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Evans potentials and the Riesz decomposition

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Abstract. A superharmonic function u on a parabolic Riemannian manifold M is shown to admit the Riesz decomposition $u = h + (1/c_d)\int_M e(\cdot, y)d\mu(y)$ on M into the harmonic function h on M and the Evans potential of an Evans kernel $e(x, y)$ on M and of the Borel measure $\mu := -\Delta u \geqq 0$ on M multiplied by a certain constant $1/c_d$ if and only if $m(t^2, u) - 2m(t, u) = \mathcal{O}(1)$ $(t \to +\infty)$, where $m(t, u)$ is the spherical mean over the sphere of radius t all induced by the above chosen Evans kernel $e(x, y)$ on M.

1. Introduction

Any superharmonic function u on the plane $\mathbf{R}^2$ has the positive distribution $-\Delta u \geqq 0$ on $\mathbf{R}^2$ so that $\mu := -\Delta u$ is a Borel measure on $\mathbf{R}^2$. When and only when the above μ satisfies

$$\int_{\mathbf{R}^2} \log(1 + |y|)d\mu(y) < +\infty, \tag{1.1}$$

the logarithmic potential $l(\cdot, \mu) := \int_{\mathbf{R}^2} l(\cdot, y)d\mu(y)$ is well defined and gives a special superharmonic function on $\mathbf{R}^2$, where $l(x, y)$ is the logarithmic kernel on $\mathbf{R}^2$ given by $l(x, y) = \log(1/|x - y|)$ $(x, y \in \mathbf{R}^2)$, and in this case u is said to admit the *Riesz decomposition* $u = h + (1/c_2)l(\cdot, \mu)$ on $\mathbf{R}^2$, i.e.

$$u(x) = h(x) + \frac{1}{c_2}\int_{\mathbf{R}^2} \log \frac{1}{|x - y|}\, d\mu(y) \qquad (x \in \mathbf{R}^2), \tag{1.2}$$

where h is a harmonic function on $\mathbf{R}^2$ and $c_2 = \sigma_2 = 2\pi$ and, in general, $c_d = (d - 2)\sigma_d$ $(d \geqq 3)$ with the Euclidean area σ_d of the d-dimensional Euclidean unit sphere. A few years ago Premalatha [6] gave a characterization for u to admit the Riesz decomposition on $\mathbf{R}^2$ in terms of the circle means of u as follows: u admits the Riesz decomposition (1.2) on $\mathbf{R}^2$ if and only if

$$m(t^2, u) - 2m(t, u) = \mathcal{O}(1) \qquad (t \to +\infty), \tag{1.3}$$

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where $m(t, f) = (1/c_2)\int_{|x|=t} f(x)d\theta(x)$ is the circle mean of a function f over the circle $|x| = t > 0$ with $t\,d\theta$ the arc length on the circle $|x| = t$. Recently Kitaura and Mizuta [2] generalized the Premalatha result to superbiharmonic functions on $\mathbf{R}^d$ $(d \geqq 2)$ from the view points of the interplay between the Riesz decomposition and the spherical mean. In the same paper [2] they gave another proof to the Premalatha theorem and also gave a result of the same intention as the above Premalatha result to superharmonic functions on $\mathbf{R}^d$ $(d \geqq 3)$ as well on replacing the logarithmic kernel $l(x, y)$ by Newtonian kernel $n(x, y)$ given by $n(x, y) := 1/|x - y|^{d-2}$. Specifically they prove that a superharmonic function u on $\mathbf{R}^d$ $(d \geqq 3)$ admits the Riesz decomposition $u = h + (1/c_d)n(\cdot, \mu)$ on $\mathbf{R}^d$ with a harmonic function h on $\mathbf{R}^d$ and with the Borel measure $\mu := -\Delta u \geqq 0$ if and only if $m(t^2, u) - 2^{2-d}m(t, u) = \mathcal{O}(1)$ $(t \to +\infty)$.

The purpose of this paper is to generalize the above Premalatha theorem on $\mathbf{R}^2$ to parabolic Riemannian manifolds M of dimensions not only $d = 2$ but also $d \geqq 3$. Here a Riemannian manifold M is said to be *hyperbolic* (*parabolic*, resp.) if the Green kernel $g(x, y)$ exists (does not exist, resp.) on M so that $\mathbf{R}^2$ is parabolic and all $\mathbf{R}^d$ $(d \geqq 3)$ are hyperbolic. In compensation of the nonexistence of the Green kernel the parabolicity of M is characterized by the existence of an *Evans kernel* $e(x, y)$ on M which reduces to the logarithmic kernel when $M = \mathbf{R}^2$. The precise definition of e will be given in §2 below but roughly $e(x, y)$ is a symmetric function on $M \times M$ such that $e(\cdot, y)$ is a harmonic function on $M \backslash \{y\}$ having the positive singularity at y and a negative singularity at the point ∞_M at infinity of M. The most simple examples of parabolic manifolds are given by subtracting compact subsets of capacity zero from compact manifolds, e.g., $\mathbf{R}^2$ is obtained by subtracting one point from the 2-dimensional sphere so that it is a typical parabolic manifold with its Evans kernel $e(x, y) = l(x, y)$. We will give the following theorem as the main result of this paper: a superharmonic function u on M admits the Riesz decomposition

