HIROSHIMA MATH. J.
1 (1971), 253-271

K_Λ-Rings of Lens Spaces $L^n(4)$

Teiichi KOBAYASHI and Masahiro SUGAWARA

(Received September 20, 1971)

§1. Introduction

Let $L^n(k)=L^n(k; 1,\cdots, 1)$ be the $(2n+1)$-dimensional standard lens space mod k, where n and k are positive integers and $k\geqq 2$. Denote by Λ the field R of the real numbers or C of the complex numbers. The structure of K_Λ-rings of $L^n(k)$ is determined by J. F. Adams [1] when $k=2$ ($L^n(2)$ is the real projective space), and by T. Kambe [5] when k is an odd prime.

The purpose of this note is to determine the structure of $\tilde{K}_\Lambda(L^n(k))$ for the case $k=4$. We use K or KO instead of K_C or K_R.

Let η be the canonical complex line bundle over $L^n(k)$, and set

$$\sigma=\eta-1 \in \tilde{K}(L^n(k)).$$

Then, we have the following theorem[1]:

THEOREM A. (4.6)

$$\tilde{K}(L^n(4))\cong Z_{2^{n+1}}\oplus Z_{2^{[n/2]}}\oplus Z_{2^{[(n-1)/2]}},$$

and the direct summands are generated by the three elements

$$\sigma,\quad \sigma^2+2\sigma,\quad \sigma^3+2\sigma^2+2^{n/2+1}\sigma \qquad (\textit{if } n \textit{ is even}),$$

$$\sigma,\quad \sigma^2+2\sigma+2^{[n/2]+1}\sigma,\quad \sigma^3+2\sigma^2 \qquad (\textit{if } n \textit{ is odd}),$$

respectively. The multiplicative structure is given by

$$\sigma^4=-4\sigma^3-6\sigma^2-4\sigma,\quad \sigma^{n+1}=0.$$

Let ρ be the non-trivial (real) line bundle over $L^n(4)$ and set $\kappa=\rho-1 \in \widetilde{KO}(L^n(4))$. Let $r\sigma \in \widetilde{KO}(L^n(4))$ denote the real restriction of σ.

THEOREM B. (5.3, 5.6, 5.13, 5.18, 6.1, 6.7)

---

1) According to N. Mahammed [8], it is announced that

$$K(L^n(k))\cong Z[\eta]/<(\eta-1)^{n+1}, \eta^k-1>$$

for any k.

$$\widetilde{KO}(L^n(4)) \cong \begin{cases} Z_{2^{n+1}} \oplus Z_{2^{n/2}} & \textit{for even } n>0, \\ Z_{2^n} \oplus Z_{2^{[n/2]+1}} & \textit{for } n\equiv 1 \bmod 4, \\ Z_{2^n} \oplus Z_{2^{[n/2]}} & \textit{for } n\equiv 3 \bmod 4, \end{cases}$$

and the first summand is generated by $r\sigma$ *and the second by* $\kappa+2^{[n/2]}r\sigma$, *where it is able to replace the last element by* κ *if* $n\equiv 1 \bmod 4$.

The multiplicative structure in $\widetilde{KO}(L^n(4))$ *is given by*

$$(r\sigma)^2=-4r\sigma+2\kappa, \quad \begin{cases} (r\sigma)^{[n/2]+1}=0 & \textit{if } n\not\equiv 1 \bmod 4, \\ (r\sigma)^{[n/2]+2}=0 & \textit{if } n\equiv 1 \bmod 4; \end{cases}$$

$$\kappa^2=\kappa\cdot r\sigma=-2\kappa, \qquad \kappa^{[n/2]+2}=0.$$

We can calculate the order of $(r\sigma)^i$ by the above theorems, and apply the γ^i-operation to the problem of the immersion and the embedding of $L^n(4)$ in Euclidean space by making use of the method of M. F. Atiyah (cf. [2] and [5]).

THEOREM C. *$L^n(4)$ cannot be immersed in $R^{2n+2L(n,4)}$, and $L^n(4)$ cannot be embedded in $R^{2n+2L(n,4)+1}$, where*

$$L(n,4)=\begin{cases} \max\left\{i \,\middle|\, 1\leqq i\leqq \left[\frac{n}{2}\right],\ \binom{n+i}{i}\not\equiv 0 \bmod 2^{n-2i+2}\right\} & \textit{if } n\equiv 1 \bmod 2, \\ \max\left\{i \,\middle|\, 1\leqq i\leqq \left[\frac{n}{2}\right],\ \binom{n+i}{i}\not\equiv 0 \bmod 2^{n-2i+3}\right\} & \textit{if } n\equiv 0 \bmod 2. \end{cases}$$

In §2, we recall the cohomology groups of $L^n(k)$. In §3, we consider the element $\sigma(1)=\sigma^2+2\sigma=\eta^2-1 \in \tilde{K}(L^n(4))$, and establish the following formulas:

$$cr\sigma=2\sigma+\sigma(1)+\sigma(1)\sigma, \qquad c\kappa=\sigma(1),$$

where c: $\widetilde{KO}(L^n(4))\to \tilde{K}(L^n(4))$ is the complexification (Lemmas 3.10–11). Theorem A is proved in §4 by means of the relations:

$$(\sigma+1)^4=1, \qquad \sigma^{n+1}=0,$$

and by using the Atiyah-Hirzebruch spectral sequences (cf. [3]). Moreover, we verify that the elements σ^i and $\sigma(1)^i\sigma^j (i\geqq 1)$ in $\tilde{K}(L^n(4))$ are of order 2^{2+n-i} and $2^{1+[(n+1-2i-j)/2]}$ respectively (Cor. 4.7, Th. 4.8).

The proofs of Theorem B are carried out in §§5–6. The additive structure of $\widetilde{KO}(L^n(4))$ is determined in §5, by making use of the complexification c and Theorem A. The multiplicative structure of $\widetilde{KO}(L^n(4))$ is determined in §6. In the final section, we give the proof of Theorem C and discuss the immersion problem for $L^n(k)$.

The K_Λ-rings of $L^n(p^2)$, for p an odd prime, will be considered in a forthcoming paper [6].

§2. Cohomology groups of $L^n(k)$

Let S^{2n+1} be the unit $(2n+1)$-sphere in the complex $(n+1)$-space C^{n+1}, and γ be the rotation of S^{2n+1} given by

$$\gamma(z_0, z_1, \cdots, z_n)=(e^{2\pi i/k}z_0, e^{2\pi i/k}z_1, \cdots, e^{2\pi i/k}z_n).$$

Then γ generates the topological transformation group Z_k of S^{2n+1}, and the standard lens space mod k is

$$L^n(k)=S^{2n+1}/Z_k.$$

As is well-known, $L^n(k)$ has a cell structure

$$L^n(k)=e^0\cup e^1\cup\cdots\cup e^{2n}\cup e^{2n+1} \tag{2.1}$$

and its cohomology groups are given by

$$H^i(L^n(k); Z)\cong\begin{cases} Z_k & \text{for } i=2, 4, \cdots, 2n \\ Z & \text{for } i=0, 2n+1 \\ 0 & \text{otherwise,}\end{cases}$$

$$H^i(L^n(2l); Z_2)\cong Z_2 \qquad \text{for } 0\leqq i\leqq 2n+1.$$

Let Δ: $H^1(L^n(k); Z_2)\to H^2(L^n(k); Z)$ be the Bockstein homomorphism associated with the coefficient sequence: $0\to Z\to Z\to Z_2\to 0$. If $k=2l$, we have the following lemma easily.

