Hopf algebra if, moreover, there exist an anti-automorphism $S$ of the algebra $A$ and elements $\alpha$ and $\beta$ of $A$ such that, for all $a \in A$, we have:*

(2.5) $\sum S(a_1)\alpha a_2 = \varepsilon(a)\alpha$ *and* $\sum a_1\beta S(a_2) = \varepsilon(a)\beta$,

(2.6) $\sum X^1\beta S(X^2)\alpha X^3 = 1$ *and* $\sum S(x^1)\alpha x^2\beta S(x^3) = 1$,

*where $\Phi = \sum X^1 \otimes X^2 \otimes X^3$, $\Phi^{-1} = \sum x^1 \otimes x^2 \otimes x^3$ (formal notation) and we used the $\Sigma$-notation : $\Delta(a) = \sum a_1 \otimes a_2$. In this case, $S$ is called the antipode of $A$.*

Let us note that every Hopf algebra with bijective antipode is a quasi-Hopf algebra with $\Phi = 1 \otimes 1 \otimes 1$ and $\alpha = \beta = 1$.

We note the following two consequences of the definitions of $S, \alpha, \beta$: $\varepsilon(\alpha)\varepsilon(\beta) = 1$, $\varepsilon \circ S = \varepsilon$. Moreover, (2.3) and (2.4) imply $(\varepsilon \otimes I \otimes I)(\Phi) = (I \otimes I \otimes \varepsilon)(\Phi) = 1$.

**Definition 2.2.** *If $A$ is a finite dimensional quasi-Hopf algebra, an element $\lambda \in A$ satisfying the condition $a\lambda = \varepsilon(a)\lambda$ for all $a \in A$ will be called a left integral for $A$. The space of left integrals will be denoted by $\int$.*

**Proposition 2.3.** *If $A$ is a finite dimensional quasi-Hopf algebra, then $\int \neq 0$.*

*Proof :* Let $\{e_1, ..., e_n\}$ be a basis in $A$ and $\{e^1, ..., e^n\}$ the dual basis in $A^*$. For any element $b \in A$ we define the following element in $A$:

$$t(b) = \sum\langle e^i, \beta S(\alpha X^3)S^2(X^2(e_i)_2)b\rangle X^1(e_i)_1$$

where, if $p \in A^*$ and $a \in A$ we denoted by $\langle p, a\rangle = p(a)$. We shall prove that

$$at(b) = \varepsilon(a)t(b)$$

for all $a \in A$. Indeed, if $a \in A$, we calculate:

$$\varepsilon(a)t(b) = \sum\langle e^i, a_1\beta S(a_2)S(\alpha X^3)S^2(X^2(e_i)_2)b\rangle X^1(e_i)_1$$

$$= \sum\langle e^i, a_1e_j\rangle\langle e^j, \beta S(a_2)S(\alpha X^3)S^2(X^2(e_i)_2)b\rangle X^1(e_i)_1$$

$$= \sum\langle e^j, \beta S(a_2)S(\alpha X^3)S^2(X^2(a_1)_2(e_j)_2)b\rangle X^1(a_1)_1(e_j)_1$$

$$= \sum\langle e^j, \beta S(\alpha X^3a_2)S^2(X^2(a_1)_2(e_j)_2)b\rangle X^1(a_1)_1(e_j)_1$$

$$= \sum \langle e^j, \beta S(\alpha(a_2)_2 X^3) S^2((a_2)_1 X^2 (e_j)_2) b\rangle a_1 X^1 (e_j)_1 \qquad (by\ (2.1))$$
$$= \sum \langle e^j, \beta S(X^3) S(S((a_2)_1)\alpha(a_2)_2) S^2(X^2(e_j)_2) b\rangle a_1 X^1 (e_j)_1$$
$$= \sum \langle e^j, \beta S(X^3) S(\alpha) S^2(X^2(e_j)_2) b\rangle a X^1 (e_j)_1 \qquad (by\ (2.5))$$
$$= at(b) \quad q.e.d.$$

Put $h_j = t(e_j)$ for all $j = 1, ..., n$. We shall prove that

$$\sum \langle e^j, S(h_j \beta)\rangle = \varepsilon(\beta)$$

and since $\varepsilon(\beta) \neq 0$ it follows that at least one of the elements $h_j$ is non zero, so $\int \neq 0$.

Indeed, we have:

$$\sum \langle e^j, S(h_j \beta)\rangle =$$
$$= \sum \langle e^i, \beta S(\alpha X^3) S^2(X^2(e_i)_2) e_j\rangle \langle e^j, S(X^1(e_i)_1 \beta)\rangle$$
$$= \sum \langle e^i, \beta S(\alpha X^3) S^2(X^2(e_i)_2) S(X^1(e_i)_1 \beta)\rangle$$
$$= \sum \langle e^i, \beta S(\alpha X^3) S(X^1(e_i)_1 \beta S((e_i)_2) S(X^2))\rangle$$
$$= \sum \varepsilon(e_i) \langle e^i, \beta S(\alpha X^3) S(X^1 \beta S(X^2))\rangle \qquad (by\ (2.5))$$
$$= \sum \varepsilon(e_i) \langle e^i, \beta S(X^1 \beta S(X^2) \alpha X^3)\rangle$$
$$= \sum \varepsilon(e_i) \langle e^i, \beta\rangle \qquad (by\ (2.6))$$
$$= \varepsilon(\beta) \quad q.e.d.$$

**Remark 2.4.** *The proof of the proposition provides us with the k-linear map $t : A \to \int$. If we denote the inclusion of $\int$ into $A$ by $i : \int \to A$, then $t \circ i = id$.*

**Remark 2.5.** *If we denote the space of right integrals of $A$ by $\int_r$, then the fact that $S$ is an algebra anti-automorphism entails that for a left integral $\lambda$, $S(\lambda)$ is a right integral, so we also have $\int_r \neq 0$.*

**Remark 2.6.** If $H$ is a finite dimensional Hopf algebra, $\lambda \in H^*$ a left integral and $\Lambda \in H$ a right integral, then D. Radford proved that $\lambda \otimes \Lambda$ is a left and right integral for the quantum double $D(H)$, cf. [10], Th.4. Recently the quantum double has been generalized for quasi-Hopf algebras, see [7], [4], [5], as follows: if $A$ is a finite dimensional quasi-Hopf algebra, then $D(A)$ is a quasitriangular quasi-Hopf algebra having $A^* \otimes A$ as underlying linear space and $A$ is a sub quasi-Hopf algebra of $D(A)$. It would be interesting to find a relation between $\int_A$ and $\int_{D(A)}$ similar to Radford's result (such a relation was proved in [1] for the quasi-Hopf algebra $D^\omega(H)$).

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