$$u = h + \frac{1}{c_d}\int_M e(\cdot, y)d\mu(y) \tag{1.4}$$

on M, where h is a harmonic function on M and $\mu := -\Delta u \geqq 0$ is a Borel measure on M, if and only if

$$m(t^2, u) - 2m(t, u) = \mathcal{O}(1) \qquad (t \to +\infty). \tag{1.5}$$

Here $m(t, f)$ is the spherical mean of a function f over the sphere $r = t$ $(t > 0)$ induced by, what we call, the polar system (r, θ) with center $o \in M$ of radius function r and a measure θ determined by

$$\begin{cases} r = \exp(-e(\cdot, o)), \\ d\theta = -*de(\cdot, o) \end{cases} \tag{1.6}$$

so that the mean $m(t,u)$ over the sphere $C_t := \{x \in M : r(x) = t\}$ is given by

$$m(t,u) := \frac{1}{c_d}\int_{C_t} u(x)d\theta(x). \tag{1.7}$$

2. Preliminaries and the main result

Let M be an orientable and connected manifold of class C^∞ whose dimension $d \geqq 2$. The local coordinate of a point $x \in M$ is denoted by $(x^1, x^2, \dots, x^d)$. Following the convention of the tensor analysis we use the Einstein convention: whenever an index $i \in \{1, \dots, d\}$ appears both in the upper and lower position, it is understood that summation for $i = 1, \dots, d$ is carried out (cf. e.g. [8]). To make M a Riemannian manifold we give the metric squared by $a_{ij}\, dx^i dx^j$ on M by a C^∞ covariant tensor $(a_{ij}(x))$ of order 2 which is strictly positive definite symmetric matrix at each point $x \in M$. The induced volume is denoted by λ, i.e. $d\lambda(x) := \sqrt{a(x)}\, dx^1 \dots dx^d$ with $a(x) := \det(a_{ij})$, and the Laplace-Beltrami operator $-\Delta = d\delta + \delta d$ is

$$-\Delta u(x) := -\frac{1}{\sqrt{a(x)}}\frac{\partial}{\partial x^i}\left(\sqrt{a(x)}a^{ij}(x)\frac{\partial u(x)}{\partial x^j}\right) \tag{2.1}$$

in terms of local coordinate $x = (x^1, \dots, x^d)$ with $(a^{ij}(x))$ which is the contravariant tensor $(a_{ij}(x))^{-1}$. We denote by $H(M)$ the class of *harmonic* functions u on M, where u are C^∞ functions satisfying $-\Delta u = 0$ on M. We also denote by ${}^dH(M)$ the subspace of $L^1_{loc}(M, \lambda)$ consisting of u with $-\Delta u = 0$ on M in the distributional sense. It is clear that $H(M) \subset {}^dH(M)$ but actually the *Weyl lemma* says that

$$H(M) = {}^dH(M) \tag{2.2}$$

in the sense that for any $u \in {}^dH(M)$ there is a $\hat{u} \in H(M)$ such that $u = \hat{u}$ λ-a.e. on M (cf. e.g. [1]).

A function u on M is *superharmonic* on M if the following three conditions are satisfied: u is lower semicontinuous on M; u is a mapping from M to $(-\infty, +\infty]$ but $u \not\equiv +\infty$ on M; u is harmonically concave in the sense that $H_u^V \leqq u$ on every small parametric ball V, where H_u^V is the Perron-Wiener-Brelot solution of the Dirichlet problem for $-\Delta u = 0$ on V with the boundary data u on ∂V. We denote by $S(M)$ the class of superharmonic functions on M and by ${}^dS(M)$ the subspace of $L^1_{loc}(M, \lambda)$ consisting of u with $-\Delta u \geqq 0$ on M in the distributional sense. As already stated in §1 let σ_d be the surface area of the Euclidean unit sphere in the d-dimensional Euclidean space $\mathbf{R}^d$ $(d \geqq 2)$ and we set $c_d = (d-2)\sigma_d$ for $d \geqq 3$ and $c_2 = \sigma_2 = 2\pi$. The *Green kernel* $g(x, y; M)$ on M is the minimal positive solution of the Poisson equation

$$-\Delta g(\cdot, y; M) = c_d \delta_y$$

on M, where δ_y is the Dirac measure supported by an arbitrarily fixed point $y \in M$ so that $g(\cdot, y; M) \in {}^d S(M)$. The Green kernel may or may not exist on M. Hereafter we assume that M is not compact. Then M is said to be *hyperbolic* (*parabolic*, resp.) if M carries (does not carry, resp.) the Green kernel. If W is a regular subregion of M, then W, as a Riemannian manifold, is hyperbolic and $g(\cdot, y; W)$ $(y \in W)$ has vanishing boundary values on ∂W. In this case we have the Poisson representation for $u \in H(W) \cap C(\overline{W})$:

$$u(y) = -\frac{1}{c_d}\int_{\partial W} u(x) * dg(x, y; M) \qquad (y \in W). \tag{2.3}$$