LEMMA 2.2. $$\Delta x=l y,$$

where x and y are generators of $H^1(L^n(2l); Z_2)\cong Z_2$ and $H^2(L^n(2l); Z)\cong Z_{2l}$, respectively.

Let P be a single point, then it is well-known that K- and KO-groups of P are given by

$$K^{-p}(P)\cong Z\ (p \text{ even}), \quad \cong 0\ (p \text{ odd});$$

$$KO^{-p}(P)\cong Z\ (p\equiv 0, 4 \bmod 8), \quad \cong Z_2\ (p\equiv 1, 2 \bmod 8),$$

$$\cong 0\ (\text{otherwise}).$$

Let CP^n be the n-dimensional complex projective space, and

$$\pi: L^n(k) \rightarrow CP^n = S^{2n+1}/S^1$$

be the natural projection. Then

LEMMA 2.3. *$\pi^*: \tilde{H}^p(CP^n; K^{-p}(P)) \rightarrow \tilde{H}^p(L^n(k); K^{-p}(P))$ is an epimorphism.*

PROOF. It is trivial for odd p. For even p, the result follows from the Gysin exact sequence. q. e. d.

The following are easy.

$$(2.4) \qquad \tilde{H}^p(L^n(k); K^{-q}(P)) \cong \begin{cases} Z_k & \text{for } p \text{ and } q \text{ even, } 0 < p \leqq 2n, \\ 0 & \text{otherwise,} \end{cases}$$

$$(2.5) \qquad \tilde{H}^p(L^n(2l); KO^{-p}(P)) \cong \begin{cases} Z_{2l} & \text{for } p \equiv 0, 4 \bmod 8,\ 0 < p \leqq 2n, \\ Z_2 & \text{for } p \equiv 1, 2 \bmod 8,\ 0 < p \leqq 2n+1, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 2.6. *The induced homomorphism*

$$i^*: \tilde{H}^p(L^{n+1}(2l); KO^{-p}(P)) \rightarrow \tilde{H}^p(L^n(2l); KO^{-p}(P))$$

by the inclusion i is an epimorphism for any p.

Consider the $2n$-skeleton

$$(2.7) \qquad L_0^n(k) = e^0 \cup e^1 \cup \cdots \cup e^{2n}$$

of the CW-complex $L^n(k)$ of (2.1). Then

$$(2.8) \qquad L^n(k)/L_0^n(k) = S^{2n+1}, \quad L_0^{n+1}(k)/L_0^n(k) = S^{2n+1} \cup_k e^{2n+2},$$

where the attaching map k: $S^{2n+1} \rightarrow S^{2n+1}$ means the map of degree k.

Also $\tilde{H}^i(L_0^n(k); Z) \cong Z_k$ for $i = 2, 4, \cdots, 2n$ and $\cong 0$ otherwise, and $\tilde{H}^i(L_0^n(2l); Z_2) \cong Z_2$ for $0 < i \leqq 2n$. Furthermore, we have

$$(2.9) \qquad \tilde{H}^p(L_0^n(2l); KO^{-p}(P)) \cong \tilde{H}^p(L^n(2l); KO^{-p}(P))$$

for $n \not\equiv 0 \bmod 4$.

§3. Non-trivial line bundle over $L^n(k)$

Let ρ be the non-trivial line bundle over $L^n(2l)$, i. e., ρ is the line bundle such that the first Stiefel-Whitney class $w_1(\rho) \in H^1(L^n(2l); Z_2) \cong Z_2$ is non-zero.

LEMMA 3.1. *The Euler class $\chi(2\rho)$ of the two-fold Whitney sum 2ρ of ρ is non-zero.*

PROOF. By the relation between the Euler class and Bockstein homomorphism and by Lemma 2.2, we have $\chi(2\rho) = \Delta(w_1(\rho)) = ly \neq 0$. q. e. d.

Let η be the canonical complex line bundle over $L^n(k)$. The first Chern class $c_1(\eta)$ is a generator of $H^2(L^n(k); Z) \cong Z_k$. Let

$$c: \widetilde{KO}(X) \to \tilde{K}(X), \quad r: \tilde{K}(X) \to \widetilde{KO}(X)$$

be the complexification and the real restriction respectively. Then it is well-known that

$$(3.2) \qquad rc = 2, \quad cr = 1 + t,$$

where t denotes the complex conjugation (cf. [1]).

PROPOSITION 3.3. *For $L^n(2l)$, we have*

$$c\rho = \eta^l = \eta \otimes \cdots \otimes \eta \quad (l\text{-fold tensor product}).$$

PROOF. By Lemma 3.1 and (3.2), $\chi(rc\rho) \neq 0$. Thus $c\rho$ is non-trivial.

Denote by C the total Chern class. Then, by (3.2), $C(cr(c\rho)) = C(c\rho \oplus tc\rho) = C(c\rho)C(tc\rho) = (1 + c_1(c\rho))\,(1 - c_1(c\rho)) = 1 - c_1(c\rho)^2$, while $C(cr(c\rho)) = C(c(2\rho)) = C(2c\rho) = (1 + c_1(c\rho))^2 = 1 + 2c_1(c\rho) + c_1(c\rho)^2$. Therefore we obtain $2c_1(c\rho) = 0$. Since complex line bundles are classified by the first Chern classes, the relation $c\rho = \eta^l$ follows from $c_1(c\rho) = lc_1(\eta) = c_1(\eta^l)$. q. e. d.

Let X and Y be finite CW-complexes and $f: Y \to X$ be a map. Let $\{E_r^{p,q}\}$ be the Atiyah-Hirzebruch spectral sequence for $\tilde{K}_\Lambda(X)$, i.e., $E_2^{p,q} \cong \tilde{H}^p(X; K_\Lambda^q(P))$ and $E_\infty^{p,-p}$ is the graded group associated to $\tilde{K}_\Lambda(X)$, and also $\{'E_r^{p,q}\}$ be that for $\tilde{K}_\Lambda(Y)$ (cf. [3]). Then

PROPOSITION 3.4. *Assume that there is an integer $r \geqq 2$ such that $E_r^{p,-p} = E_{r+1}^{p,-p} = \cdots = E_\infty^{p,-p}$ and $f_r^*: E_r^{p,-p} \to {'E_r^{p,-p}}$ is an epimorphism for any p. Then the induced homomorphism $f^!: \tilde{K}_\Lambda(X) \to \tilde{K}_\Lambda(Y)$ is an epimorphism.*

PROOF. By the assumptions it follows that $f_\infty^*: E_\infty^{p,-p} \to {'E_\infty^{p,-p}}$ is an epimorphism for each p. Then we have the result by the five lemma. q.e.d.