An important property of the class $S(M)$ is

$$S(M) \subset \mathrm{mc}(M), \tag{2.4}$$

where $\mathrm{mc}(M)$ is the class of *mean continuous* functions f on M characterized by

$$f(y) = \lim_{r \downarrow 0} \frac{1}{\lambda(B(y,r))}\int_{B(y,r)} f(x) d\lambda(x)$$

at any point $y \in M$, where $B(y,r)$ denotes the geodesic ball with radius $r > 0$ centered at y (cf. e.g. [5]). This implies that if $u_1 = u_2$ λ-a.e. on M for u_1 and u_2 in $S(M)$, then $u_1 \equiv u_2$ on M. It is not too difficult to see that $S(M) \subset {}^d S(M)$. Pick a $u \in {}^d S(M)$. Then $\mu := -\Delta u \geqq 0$ is a Borel measure on M and in particular on any parametric ball V. Hence

$$-\Delta\left(u - \frac{1}{c_d}\int_V g(\cdot, y; V) d\mu(y)\right) = \mu - \frac{1}{c_d}(c_d \mu) = 0$$

on V shows, by the Weyl lemma, there is an $h \in H(V)$ such that

$$h = u - \frac{1}{c_d}\int_V g(\cdot, y; V) d\mu(y)$$

λ-a.e. on V, or that two superharmonic functions u and $h + (1/c_d)\int_V g(\cdot, y; V) d\mu(y)$ on V coincide λ-a.e. on V and therefore they are identical everywhere on V and

$$u = h + \frac{1}{c_d}\int_V g(\cdot, y; V) d\mu(y) \in S(V).$$

Thus we deduce the *Weyl lemma* in the superharmonic version:

$$S(M) = {}^dS(M). \tag{2.5}$$

Since we use this repeatedly later, we once more here stress this fact: If u_1 and u_2 are superharmonic on M and $-\Delta(u_1 - u_2) = 0$ on M in the distributional sense, then $u_1 - u_2 =: h \in H(M)$. Viewing $u \in S(M)$ ($u \in {}^dS(M)$, resp.) as the definition for u to be superharmonic it may be impressive to call the former (the latter, resp.) the *axiomatical* (*distributional*, resp.) definition of superharmonicity, and (2.5) may be expressed as the equivalence of the axiomatic and distributional definitions of superharmonicity. Let V be a parametric ball around $y \in M$ and f be a harmonic funtion on V. We say that f has a *positive singularity* at y if $f - cg(\cdot, y; V) \in H(V)$ for a positive constant $c > 0$ but in this paper we always take $c = 1$. This definition does not depend upon V. Let W be a regular subregion of M such that $M \backslash \overline{W}$ is connected. Here regular subregions are supposed to be relatively compact. We say that a function s on $M \backslash \overline{W}$ is said to be a *positive singularity* at $\infty = \infty_M$, the point at infinity of M or in another term the Alexandroff point of M, if

$$\lim_{x \to \infty_M} s(x) = +\infty.$$

The existence of such an s is assured if and only if M is parabolic ([3], cf. also [9]). In the cited paper the existence result is only stated and proved for the case $d = 2$ but it can easily be generalised verbatim to the case $d \geqq 3$. Then we say that a harmonic function f on $M \backslash \overline{W}$ has a *negative singularity* $-s$ at ∞_M if $f + s$ is a bounded harmonic function on $M \backslash \overline{W_0}$ with some regular subregion W_0 containing $\overline{W}$. The definition does not depend upon the choice of W once s is fixed. Suppose that M is parabolic. We fix a positive singularity s throughout the paper.

An *Evans kernel* $e(\cdot\,,\cdot)$ on M is a continuous mapping from $M \times M$ to $(-\infty, +\infty]$ satisfying the following four conditions: $e(\cdot, y) \in H(M \backslash \{y\})$; $e(\cdot, y)$ has a positive singularity at y for every $y \in M$; $e(\cdot, y)$ has a fixed negative singularity $-s$ for every $y \in M$; $e(\cdot\,,\cdot)$ is symmetric, i.e. $e(x, y) = e(y, x)$ for every x and y in M ([4], cf. also [7], [8]). As remarked above, the result in the reference just cited is stated and proved for the case of dimension $d = 2$ but again its generalization to any dimension $d \geqq 3$ is easy and straightforward. Let V be a parametric ball about a point $y \in M$. Then, since $e(\cdot, y) - g(\cdot, y; V) \in H(V)$, its flux across ∂V is zero and hence by (2.3) we have

$$\int_{\partial V} *de(\cdot, y) = \int_{\partial V} *dg(\cdot, y; V) = -c_d\left(\frac{1}{c_d}\int_{\partial V} 1 * dg(\cdot, y; V)\right) = -c_d 1$$

so that for any $y \in M$ we have

$$\int_{\partial V} *de(\cdot, y) = -c_d. \tag{2.6}$$

For a parabolic Riemannian manifold M we fix a point $o \in M$ and an Evans kernel $e(x, y)$ on M. A *polar system* $(r, d\theta)$ of *center* o on M induced by $e(x, y)$ is the pair $(r, d\theta)$ of the function r, called the *radius function*, given by (1.6): $r = \exp(-e(\cdot, o))$ and Borel measures $d\theta$ on every "sphere"

$$C_t := \{x \in M : r(x) = t\} \qquad (t > 0)$$

given also by (1.6): $d\theta = - * de(\cdot, o)$. In view of (2.6) we have

$$\int_{C_t} d\theta = c_d \qquad (t > 0). \tag{2.7}$$

The "spherical mean" $m(t, f)$ of a function f on C_t over the "sphere" C_t $(t > 0)$ is given by

$$m(t, f) = \frac{1}{c_d} \int_{C_t} f(x) d\theta. \tag{2.8}$$

We denote by B_t the "ball" $B_t := \{x \in M : r(x) < t\}$ $(t > 0)$ so that $C_t = \partial B_t$. Since $y \mapsto \int_{C_t} e(x, y) d\lambda(x)$ is a finitely continuous function on M for any $t > 0$, the Fubini theorem assures that, for any Borel measure ν on M,

$$\int_{B_\tau} \left(\int_{B_t} |e(x, y)| d\nu(y) \right) d\lambda(x) = \int_{B_t} \left(\int_{B_\tau} |e(x, y)| d\lambda(x) \right) d\nu(y) < +\infty$$

for every $\tau > 0$ and therefore $\int_{B_t} |e(x, y)| d\nu(y) < +\infty$ λ-a.e. $x \in M$ for every $t > 0$. Observe that there is a constant $K > 1$ such that

$$K^{-1} \int_{M \setminus B_{2t}} |e(o, y)| d\nu(y) \leqq \int_{M \setminus B_{2t}} |e(x, y)| d\nu(y) \leqq K \int_{M \setminus B_{2t}} |e(o, y)| d\nu(y)$$

for every $x \in B_t$. Thus $\int_M |e(\cdot, y)| d\nu(y) < +\infty$ λ-a.e. on M if and only if $\int_{M \setminus B_t} |e(o, y)| d\nu(y) < +\infty$ for some $t > 0$. Clearly $x \mapsto \int_M e(x, y) d\nu(y)$ defines a superharmonic function on M if and only if $\int_M |e(\cdot, y)| d\nu(y) < +\infty$ λ-a.e. on M. Hence we can say the following:

PROPOSITION 2.1. *The function $\int_M e(\cdot, y) d\nu(y)$ is superharmonic on M if and only if*

$$\int_M \log(1 + r(y)) d\nu(y) < +\infty. \tag{2.9}$$

We call $\int_M e(\cdot, y)dv(y)$ an *Evans potential* with the measure v. If we write the notation $\int_M e(\cdot, y)dv(y)$, then it is always understood that (2.9) is satisfied so that $\int_M e(\cdot, y)dv(y)$ is an Evans potential on M. The purpose of this paper is to prove the following result.

THEOREM. *A superharmonic function u on M admits the Riesz decomposition*

$$u(x) = h(x) + \frac{1}{c_d}\int_M e(x, y)d\mu(y) \qquad (x \in M), \tag{2.10}$$

where h is a harmonic function on M and $\mu := -\Delta u \geqq 0$ is a Borel measure on M, if and only if

$$m(t^2, u) - 2m(t, u) = \mathcal{O}(1) \qquad (t \to \infty). \tag{2.11}$$

3. The mean of Evans kernel

Before proceeding to the proof of the theorem we derive the following formula which is a well known elementary knowledge for the case of the logarithmic kernel on $\mathbf{R}^2$. Namely, we prove the following formula.

PROPOSITION 3.1. *For any point $y \in M$ and any positive number $t \in (0, +\infty)$ it holds that*

$$m(t, e(\cdot, y)) = -\max\{\log r(y), \log t\}. \tag{3.1}$$

PROOF. Let $\beta_\varepsilon := \{x \in M : e(x, y) > 1/\varepsilon\}$ and $\gamma_\varepsilon = \partial\beta_\varepsilon$ $(\varepsilon > 0)$. For any $u \in H(\beta_\varepsilon) \cap C(\overline{\beta_\varepsilon})$ we have the Poisson representation (2.3):

$$u(y) = -\frac{1}{c_d}\int_{\gamma_\varepsilon} u * dg(\cdot, y; \beta_\varepsilon).$$

Since $e(\cdot, y) = g(\cdot, y; \beta_\varepsilon) + 1/\varepsilon$, we see that $*de(\cdot, y) = *dg(\cdot, y; \beta_\varepsilon)$ and therefore, by the above, we have

$$\int_{\gamma_\varepsilon} u * de(\cdot, y) = -c_d u(y). \tag{3.2}$$

In particular we once more see (cf. (2.6)) on taking $u = 1$ in (3.2) that

$$\int_{\gamma_\varepsilon} *de(\cdot, y) = -c_d. \tag{3.3}$$

To prove (3.1) we first consider the case $r(y) > t$. Then $e(\cdot, y) \in H(B_t) \cap C(\overline{B_t})$ and we can apply (3.2) to deduce that

$$m(t, e(\cdot, y)) = -\frac{1}{c_d}\int_{C_t} e(\cdot, y) * de(\cdot, o) = e(o, y) = e(y, o) = -\log r(y).$$

Hence (3.1) is established for the case $r(y) > t$.