LEMMA 3.5. *$\pi^!$: $\tilde{K}(CP^n)\to\tilde{K}(L^n(k))$ is an epimorphism, where π is the natural projection.*

PROOF. Since the Atiyah-Hirzebruch spectral sequence for $\tilde{K}(CP^n)$ is trivial, we have the desired result by Lemma 2.3 and Prop. 3.4. q.e.d.

Let $\sigma=\eta-1\in\tilde{K}(L^n(k))$ denote the stable class of η. Then we have

LEMMA 3.6. *In $\tilde{K}(L^n(k))$, it holds*

$$(3.7)\qquad (\sigma+1)^k=1,\quad \sigma^{n+1}=0.$$

Furthermore, the elements $\sigma, \sigma^2,\cdots,\sigma^{k-1}$ generate $\tilde{K}(L^n(k))$ additively.

PROOF. The first equality of (3.7) follows from $c_1(\eta^k)=kc_1(\eta)=0$ in $H^2(L^n(k))\cong Z_k$.

Consider the canonical complex line bundle over CP^n and denote it also by η, then $\pi^!\eta=\eta$. Furthermore, it is well-known that the ring $\tilde{K}(CP^n)$ is generated by the element $\eta-1$ and $(\eta-1)^{n+1}=0$ (e.g. [1, Th. 7.2]). Thus we have the lemma using Lemma 3.5. q.e.d.

Denote by $\#A$ the number of the elements of a finite set A.

LEMMA 3.8. $$\#\tilde{K}(L^n(k))=k^n.$$

PROOF. Let $\{E_r^{p,q}\}$ be the Atiyah-Hirzebruch spectral sequence for $\tilde{K}(L^n(k))$. Then $E_2^{p,-q}\cong\tilde{H}^p(L^n(k); K^{-q}(P))$ is given by (2.4). Therefore this spectral sequence is trivial and the lemma follows. q.e.d.

Henceforth, we consider the case $k=4$. Put

$$\sigma(1)=(\sigma+1)^2-1=\sigma^2+2\sigma\in\tilde{K}(L^n(4)).$$

The relation $(\sigma+1)^4=1$ of (3.7) is equivalent to $(\sigma(1)+1)^2=1$, and so we have

$$(3.9)\qquad \sigma(1)^{i+1}=(-1)^i2^i\sigma(1)\qquad \text{for } i\geqq 0.$$

LEMMA 3.10. $cr\sigma=\sigma^2/(\sigma+1)=2\sigma+\sigma(1)+\sigma(1)\sigma$.

PROOF. The first equality is proved in the proof of [5, Lemma (3.5), ii)]. The second follows from (3.7) and (3.9). q.e.d.

Let $\kappa=\rho-1\in\widetilde{KO}(L^n(4))$ denote the stable class of ρ. Then we have

LEMMA 3.11. $$c\kappa=\sigma(1).$$

PROOF. By Prop. 3.3, $c\rho=\eta^2$. Therefore $c\kappa=\eta^2-1=\sigma(1)$. q.e.d.

§4. The structure of $\tilde{K}(L^n(4))$

LEMMA 4.1. $2^{i+2}\sigma^{n-i}=0$ *for* $i=0, 1,\dots, n-1$.

PROOF. Multiplying σ^{n-1} to the relation

$$\sigma^4+4\sigma^3+6\sigma^2+4\sigma=0, \tag{4.2}$$

we have $4\sigma^n=0$, because $\sigma^{n+i}=0$ for $i>0$ by (3.7). Assume that $2^{i+2}\sigma^{n-i}=0$ for $0\leqq i<n-1$. Multiplying $2^{i+1}\sigma^{n-i-2}$ to the equation (4.2), we have

$$2^{i+1}\sigma^{n-i+2}+2^{i+3}\sigma^{n-i+1}+3\cdot 2^{i+2}\sigma^{n-i}+2^{i+3}\sigma^{n-i-1}=0.$$

By the assumption, we have $2^{i+3}\sigma^{n-i-1}=0$. q.e.d.

LEMMA 4.3. *For* $i=0, 1,\dots, n-2$,

$$2^{i+1}\sigma^{n-i}=2^{i+2}\sigma^{n-i-1}=-2^{i+2}\sigma^{n-i-1},\quad 2^{i+1}\sigma(1)\sigma^{n-i-2}=0.$$

PROOF. If we multiply $2^i\sigma^{n-i-2}$ to the equation (4.2), we have

$$2^i\sigma^{n-i+2}+2^{i+2}\sigma^{n-i+1}+3\cdot 2^{i+1}\sigma^{n-i}+2^{i+2}\sigma^{n-i-1}=0.$$

By Lemma 4.1, we have the desired result. q.e.d.

LEMMA 4.4. *If* $n=2m$, *then*

$$2^m\sigma(1)=0,\quad 2^{m-1}(\sigma(1)\sigma+2^{m+1}\sigma)=0.$$

PROOF. By the definition of $\sigma(1)$, (3.7) and Lemma 4.1, we have

$$\sigma(1)^{m+1}=(\sigma^2+2\sigma)^{m+1}=\sum_{i=0}^{m+1}\binom{m+1}{i}2^i\sigma^{n-i+2}=0.$$

Thus the first result follows from (3.9).

Next, by the definition of $\sigma(1)$, (3.9) and Lemma 4.3, we have

$$\begin{aligned}2^{m-1}\sigma(1)\sigma&=(-1)^{m-1}\sigma(1)^m\sigma=(-1)^{m-1}\sigma(1)(\sigma^2+2\sigma)^{m-1}\sigma\\&=(-1)^{m-1}\sum_{i=0}^{m-1}\binom{m-1}{i}2^i\sigma(1)\sigma^{n-i-1}\\&=(-1)^{m-1}\sigma(1)\sigma^{n-1}=2\sigma^n=-2^n\sigma.\end{aligned}$$

Therefore we have the second result. q.e.d.

The following lemma is verified quite similarly as the above lemma.

LEMMA 4.5. *If* $n=2m+1$, *then*

$$2^m(\sigma(1)+2^{m+1}\sigma)=0,\quad 2^m\sigma(1)\sigma=0.$$

The following theorem is one of our main theorems.

THEOREM 4.6.

$$\tilde{K}(L^n(4)) \cong Z_{2^{n+1}} \oplus Z_{2^m} \oplus Z_{2^{m-1}}, \qquad \textit{for } n=2m>0,$$

whose direct summands are generated by σ, $\sigma(1)$ *and* $\sigma(1)\sigma+2^{m+1}\sigma$ *respectively.*

$$\tilde{K}(L^n(4)) \cong Z_{2^{n+1}} \oplus Z_{2^m} \oplus Z_{2^m}, \qquad \textit{for } n=2m+1,$$

whose direct summands are generated by σ, $\sigma(1)+2^{m+1}\sigma$ *and* $\sigma(1)\sigma$ *respectively.*
The multiplicative structure is given by

$$\sigma^4=-4\sigma^3-6\sigma^2-4\sigma, \quad \sigma^{n+1}=0.$$

PROOF. According to Lemma 3.6, we see that the elements σ, σ^2 and σ^3 generate $\tilde{K}(L^n(4))$ additively. Thus it is clear that σ, $\sigma(1)$ and $\sigma(1)\sigma+2^{m+1}\sigma$ (or σ, $\sigma(1)+2^{m+1}\sigma$ and $\sigma(1)\sigma$) generate $\tilde{K}(L^n(4))$ additively. Then our results follow from Lemmas 4.4–5, 3.8 and (3.7). q.e.d.

COROLLARY 4.7. *The element* $\sigma^i \in \tilde{K}(L^n(4))$ *is of order* 2^{n-i+2} *for* $1 \leqq i \leqq n$, *and* $\sigma^{n+1}=0$.

PROOF. Th. 4.6 shows that the element σ is of order 2^{n+1}. Suppose that σ^i is of order 2^{n-i+2} for $1 \leqq i < n$. By Lemma 4.3, $2^{n-i}\sigma^{i+1}=2^{n-i+1}\sigma^i \neq 0$. On the other hand, $2^{n-i+1}\sigma^{i+1}=0$ by Lemma 4.1. Thus the order of σ^{i+1} is equal to 2^{n-i+1}. q.e.d.

THEOREM 4.8. *The element* $\sigma(1)^i\sigma^j \in \tilde{K}(L^n(4))$ *is of order* $2^{1+[(n+1-2i-j)/2]}$ *for any* i, j *with* $1 \leqq i \leqq 1+[(n-j-1)/2]$.

PROOF. Since $\sigma(1)^i\sigma^j=(-1)^{i-1}2^{i-1}\sigma(1)\sigma^j$ by (3.9), it is sufficient to prove that

(4.9) $\quad \sigma(1)\sigma^j$ *is of order* $2^{1+[(n-j-1)/2]}$ *for* $0 \leqq j < n$.

Put $[(n-j-1)/2]=h$. Then $j=n-2h-1$ or $j=n-2h-2$. In order to prove (4.9), it is sufficient to show

$$2^{h+1}\sigma(1)\sigma^{n-2h-2}=0, \quad 2^h\sigma(1)\sigma^{n-2h-1} \neq 0.$$

Now, by (3.9), Lemma 4.1 and (3.7), we have

$$\begin{aligned} 2^{h+1}\sigma(1)\sigma^{n-2h-2} &= (-1)^{h+1}\sigma^{n-2h-2} \\ &= (-1)^{h+1}(\sigma^2+2\sigma)^{h+2}\sigma^{n-2h-2} \\ &= (-1)^{h+1}\sum_{k=0}^{h+2}\binom{h+2}{k}2^k\sigma^{n-k+2}=0. \end{aligned}$$

Similarly, we have

$$\begin{aligned} 2^h\sigma(1)\sigma^{n-2h-1} &= (-1)^h\sigma(1)^{h+1}\sigma^{n-2h-1} \\ &= (-1)^h\sigma(1)(\sigma^2+2\sigma)^h\sigma^{n-2h-1} \\ &= (-1)^h \sum_{k=0}^{h} \binom{h}{k} 2^k\sigma(1)\sigma^{n-k-1} \\ &= (-1)^h\sigma(1)\sigma^{n-1} = (-1)^h 2\sigma^n \neq 0 \end{aligned}$$

using Lemma 4.3 and Cor. 4.7. Therefore, we have (4.9). q.e.d.

The following two corollaries are immediate consequences.

COROLLARY 4.10. *The element* $\sigma(1)^i$ *is of order* $2^{1+[(n+1-2i)/2]}$ *for* $1 \leqq i \leqq [(n+1)/2]$, *and* $\sigma(1)^{1+[(n+1)/2]}=0$.

COROLLARY 4.11. *The element* $\sigma(1)\sigma$ *is of order* $2^{[n/2]}$.

Finally, we notice that

$$j^! : \tilde{K}(L^n(4)) \cong \tilde{K}(L_0^n(4)), \tag{4.12}$$

where $L_0^n(4)$ is the subcomplex of (2.7) and j is the inclusion, and hence the above results hold also for $L_0^n(4)$ taking into account the element $\sigma = j^!\sigma$. In fact, consider the Puppe exact sequence (cf. [3, Prop. 1.4])

$$\tilde{K}(S^{2n+1}) \longrightarrow \tilde{K}(L^n(4)) \xrightarrow{j^!} \tilde{K}(L_0^n(4)) \longrightarrow \tilde{K}^1(S^{2n+1})$$

of $L^n(4)/L_0^n(4) = S^{2n+1}$ of (2.8), where the first term is 0 and the last term is Z. Then, (4.12) follows from the fact that $\tilde{K}(L_0^n(4))$ is finite, which is seen similarly as Lemma 3.8.

§5. The additive structure of $\widetilde{KO}(L^n(4))$

LEMMA 5.1. $\widetilde{KO}(L^n(4))$ *has only* 2-*component, and*

$$\#\widetilde{KO}(L^n(4)) \leqq \begin{cases} 2^{6t+1} & \text{for } n=4t, \\ 2^{6t+2} & \text{for } n=4t+1, \\ 2^{6t+4} & \text{for } n=4t+2, \\ 2^{6t+4} & \text{for } n=4t+3. \end{cases}$$

PROOF. We make use of the Atiyah-Hirzebruch spectral sequence for $\widetilde{KO}(L^n(4))$. The terms $E_2^{p,-p} \cong \tilde{H}^p(L^n(4); KO^{-p}(P))$ are given by (2.5), and so we have the desired result. q.e.d.

Case 1. $n=4t+3$.

LEMMA 5.2. *For the case* $n=2s+1>1$, *the elements* $cr\sigma$ *and* $c(\kappa+2^{[n/2]}r\sigma)$ *of* $\tilde{K}(L^n(4))$ *are of order* 2^n *and* $2^{[n/2]}$, *respectively, and these elements generate a subgroup* $Z_{2^n}\oplus Z_{2^{[n/2]}}$ *of* $\tilde{K}(L^n(4))$.

PROOF. We notice that $1\pm 2^{[n/2]}$ is odd by the assumption $n>1$. Using Lemmas 3.10–11 and Th. 4.6, we have

$$cr\sigma=2(1-2^{[n/2]})\sigma+(\sigma(1)+2^{[n/2]+1}\sigma)+\sigma(1)\sigma,$$

$$c(\kappa+2^{[n/2]}r\sigma)=-2^n\sigma+(1+2^{[n/2]})(\sigma(1)+2^{[n/2]+1}\sigma).$$

Therefore, the order of these elements are 2^n and $2^{[n/2]}$ by Th. 4.6.

Suppose $\alpha cr\sigma+\beta c(\kappa+2^{[n/2]}r\sigma)=0$. Then,

$$2(1-2^{[n/2]})\alpha-2^n\beta\equiv 0 \bmod 2^{n+1},$$

$$\alpha+(1+2^{[n/2]})\beta\equiv 0, \quad \alpha\equiv 0 \bmod 2^{[n/2]}$$

by the above equalities and Th. 4.6. These congruences imply that $\alpha\equiv 0 \bmod 2^n$ and $\beta\equiv 0 \bmod 2^{[n/2]}$, and so we have the desired results. q.e.d.