Next we consider the case $r(y) < t$. We choose $\varepsilon > 0$ so small as to satisfy $\overline{\beta_\varepsilon} \subset B_t \backslash \overline{B_\varepsilon}$ so that $e(\cdot, o) \in H(\beta_\varepsilon) \cap C(\overline{\beta_\varepsilon})$. Hence by (3.2) we have

$$\int_{\gamma_\varepsilon} e(\cdot, o) * de(\cdot, y) = -c_d e(y, o).$$

We also have

$$\int_{\gamma_\varepsilon} e(\cdot, y) * de(\cdot, o) = \frac{1}{\varepsilon}\int_{\gamma_\varepsilon} *de(\cdot, o) = 0.$$

From the above two displayed identities it follows that

$$\int_{\gamma_\varepsilon} (e(\cdot, y) * de(\cdot, o) - e(\cdot, o) * de(\cdot, y)) = c_d e(y, o). \tag{3.4}$$

Similarly, in view of $e(\cdot, y) \in H(B_\varepsilon) \cap C(\overline{B_\varepsilon})$, we have

$$\int_{C_\varepsilon} e(\cdot, y) * de(\cdot, o) = -c_d e(o, y) = -c_d e(y, o)$$

and we also have

$$\int_{C_\varepsilon} e(\cdot, o) * de(\cdot, y) = (-\log \varepsilon)\int_{C_\varepsilon} *de(\cdot, y) = 0.$$

From the above two displayed identities it follows that

$$\int_{C_\varepsilon} (e(\cdot, y) * de(\cdot, o) - e(\cdot, o) * de(\cdot, y)) = -c_d e(y, o). \tag{3.5}$$

By the definition (2.8) we have

$$\int_{C_t} e(\cdot, y) * de(\cdot, o) = -c_d m(t, e(\cdot, y)).$$

Since $e(\cdot, y) \in H(B_t \backslash \overline{\beta_\varepsilon}) \cap C(\overline{B_t} \backslash \beta_\varepsilon)$, we have $\int_{C_t - \gamma_\varepsilon} *de(\cdot, y) = 0$ and thus by using (3.3)

$$\begin{aligned}\int_{C_t} e(\cdot, o) * de(\cdot, y) &= (-\log t)\int_{C_t} *de(\cdot, y) \\ &= (-\log t)\int_{\gamma_\varepsilon} *de(\cdot, y) = c_d \log t.\end{aligned}$$

Therefore we can conclude that

$$\int_{C_t}(e(\cdot,y)*de(\cdot,o)-e(\cdot,o)*de(\cdot,y))=-c_d m(t,e(\cdot,y))-c_d\log t. \tag{3.6}$$

Put $D:=B_t\backslash(\overline{B_\varepsilon}\cup\overline{\beta_\varepsilon})$. Then both of $e(\cdot,y)$ and $e(\cdot,o)$ are harmonic on $\bar{D}$ and by applying Stokes formula we see that

$$\begin{aligned}
&\int_{C_t-C_\varepsilon-\gamma_\varepsilon}(e(\cdot,y)*de(\cdot,o)-e(\cdot,o)*de(\cdot,y))\\
&\quad=\int_{\partial D}(e(\cdot,y)*de(\cdot,o)-e(\cdot,o)*de(\cdot,y))\\
&\quad=\int_{D}d(e(\cdot,y)*de(\cdot,o)-e(\cdot,o)*de(\cdot,y))\\
&\quad=\int_{D}(e(\cdot,y)\Delta e(\cdot,o)-e(\cdot,o)\Delta e(\cdot,y))=0.
\end{aligned}$$

Hence by replacing each of $\int_{C_t}$, $\int_{C_\varepsilon}$, and $\int_{\gamma_\varepsilon}$ of $e(\cdot,y)*de(\cdot,o)-e(\cdot,o)*de(\cdot,y)$ by each of (3.6), (3.5), and (3.4), respectively, we conclude that

$$-(c_d m(t,e(\cdot,y)+c_d\log t)-(-c_d e(\cdot,y))-c_d e(\cdot,y)=0,$$

or, $m(t,e(\cdot,y))=-\log t$ and a fortiori (3.1) is also valid for the case $r(y)<t$.

Finally we treat the case $r(y)=t$ or equivalently $y\in C_t$. Before proceeding to the proof of (3.1): $m(t,e(\cdot,y))=-\log r(y)=-\log t$ in this case, we recall the following well known general result. Consider a regular subregion $\Omega\subset M$ and fix a point y in the boundary $\partial\Omega$ of Ω. Let u be a nonnegative harmoic function on Ω such that u has vanishing boundary values on $\partial\Omega\backslash\{y\}$. Then either $u\equiv 0$ on Ω or u is a minimal positive harmonic function (i.e. a constant multiple of the Martin kernel associated with y). In this situation the former (i.e. $u\equiv 0$ on Ω) is the case if and only if

$$u(x)=\mathcal{O}(g(x,y;\Omega'))\qquad(x\in\bar{\Omega}\backslash\{y\},x\to y) \tag{3.7}$$