THEOREM 5.3. *If* $n=4t+3$, *we have*

$$\widetilde{KO}(L^n(4))\cong Z_{2^n}\oplus Z_{2^{[n/2]}},$$

where the direct summands are generated by $r\sigma$ *and* $\kappa+2^{[n/2]}r\sigma$, *respectively.*

PROOF. For the homomorphism $c\colon \widetilde{KO}(L^n(4))\to\tilde{K}(L^n(4))$,

$$2^{6t+4}\geqq\#\widetilde{KO}(L^n(4))\geqq\#\operatorname{Im} c\geqq\#(Z_{2^n}\oplus Z_{2^{[n/2]}})=2^{6t+4},$$

by the above two lemmas. Therefore, $\#\widetilde{KO}(L^n(4))=2^{6t+4}$, Im c is the subgroup of Lemma 5.2 and c is monomorphic, and so we have the theorem. q.e.d.

COROLLARY 5.4. *The complexification*

$$c\colon \widetilde{KO}(L^{4t+3}(4))\to\tilde{K}(L^{4t+3}(4))$$

is a monomorphism.

Case 2. $n=4t+2$.

Let $i\colon L^n(4)\to L^{n+1}(4)$ be the inclusion.

LEMMA 5.5. *If* $n=4t+2$, $i^!\colon \widetilde{KO}(L^{n+1}(4))\to\widetilde{KO}(L^n(4))$ *is an isomorphism.*

PROOF. Consider the Puppe exact sequence (cf. [3, Prop. 1.4]):

$$\widetilde{KO}(L^{n+1}(4)/L^n(4))\longrightarrow\widetilde{KO}(L^{n+1}(4))\xrightarrow{i^!}\widetilde{KO}(L^n(4))\longrightarrow\widetilde{KO}^1(L^{n+1}(4)/L^n(4)).$$

It is easily seen that the first term is $\widetilde{KO}(S^{8t+6}\cup e^{8t+7})\cong 0$ and the last term is $\widetilde{KO}(S^{8t+5}\cup e^{8t+6})\cong 0$. Hence $i^!$ is an isomorphism. q.e.d.

By the above lemma and Th. 5.3, we have

THEOREM 5.6. *If* $n=4t+2$,

$$\widetilde{KO}(L^n(4))\cong Z_{2^{n+1}}\oplus Z_{2^{n/2}},$$

where the first summand is generated by $r\sigma$, *and the second by* $\kappa+2^{n/2}r\sigma$.

COROLLARY 5.7. *If* $n=4t+2$ *or* $4t+3$, *then* $\widetilde{KO}(L^n(4))$ *is generated by the elements* $r\sigma$ *and* κ, *and the order of* κ *is equal to* $2^{[n/2]+1}$.

Case 3. $n=4t+1$.

Consider the following commutative diagram

$$\text{(5.8)}\qquad \begin{array}{ccccc} & & \widetilde{KO}(L^{4t+2}(4)) & \xrightarrow{i^!} & \widetilde{KO}(L^{4t+1}(4)) \\ & & \downarrow{\scriptstyle j'^!} & & \downarrow{\scriptstyle j^!} \\ \widetilde{KO}(S^{8t+3}\bigcup_4 e^{8t+4}) & \longrightarrow & \widetilde{KO}(L_0^{4t+2}(4)) & \xrightarrow{i'^!} & \widetilde{KO}(L_0^{4t+1}(4)) \end{array}$$

where i, i', j and j' are the inclusions, and the lower sequence is the Puppe exact sequence of (2.8). Then we have the following lemmas.

LEMMA 5.9. *$i^!$ is epimorphic.*

PROOF. Let $\{E_r^{p,q}\}$ and $\{'E_r^{p,q}\}$ be the spectral sequence for $\widetilde{KO}(L^{4t+2}(4))$ and $\widetilde{KO}(L^{4t+1}(4))$, respectively. Then, i^*: $E_2^{p,-p}\to 'E_2^{p,-p}$ is epimorphic by Lemma 2.6, and $E_2^{p,-p}=\cdots=E_\infty^{p,-p}$ by Th. 5.6 and Lemma 5.1. Therefore, we have the lemma by Prop. 3.4. q.e.d.

LEMMA 5.10. *$j'^!$ is an isomorphism.*

PROOF. Consider the Puppe exact sequence

$$\widetilde{KO}(S^{8t+5})\longrightarrow\widetilde{KO}(L^{4t+2}(4))\xrightarrow{j'^!}\widetilde{KO}(L_0^{4t+2}(4))\longrightarrow\widetilde{KO}^1(S^{8t+5})$$

of (2.8), where $\widetilde{KO}(S^{8t+5})\cong 0$ and $\widetilde{KO}^1(S^{8t+5})\cong Z$. We see that $\widetilde{KO}(L_0^{4t+2}(4))$ is finite similarly as Lemma 5.1 by (2.9), and so $j'^!$ is isomorphic. q.e.d.

LEMMA 5.11. *$i'^!$ is epimorphic and* $\#\widetilde{KO}(L_0^{4t+1}(4))=2^{6t+2}$.

PROOF. The Puppe exact sequence of $(S^{8t+3}\bigcup_4 e^{8t+4})/S^{8t+3}=S^{8t+4}$ is the following

$$\widetilde{KO}(S^{8t+4})\xrightarrow{\times 4}\widetilde{KO}(S^{8t+4})\longrightarrow\widetilde{KO}(S^{8t+3}\bigcup_4 e^{8t+4})\longrightarrow\widetilde{KO}(S^{8t+3}),$$

since the degree of the attaching map is 4. Therefore $\widetilde{KO}(S^{8t+3}\bigcup_4 e^{8t+4})\cong Z_4$.

On the other hand, $\#\widetilde{KO}(L_0^{4t+2}(4))=2^{6t+4}$ by the above lemma, and so we can prove that $i'^!$ is epic similarly as Lemma 5.9. Also we have $\#\widetilde{KO}(L_0^{4t+1}(4))\leqq 2^{6t+2}$ similarly as Lemma 5.1 by (2.9), and so the lemma. q.e.d.

LEMMA 5.12. *$j^!$ is isomorphic and $\#\widetilde{KO}(L^{4t+1}(4))=2^{6t+2}$.*

PROOF. $\#\widetilde{KO}(L^{4t+1}(4))\leqq 2^{6t+2}$ by Lemma 5.1, and $j^!$ is epic by the commutativity of (5.8) and Lemmas 5.10–11. Therefore we have the desired results by the above lemma. q.e.d.

Now, we have the following

THEOREM 5.13. *If $n=4t+1$, then*

$$\widetilde{KO}(L^n(4))\cong Z_{2^n}\oplus Z_{2^{[n/2]+1}},$$

and the first summand is generated by $r\sigma$ and the second by κ, where the latter can be replaced by $\kappa+2^{[n/2]}r\sigma$.