for some regular region Ω' containing y. There are many ways considered to see this but one simple way is to use the fact that surface measures on regular surfaces are measures of Kato class. Now we return to the proof of (3.1) when $r(y)=t$. To make notations simple let $v:=e(\cdot,y)$ with $y\in C_t$ and $v_n=\min\{v,n\}$ for $n=1,2,\dots$. We denote by $h_n\in H(B_t)\cap C(\overline{B_t})$ such that $h_n|C_t=v_n|C_t$. Then

$$h_1\leqq h_2\leqq\cdots\leqq h_n\leqq h_{n+1}\leqq\cdots\leqq v$$

on $\overline{B_t}$. Thus $u := v - \lim_{n\to\infty} h_n \geqq 0$ and belongs to $H(B_t) \cap C(\overline{B_t})$ with $u|C_t\backslash\{y\} = 0$. Since

$$0 \leqq u = v - \lim_{n\to\infty} h_n \leqq v - h_1 \leqq g(\cdot, y; B_{2t}) + \sup_{C_{2t}} v + h_1$$

on $B_t\backslash\{y\}$, we see that

$$u(x) = \mathcal{O}(g(x, y; B_{2t})) \qquad (x \in \overline{B_t}\backslash\{y\}, x \to y).$$

Then, by the above remark related to (3.7), we have $u(x) \equiv 0$ on $\overline{B_t}$ so that $v = \lim_{n\to\infty} h_n$ and in particular

$$e(o, y) = \lim_{n\to\infty} h_n(o). \tag{3.8}$$

By (3.2) we see that

$$\int_{C_t} h_n * de(\cdot, o) = -c_d h_n(o) \qquad (n = 1, 2, \dots).$$

By the Fatou lemma we deduce on making $n \uparrow \infty$ in the above that

$$\int_{C_t} e(\cdot, y) * de(\cdot, o) = -c_d e(o, y)$$

and a fortiori $m(t, e(\cdot, y)) = e(o, y) = e(y, o) = -\log r(y) = -\log t$.

4. Proof of the theorem

We now prove our main theorem of this paper stated at the end of §2. In addition to the Evans kernel $e(x, y)$ fixed in advance throughout this paper we also consider the kernel $e_0(x, y)$ given by

$$e_0(x, y) := \begin{cases} e(x, y) - e(o, y) & (r(y) \geqq 1), \\ e(x, y) & (r(y) < 1). \end{cases}$$

Observe that $-\Delta e(\cdot, y) = c_d \delta_y$, with the Dirac measure δ_y supported at $\{y\}$, so that $-\Delta e_0(\cdot, y) = c_d \delta_y$ too. Fix a $u \in S(M)$ and we denote by $\mu := -\Delta u \geqq 0$ its associated Borel measure on M. For any $t > 0$ we set $B_t := \{x \in M : r(x) < t\}$ and $C_t := \partial B_t$ as before. Then we have

$$u(x) = \frac{1}{c_d} \int_{B_t} e_0(x, y) d\mu(y) + h_t(x) \qquad (x \in B_t) \tag{4.1}$$

for every $t > 0$, where $h_t \in H(B_t)$ depends upon the choice of $t > 0$. To see (4.1), we set

$$v(x) := u(x) - \frac{1}{c_d}\int_{B_t} e_0(x,y)\,d\mu(y)$$

on B_t. Then

$$-\Delta v = -\Delta u - \left(-\Delta\left(\frac{1}{c_d}\int_{B_t} e_0(x,y)\,d\mu(y)\right)\right) = \mu - \frac{1}{c_d}(c_d\mu) = 0$$

shows, by (2.2), that v is harmonic on B_t. Since it depends upon t, we denote it by $h_t \in H(B_t)$ and thus (4.1) is deduced. We observe here that there is a constant a independent of $t \geqq 1$ such that

$$h_t(o) = a \qquad (t \geqq 1), \tag{4.2}$$

i.e. $h_t(o)$ does not depend on $t \geqq 1$ although the function h_t itself does. This can be seen as follows. Observe that

$$\begin{aligned} h_t(x) &= u(x) - \frac{1}{c_d}\int_{B_t} e_0(x,y)\,d\mu(y) \\ &= \left(u(x) - \frac{1}{c_d}\int_{B_1} e_0(x,y)\,d\mu(y)\right) - \frac{1}{c_d}\int_{B_t\setminus B_1} e_0(x,y)\,d\mu(y) \\ &= h_1(x) - \frac{1}{c_d}\int_{B_t\setminus B_1}(e(x,y) - e(o,y))\,d\mu(y) \end{aligned}$$

for any $t \geqq 1$ and for any $x \in B_1$. In view of the continuity of $e(x,y)$ for $(x,y) \in B_1 \times (B_t\setminus B_1)$, on taking the limit as $x \to o$ in

$$h_t(x) - h_1(x) = -\frac{1}{c_d}\int_{B_t\setminus B_1}(e(x,y) - e(o,y))\,d\mu(y),$$

we deduce that $h_t(o) - h_1(o) = 0$ so that $h_t(o) = h_1(o)$ does not depend upon $t \geqq 1$. Hence we have established (4.2) and thus we see that

$$m(t,u) = \frac{1}{c_d}\int_{B_t} m(t, e_0(\cdot,y))\,d\mu(y) + a \qquad (t \geqq 1) \tag{4.3}$$

for a constant a independent of $t \geqq 1$. In fact, on taking $m(t,\cdot)$ of both sides of (4.1) and using the Fubini theorem we obtain

$$m(t,u) = \frac{1}{c_d}\int_{B_t} m(t, e_0(\cdot,y))\,d\mu(y) + m(t,h_t) \qquad (t \geqq 1).$$

By the Gauss mean value theorem (cf. (3.2)), $m(t,h_t) = h_t(o)$ and hence (4.2) assures that $m(t,h_t) = a$ is a constant independent of $t \geqq 1$, which proves (4.3) as desired.

First we assume (2.11): $m(t^2, u) - 2m(t, u) = \mathcal{O}(1)$ $(t \to \infty)$ and we will prove (2.10): the Riesz decomposition of u. Now we maintain the validity of

$$m(t, u) = \frac{1}{c_d}\left(\log \frac{1}{t}\right)\mu(B_t) + \frac{1}{c_d}\int_{B_t \setminus B_1} \log \frac{r(y)}{t}\, d\mu(y) + a, \tag{4.4}$$

for $t \geqq 1$. In fact, we divide the integration over B_t on the right hand side of (4.3) into those over B_1 and $B_t \setminus B_1$ so that

$$m(t, u) = \frac{1}{c_d}\int_{B_1} m(t, e_0(\cdot, y))d\mu(y) + \frac{1}{c_d}\int_{B_t \setminus B_1} m(t, e_0(\cdot, y))d\mu(y) + a.$$

By (3.1), we see that

$$m(t, e_0(\cdot, y)) = m(t, e(\cdot, y)) = \log \frac{1}{t} \qquad (y \in B_1)$$

and also we see that

$$m(t, e_0(\cdot, y)) = m(t, e(\cdot, y)) - m(t, e(o, y)) = \log \frac{1}{t} - \log \frac{1}{r(y)} \qquad (y \in B_t \setminus B_1).$$

Thus we infer that

$$\begin{aligned} m(t, u) &= \frac{1}{c_d}\int_{B_1} \log \frac{1}{t}\, d\mu(y) + \frac{1}{c_d}\int_{B_t \setminus B_1} \log \frac{r(y)}{t}\, d\mu(y) + a \\ &= \frac{1}{c_d}\left(\log \frac{1}{t}\right)\mu(B_1) + \frac{1}{c_d}\int_{B_t \setminus B_1} \log \frac{r(y)}{t}\, d\mu(y) + a, \end{aligned}$$

i.e. (4.4) is deduced. Replacing t by t^2 in (4.4) we obtain

$$m(t^2, u) = \frac{2}{c_d}\left(\log \frac{1}{t}\right)\mu(B_1) + \frac{1}{c_d}\int_{B_{t^2} \setminus B_1} \log \frac{r(y)}{t^2}\, d\mu(y) + a.$$

Subtracting sides by sides from the above, the following

$$2m(t, u) = \frac{2}{c_d}\left(\log \frac{1}{t}\right)\mu(B_1) + \frac{1}{c_d}\int_{B_t \setminus B_1} \log \frac{r(y)^2}{t^2}\, d\mu(y) + 2a$$

implies, by considering $B_{t^2} \setminus B_1 = (B_{t^2} \setminus B_t) \cup (B_t \setminus B_1)$, that $m(t^2, u) - 2m(t, u)$ is

$$\begin{aligned} &\left(\frac{1}{c_d}\int_{B_{t^2} \setminus B_t} \log \frac{r(y)}{t^2}\, d\mu(y) + \frac{1}{c_d}\int_{B_t \setminus B_1}\left(-\log r(y) + \log \frac{r(y)^2}{t^2}\right)d\mu(y)\right) \\ &\quad - \frac{1}{c_d}\int_{B_t \setminus B_1} \log \frac{r(y)^2}{t^2}\, d\mu(y) - a. \end{aligned}$$

Therefore finally we see that

$$m(t^2,u) - 2m(t,u) = \frac{1}{c_d}\int_{B_{t^2}\setminus B_t} \log\frac{r(y)}{t^2}\,d\mu(y) + \frac{1}{c_d}\int_{B_t\setminus B_1} \log\frac{1}{r(y)}\,d\mu(y) - a.$$

Since both of the first and the second terms on the right hand side of the above are negative, the assumption $m(t^2,u) - 2m(t,u) = \mathcal{O}(1)$ $(t \to \infty)$ assures that

$$\int_{B_t\setminus B_1} \log r(y)d\mu(y) = \mathcal{O}(1) \qquad (t \to \infty),$$

which in turn implies (2.9) and, by Proposition 2.1, $\int_M e(\cdot, y)d\mu(y)$ is superharmonic on M. Hence both of u and $\int_M e(\cdot, y)d\mu(y)$ belong to $S(M)$ and

$$-\Delta\left(u - \frac{1}{c_d}\int_M e(\cdot, y)d\mu(y)\right) = \mu - \frac{1}{c_d}(c_d\mu) = 0.$$

Then by (2.2) we can conclude that

$$u - \frac{1}{c_d}\int_M e(\cdot, y)d\mu(y) =: h \in H(M),$$

which shows that (2.10) is assured.