PROOF. By Th. 5.6, the equality

$$2^{2t+1}\kappa+2^{4t+2}r\sigma=0$$

holds in $\widetilde{KO}(L^{4t+2}(4))$, and so in $\widetilde{KO}(L^{4t+1}(4))$. Also,

$$2^{2t+1}\kappa+2^{4t+1}r\sigma=0 \quad \text{if } t>0$$

in $\widetilde{KO}(L^{4t+1}(4))$. In fact, the left hand side is equal to $2^{[n/2]}rc(\kappa+2^{[n/2]}r\sigma)=0$ by (3.2) and Lemma 5.2. These two equalities imply that

$$2^{4t+1}r\sigma=0,\quad 2^{2t+1}\kappa=0,\quad \text{if } t>0.$$

These hold for the case $t=0$, since $2r\sigma=rcr\sigma=r(\sigma^2/(\sigma+1))$ and $\sigma^2=0$ in $\tilde{K}(L^1(4))$ by Lemma 3.10 and (3.7).

On the other hand, $\widetilde{KO}(L^{4t+1}(4))$ is generated by $r\sigma$ and κ additively, by Cor. 5.7 and Lemma 5.9. Therefore, we have the theorem by Lemma 5.12 and the last equalities. q.e.d.

COROLLARY 5.14. *For the complexification c: $\widetilde{KO}(L^{4t+1}(4))\to\tilde{K}(L^{4t+1}(4))$,* Ker $c \cong Z_2$ *is generated by* $2^{2t}(\kappa+2^{2t}r\sigma)$, *if* $t>0$.

PROOF. This is an immediate consequence of the above theorem and Lemma 5.2. q.e.d.

Case 4. $n=4t(>0)$.

Consider the commutative diagram

$$\begin{tikzcd}
\widetilde{KO}(L^{4t+1}(4)) \arrow[r, "c"] \arrow[d, "i^!"] \arrow[dr, "j^!"] & \tilde{K}(L^{4t+1}(4)) \arrow[r, "j^!"] & \tilde{K}(L_0^{4t+1}(4)) & \tilde{K}(S^{8t+2}) \arrow[l, "p^!"'] \\
\widetilde{KO}(L^{4t}(4)) & \widetilde{KO}(L_0^{4t+1}(4)) \arrow[l, "k^!"'] \arrow[ur, "c"] & \widetilde{KO}(S^{8t+2}) \arrow[l, "p^!"'] \arrow[ur, "c"] &
\end{tikzcd}$$

where the lower sequence is the Puppe exact sequence of $L_0^{4t+1}(4)/L^{4t}(4)=S^{8t+2}$, and i, j are the inclusions.

LEMMA 5.15. $i^!$ *is an epimorphism.*

PROOF. This can be proved similarly as Lemma 5.9, using the above theorem. q.e.d.

LEMMA 5.16. *In the lower exact sequence,* $k^!$ *is epimorphic,* $p^!$ *is monomorphic and* $\#\widetilde{KO}(L^{4t}(4))=2^{6t+1}$.

PROOF. The exactness shows the lemma, since $\#\widetilde{KO}(L_0^{4t+1}(4))=2^{6t+2}$ by Lemma 5.11, $\#\widetilde{KO}(L^{4t}(4)) \leqq 2^{6t+1}$ by Lemma 5.1, and $\widetilde{KO}(S^{8t+2}) \cong Z_2$. q.e.d.

LEMMA 5.17. Ker $i^!=$ Ker c *in* $\widetilde{KO}(L^{4t+1}(4))$.

PROOF. Since the two homomorphisms $j^!$ are isomorphic by Lemma 5.12 and (4.12), it is sufficient to prove Im $p^!$ = Ker c in $\widetilde{KO}(L_0^{4t+1}(4))$. Since c: $\widetilde{KO}(S^{8t+2}) \cong Z_2 \to \tilde{K}(S^{8t+2}) \cong Z$ is 0, we have $c \circ p^!=0$ and Im $p^! \subset$ Ker c. Also, Im $p^! \cong Z_2$ by the above lemma, and Ker $c \cong Z_2$ by Cor. 5.14. Thus we have Im $p^!=$ Ker c. q.e.d.

By Th. 5.13, Lemmas 5.15, 5.17 and Cor. 5.14, we have the following

THEOREM 5.18. *If* $n=4t>0$, *then*

$$\widetilde{KO}(L^n(4)) \cong Z_{2^{n+1}} \oplus Z_{2^{n/2}},$$

where the first summand is generated by $r\sigma$ *and the second by* $\kappa+2^{n/2}r\sigma$. *Also the order of* κ *is equal to* $2^{n/2+1}$.

Thus the additive structures of $\widetilde{KO}(L^n(4))$ in Th. B of §1 are obtained completely.

In the rest of this section, we are concerned with $\widetilde{KO}(L_0^n(4))$. If $n \not\equiv 0 \bmod 4$, the induced homomorphism

$$j^! : \widetilde{KO}(L^n(4)) \to \widetilde{KO}(L_0^n(4))$$

is isomorphic, where j is the inclusion. In fact, it is proved in Lemmas 5.12 and 5.10 if $n \equiv 1, 2 \bmod 4$, and it follows immediately from the Puppe exact sequence and $\widetilde{KO}(S^{2n+1}) \cong \widetilde{KO}(S^{2n}) \cong 0$ if $n \equiv 3 \bmod 4$.

To consider the case $n \equiv 0 \bmod 4$, we use the following

LEMMA 5.19. *If $n=2s>0$, the elements $cr\sigma$ and $c(\kappa+2^{n/2}r\sigma)$ of $\tilde{K}(L^n(4))$ are of order 2^n and $2^{n/2}$, respectively, and these elements generate a subgroup $Z_{2^n} \oplus Z_{2^{n/2}}$ of $\tilde{K}(L^n(4))$.*

PROOF. By the similar way to the proof of Lemma 5.2, we have

$$cr\sigma = 2(1-2^{n/2})\sigma + \sigma(1) + (\sigma(1)\sigma + 2^{n/2+1}\sigma),$$

$$c(\kappa + 2^{n/2}r\sigma) = 2^{n/2+1}\sigma + (1+2^{n/2})\sigma(1),$$

and so the desired results, using Lemmas 3.10–11 and Th. 4.6. q.e.d.

By this lemma and Th. 5.6 and 5.18, we have immediately

COROLLARY 5.20. *For the complexification $c: \widetilde{KO}(L^{2s}(4)) \to \tilde{K}(L^{2s}(4))\ (s>0)$,* Ker $c \cong Z_2$ *is generated by $2^{2s}r\sigma$.*

Let $n=4t>0$ and consider the commutative diagram

$$\begin{array}{ccccccc}
\widetilde{KO}(S^{8t+1}) & \xrightarrow{p^!} & \widetilde{KO}(L^{4t}(4)) & \xrightarrow{j^!} & \widetilde{KO}(L_0^{4t}(4)) & \longrightarrow & \widetilde{KO}^1(S^{8t+1}) \\
\downarrow c & & \downarrow c & & & & \\
\tilde{K}(S^{8t+1}) & \xrightarrow{p^!} & \tilde{K}(L^{4t}(4)) & & & &
\end{array}$$

where the upper sequence is the Puppe exact sequence.

LEMMA 5.21. *$j^!$ is epimorphic and* Ker $j^! =$ Im $p^! =$ Ker $c \cong Z_2$ *is generated by $2^{4t}r\sigma$ in $\widetilde{KO}(L^{4t}(4))$.*

PROOF. Similarly as Lemma 5.1, we see $\#\widetilde{KO}(L_0^{4t}(4)) \leqq 2^{6t}$ by (2.9), and so $j^!$ is epimorphic since $\widetilde{KO}^1(S^{8t+1}) \cong Z$. Also, $\#\widetilde{KO}(L^{4t}(4)) = 2^{6t+1}$ by Lemma 5.16, and $\widetilde{KO}(S^{8t+1}) \cong Z_2$. Hence, the exactness shows that $\#\widetilde{KO}(L_0^{4t}(4)) = 2^{6t}$ and $p^!$ is monomorphic, and Ker $j^! =$ Im $p^! \cong Z_2$. On the other hand, by the commutativity of the diagram and $\tilde{K}(S^{8t+1}) = 0$, we have $c \circ p^! = 0$ and Im $p^! \subset$ Ker c, and so the desired results by the above corollary. q.e.d.

By this lemma, Th. 5.18 and the above considerations, we have the following

THEOREM 5.22. $$\widetilde{KO}(L_0^n(4)) \cong \widetilde{KO}(L^n(4))$$

for $n \not\equiv 0 \bmod 4$, *by the induced homomorphism* $j^!$ *of the inclusion* j.

If $n=4t>0$, *then*

$$\widetilde{KO}(L_0^n(4)) \cong Z_{2^n} \oplus Z_{2^{n/2}}$$

and the first summand is generated by $r\sigma$ *and the second by* κ (*or* $\kappa+2^{n/2}r\sigma$), *where* $r\sigma$ *and* κ *are the elements* $j^! r\sigma$ *and* $j^! \kappa$ *respectively.*

§6. The multipilcative structure of $\widetilde{KO}(L^n(4))$

We preserve the notations of the previous sections.

THEOREM 6.1. *The multiplicative structure of* $\widetilde{KO}(L^n(4))$ *is given by*

$$(r\sigma)^2 = -4r\sigma + 2\kappa, \tag{6.2}$$

$$\kappa^2 = -2\kappa = \kappa \cdot r\sigma. \tag{6.3}$$

PROOF. It is sufficient to prove these equalities for $n=4t+3$, mapping by the monomorphism c of Cor. 5.4. Now, by Lemmas 3.10–11 and (3.9), we have

$$\begin{aligned} c(r\sigma)^2 &= (cr\sigma)^2 = (2\sigma+\sigma(1)+\sigma(1)\sigma)^2 \\ &= -4(2\sigma+\sigma(1)+\sigma(1)\sigma)+2\sigma(1) = c(-4r\sigma+2\kappa), \\ c(\kappa\cdot r\sigma) &= c(\kappa)c(r\sigma) = \sigma(1)(2\sigma+\sigma(1)+\sigma(1)\sigma) \\ &= -2\sigma(1) = c(-2\kappa) \\ &= \sigma(1)^2 = (c\kappa)^2 = c(\kappa^2). \end{aligned}$$

q.e.d.

By the above theorem and the induction, we have

$$\kappa^i = (-1)^{i-1}2^{i-1}\kappa, \tag{6.4}$$

$$\begin{aligned} (r\sigma)^i &= (-1)^{i+1}2^{2i-2}r\sigma + (-1)^i(2^{2i-2}-2^{i-1})\kappa \\ &= (-1)^{i+1}2^{i-1}\{2^{i-1}+2^{[n/2]}(2^{i-1}-1)\}r\sigma \\ &\quad + (-1)^i 2^{i-1}(2^{i-1}-1)(\kappa+2^{[n/2]}r\sigma), \end{aligned} \tag{6.5}$$

for $i \geqq 1$.

Then we have the following corollaries by these equalities, Cor. 5.7 and Th. 5.3, 5.6, 5.13, 5.18.

COROLLARY 6.6. *The element* $\kappa^i \in \widetilde{KO}(L^n(4))$ *is of order* $2^{[n/2]+2-i}$ *for* $1 \leqq i \leqq [n/2]+1$, *and* $\kappa^{[n/2]+2}=0$.

COROLLARY 6.7. *The order of the element* $(r\sigma)^i$ *of* $\widetilde{KO}(L^n(4))$ *is equal to*

$$2^{n-2i+2} \text{ if } n \text{ is odd}, \quad 2^{n-2i+3} \text{ if } n \text{ is even},$$

for $1 \leqq i \leqq [n/2]$ *or* $i=[n/2]+1$ *and* $n \equiv 1 \bmod 4$. *Also*

$$(r\sigma)^{[n/2]+1}=0 \quad \text{if } n \not\equiv 1 \bmod 4,$$

$$(r\sigma)^{[n/2]+2}=0 \quad \text{if } n \equiv 1 \bmod 4.$$

§7. Applications

We study the problem of the immersion and the embedding of the lens space $L^n(k)$ in Euclidean space. The following two results are due to [2, Th. 3.3 and 4.3]. Let γ^i: $KO(X) \rightarrow KO(X)$ be the γ-operation.

(7.1) *If an* n*-dimensional differentiable manifold* M^n *is immersed in* $(n+k)$*-dimensional Euclidean space* $R^{n+k}(k>0)$, *then* $\gamma^i(n-\tau(M^n))=0$ *for all* $i>k$, *where* $\tau(M^n)$ *denotes the tangent bundle of* M^n.

(7.2) *If* M^n *is embedded in* R^{n+k}, *then* $\gamma^i(n-\tau(M^n))=0$ *for all* $i \geqq k$.

According to [10, Cor. 3.2], it is known that

$$\tau(L^n(k)) \oplus 1=(n+1)r\eta. \tag{7.3}$$

LEMMA 7.4. $\quad 2n+1-\tau(L^n(k))=-(n+1)r\sigma.$

PROOF. By (7.3), $2n+1-\tau(L^n(k))=2n+2-(n+1)r\eta=-(n+1)(r\eta-2)=-(n+1)r\sigma$. q.e.d.

Let γ_t be the operation defined by $\gamma_t(\zeta)=\sum_{i=0}^{\infty} \gamma^i(\zeta)t^i$.

LEMMA 7.5. $\quad \gamma_t(r\sigma)=1+r\sigma \cdot t-r\sigma \cdot t^2.$

PROOF. We carry out the proof in the same way as that of [5, Lemma 4.8]. q.e.d.