Conversely, we derive (2.11): $m(t^2,u) - 2m(t,u) = \mathcal{O}(1)$ $(t \to \infty)$ by assuming (2.10). Namely, we will prove that if

$$u(x) = \frac{1}{c_d}\int_M e(\cdot, y)d\mu(y) + h(x) \qquad (x \in M), \tag{4.5}$$

where $\mu := -\Delta \geqq 0$ and we understand the first term on the right hand side of the above is superharmonic on M or equivalently $\int_M \log(1 + r(y))d\mu(y) < +\infty$ (cf. (2.9)) and $h \in H(M)$, then it holds that $m(t^2,u) - 2m(t,u) = \mathcal{O}(1)$ $(t \to \infty)$. i.e. the condition (2.11). Actually we will prove a bit more:

$$\lim_{t\to\infty}(m(t^2,u) - 2m(t,u)) = -h(o). \tag{4.6}$$

Hence the condition (2.11) is equivalent to the above (4.6). For any $t > 0$ we divide M into B_t and $M\setminus B_t$ in (4.5):

$$u = \frac{1}{c_d}\int_{B_t} e(\cdot, y)d\mu(y) + \frac{1}{c_d}\int_{M\setminus B_t} e(\cdot, y)d\mu(y) + h$$

on M. Integrate both sides of the above with respect to $d\theta$ over C_t and apply the Fubini theorem. Then we obtain

$$m(t,u) = \frac{1}{c_d}\int_{B_t} m(t, e(\cdot, y))d\mu(y) + \frac{1}{c_d}\int_{M\setminus B_t} m(t, e(\cdot, y))d\mu(y) + m(t,h).$$

By (3.1) and the Gauss mean value theorem (cf. (2.3)) we deduce

$$m(t,u) = \frac{1}{c_d}\left(\log\frac{1}{t}\right)\mu(B_t) + \frac{1}{c_d}\int_{M\setminus B_t}\log\frac{1}{r(y)}\,d\mu(y) + h(o). \tag{4.7}$$

In particular, choosing $t > 1$ arbitrarily, the above (4.7) gives

$$m(t^2,u) = \frac{2}{c_d}\left(\log\frac{1}{t}\right)\mu(B_{t^2}) + \frac{1}{c_d}\int_{M\setminus B_{t^2}}\log\frac{1}{r(y)}\,d\mu(y) + h(o).$$

On dividing $M\setminus B_t$ into $M\setminus B_{t^2}$ and $B_{t^2}\setminus B_t$ in (4.7), we see that

$$\begin{aligned}2m(t,u) = {} & \frac{2}{c_d}\left(\log\frac{1}{t}\right)\mu(B_t) + \frac{2}{c_d}\int_{M\setminus B_{t^2}}\log\frac{1}{r(y)}\,d\mu(y)\\ & + \frac{2}{c_d}\int_{B_{t^2}\setminus B_t}\log\frac{1}{r(y)}\,d\mu(y) + 2h(o).\end{aligned}$$

From the above two displayed identities it follows that

$$\begin{aligned}m(t^2,u) - 2m(t,u) = {} & \frac{2}{c_d}\left(\log\frac{1}{t}\right)\mu(B_{t^2}\setminus B_t) - \frac{1}{c_d}\int_{M\setminus B_{t^2}}\log\frac{1}{r(y)}\,d\mu(y)\\ & + \frac{2}{c_d}\int_{B_{t^2}\setminus B_t}\log r(y)d\mu(y) - h(o).\end{aligned}$$

Putting the first and the third terms on the right hand side of the above identity together we deduce

$$\begin{aligned}& m(t^2,u) - 2m(t,u) - (-h(o))\\ & \quad = \frac{1}{c_d}\int_{M\setminus B_{t^2}}\log r(y)d\mu(y) + \frac{2}{c_d}\int_{B_{t^2}\setminus B_t}\log\frac{r(y)}{t}\,d\mu(y).\end{aligned}$$

Since $r(y)/t < r(y)$ by $t > 1$, we see that $1 < r(y)/t < r(y)$ on $(M\setminus B_{t^2})\cup(B_{t^2}\setminus B_t) \subset M\setminus B_t$ and a fortiori

$$\begin{aligned}& |m(t^2,u) - 2m(t,u) - (-h(o))|\\ & \quad \leqq \frac{1}{c_d}\int_{M\setminus B_{t^2}}\log r(y)d\mu(y) + \frac{2}{c_d}\int_{B_{t^2}\setminus B_t}\log\frac{r(y)}{t}\,d\mu(y)\\ & \quad \leqq \frac{3}{c_d}\int_{M\setminus B_t}\log r(y)d\mu(y) \leqq \frac{3}{c_d}\int_{M\setminus B_t}\log(1+r(y))d\mu(y).\end{aligned}$$

Here $\int_M \log(1+r(y))d\mu(y) < +\infty$ implies $\int_{M\setminus B_t}\log(1+r(y))d\mu(y) \to 0$ $(t\to\infty)$ and thus the above displayed inequalities imply the limit (4.6).

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