PROPOSITION 7.6. *For any* k, $L^n(k)$ *cannot be immersed in* $R^{2n+2L(n,k)}$, *and* $L^n(k)$ *cannot be embedded in* $R^{2n+2L(n,k)+1}$, *where*

$$L(n, k)= \max \left\{i \left| \binom{n+i}{i}(r\sigma)^i \neq 0 \right.\right\}.$$

PROOF. By Lemmas 7.4–5, we have

$$\begin{aligned}\gamma_t(2n+1-\tau(L^n(k))) &= \gamma_t(-(n+1)r\sigma)=\gamma_t(r\sigma)^{-n-1}\\ &=(1+r\sigma\cdot t-r\sigma\cdot t^2)^{-n-1}=(1+r\sigma(t-t^2))^{-n-1}\\ &=\sum_{i=0}^{\infty}\binom{-n-1}{i}(r\sigma)^i(t-t^2)^i=\sum_{i=0}^{\infty}(-1)^i\binom{n+i}{i}(r\sigma)^i(t-t^2)^i.\end{aligned}$$

Therefore, we obtain

$$\gamma^i(2n+1-\tau(L^n(k)))\neq 0 \qquad \text{for } i=2L(n, k),$$

$$\gamma^i(2n+1-\tau(L^n(k)))=0 \qquad \text{for } i>2L(n, k).$$

By (7.1–2) we have the desired results. q.e.d.

For the case $k=4$, the above proposition is Theorem C of §1, by Cor. 6.7.

The next theorem reduces the immersion problem for $L^n(k)$ to the cross-section problem for the bundle $mr\eta$ (the m-fold Whitney sum of $r\eta$).

THEOREM 7.7. *Let n and l be integers with $0<l\leqq 2n+1$. Suppose $N\geqq 2n+2$, where N is an integer such that $Nr\sigma=0$. Then there is an immersion of $L^n(k)$ in $(2n+1+l)$-dimensional Euclidean space R^{2n+1+l} if and only if the vector bundle $(N-n-1)r\eta$ has $(2N-2n-l-2)$-independent cross-sections.*

This theorem is a slight generalization of [7, I, Th. 1].

There is an integer N such that $Nr\sigma=0$, because $\widetilde{KO}(L^n(k))$ is a finite group.

PROOF. Suppose that $L^n(k)$ is immersible in R^{2n+1+l}. Let ν be a normal bundle of an immersion. Then ν is l-dimensional, and it holds that

$$\tau(L^n(k))\oplus\nu=2n+1+l.$$

Since $Nr\sigma=N(r\eta-2)=0$ by the assumption, we have by (7.3)

$$\nu+(2N-2n-2-l)=(N-n-1)r\eta \quad \text{in } KO(L^n(k)).$$

But the dimension of the bundle of both sides is greater than $2n+1$, since $N\geqq 2n+2$. So we obtain the Whitney sum decomposition: $\nu\oplus(2N-2n-2-l)=(N-n-1)r\eta$.

Conversely, assume that there exists a vector bundle α of dimension l such that $(N-n-1)r\eta=\alpha\oplus(2N-2n-2-l)$. Then $2n+1-\tau(L^n(k))=\alpha-k$

$\in \widetilde{KO}(L^n(k))$. Therefore, by the theorem of M. W. Hirsch (cf. [4, Th. 6.4] and [2, Prop. 3.2]), we see that $L^n(k)$ is immersible in R^{2n+1+l}. q.e.d.

COROLLARY 7.8. *Let p be an odd prime, and a be an integer such that $ap^{r+[(n-2)/(p-1)]} \geqq 2n+2$, where $r \geqq 1$. Then there is an immersion of $L^n(p^r)$ in $R^{2n+1+l} (0 < l \leqq 2n+1)$ if and only if the vector bundle $(ap^{r+[(n-2)/(p-1)]} - n - 1) r\eta$ has $(2ap^{r+[(n-2)/(p-1)]} - 2n - l - 2)$-independent cross-sections.*

PROOF. Since $p^{r+[(n-2)/(p-1)]} r\sigma = 0$ by [6, Th. 1.1, (ii)], the result follows from Th. 7.7. q.e.d.

Finally, we give a non-immersion theorem for $L^n(k)$.

THEOREM 7.9. *Suppose that p is an odd prime. Let $k = up^r$, where $r \geqq 1$ and $(u, p) = 1$. Let n and m be integers with $0 < m \leqq [n/2]$. Assume that the following two conditions are satisfied:*

(i) $\binom{n+m}{m} \not\equiv 0 \mod p$,

(ii) $n+m+1 \not\equiv 0 \mod p^{[(n-m-1)/(p-1)]}$.

Then $L^n(k)$ is not immersible in $R^{2n+2m+1}$.

If $u=1$ and $r=1$, this theorem coincides with [7, II, Th. C]. The assumption $m < n$ of Th. C and (6.2) in [7, II] should be $m \leqq [n/2]$.

PROOF. The natural projection $L^n(p) \to L^n(k)$ is a covering projection. Therefore, if $L^n(k)$ is immersible in R^N, then $L^n(p)$ is immersible in R^N. Thus the result is a consequence of [7, II, Th. C]. q.e.d.

The next corollaries are immediate consequences.

COROLLARY 7.10. *Assume that p is a prime >3, and that k is divisible by p. Then $L^n(k)$ is not immersible in R^{3n+1} for $n = 2p^t$, $t \geqq 1$.*

COROLLARY 7.11. *Under the assumptions of Cor. 7.10, $L^n(k)$ is not immersible in R^{3n} for $n = 2p^t + 1$, $t \geqq 1$.*

According to D. Sjerve (cf. [9]), $L^n(k)$ is immersible in $R^{2n+2[n/2]+2}$ if k is odd. This result is seen to be best possible by the above corollaries (cf. also [7, II, Cor. D–E]).

References

[1] J. F. Adams, *Vector fields on spheres*, Ann. of Math., **75** (1962), 603–632.

[2] M. F. Atiyah, *Immersions and embeddings of manifolds*, Topology, **1** (1962), 125–132.

[3] M. F. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, Proc. Symposia in Pure Math. Vol. III, Amer. Math. Soc. (1961), 7-38.
[4] M. W. Hirsch, *Immersions of manifolds*, Trans. Amer. Math. Soc., **93** (1959), 242-276.
[5] T. Kambe, *The structure of K_Λ-rings of the lens space and their applications*, J. Math. Soc. Japan, **18** (1966), 135-146.
[6] T. Kawaguchi and M. Sugawara, *K- and KO-rings of lens spaces $L^n(p^2)$ for odd prime p*, this journal, 273-286.
[7] T. Kobayashi, *Non-immersion theorems for lens spaces*, J. Math. Kyoto Univ., **6** (1966), 91-108; II, J. Sci. Hiroshima Univ., Ser. A-I, **32** (1968), 285-292.
[8] N. Mahammed, *A propos de la K-théorie des espaces lenticulaires*, C.R. Acad. Sc. Paris, **271** (1970), 639-642.
[9] D. Sjerve, *Vector bundles over orbit manifolds*, Trans. Amer. Math. Soc., **138** (1969), 97-106.
[10] R. H. Szczarba, *On tangent bundles of fibre spaces and quotient spaces*, Amer. J. Math., **86** (1964), 685-697.

Department of Mathematics,
Faculty of Science,
Hiroshima University