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On Polarizations of Certain Homogeneous Spaces

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§ 1. Introduction

It is one of the main problems in the theory of unitary representations to find a unified way of constructing all irreducible unitary representations for an arbitrary Lie group G.

Kostant ([9], [10]) has shown a very general method of constructing unitary representations, using G-homogeneous symplectic spaces and polarizations on them. Related with this, it is required to characterize and classify G-homogeneous symplectic spaces and G-invariant polarizations. It was shown by Kostant that every G-homogeneous symplectic space is diffeomorphic to a covering space of G-orbit in the dual space of the Lie algebra $\mathfrak{g}_R$ of G when G is a connected, simply connected Lie group and the 1-st and the 2-nd cohomology spaces of $\mathfrak{g}_R$ vanish (this condition is valid if $\mathfrak{g}_R$ is semisimple). The outline of Kostant's method is destribed in [7] and [10], and Kirillov has given several problems related with Kostant's works.

In this note, first we shall give an infinitesimal characterization of polarizations (Theorem 2.2). From the viewpoint of the classification of polarizations, it seems essential for us to study them in case of orbits of nilpotent elements (Theorem 2.5). Then the TDS-argument appears on the stage as a useful instrument for the investigation of polarizations of nilpotent elements. Using TDS, one can obtain polarizations of nilpotent elements in some cases (§5). But to count up all the polarizations seems to be a somewhat complicated problem. In fact, as one can see in Examples 6.2–6.4, there exist a nilpotent element with no polarizations and also a nilpotent element with many polarizations. However, when e is a nilpotent element of a special form, Proposition 5.5 will enable us to find out all polarizations of e.

§ 2. Characterization of polarizations

Let G be a connected Lie group, $\mathfrak{g}_R$ its Lie algebra and $\mathfrak{g}_R^* = \mathrm{Hom}_R(\mathfrak{g}_R, \boldsymbol{R})$ the dual vector space of $\mathfrak{g}_R$. The space $\mathfrak{g}_R^*$ has the G-module structure contragredient to the adjoint representation of G on $\mathfrak{g}_R$. For an element f in $\mathfrak{g}_R$, we denote by G^f the isotropy subgroup of G with respect to f, and by $\mathfrak{g}_R^f$ the subalgebra of $\mathfrak{g}_R$ corresponding to G^f(i.e., $\mathfrak{g}_R^f = \{X \in \mathfrak{g}_R;\ f([X, Y]) = 0$ for every $Y \in \mathfrak{g}_R\}$). Kostant [10] has shown that every G-orbit $G(f) = G/G^f$ has a canonical G-invariant symplectic structure.

Let $\mathfrak{g}$ and $\mathfrak{g}^f$ be the complexifications of $\mathfrak{g}_R$ and $\mathfrak{g}_R^f$. For a complex subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, we consider the following conditions:

i) $f([\mathfrak{p}, \mathfrak{p}])=\{0\}$,

ii) $\dim\mathfrak{p}-\dim\mathfrak{g}^f=\dim\mathfrak{g}-\dim\mathfrak{p}$,

iii) $\mathfrak{p}$ is $Ad(G)^f$-stable,

iv) $\mathfrak{p}+\sigma\mathfrak{p}$ is a complex subalgebra of $\mathfrak{g}$,

where σ denotes the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_R$.

DEFINITION 2.1. For f in $\mathfrak{g}_R^*$ and a complex subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, $\mathfrak{p}$ is called

1) *a weak polarization* (in short, *w-polarization*) *of* f if $\mathfrak{p}$ satisfies i) and ii),

2) *a polarization of* f if $\mathfrak{p}$ satisfies i)—iii),

3) *an admissible w-polarization of* f if $\mathfrak{p}$ satisfies i), ii) and iv), and

4) *an admissible polarization of* f if $\mathfrak{p}$ satisfies i)—iv).

DEFINITION 2.2. A polarization (or w-polarization) $\mathfrak{p}$ of f is called

1) *real* if $\mathfrak{p}=\sigma\mathfrak{p}$, and

2) *totally complex* if $\mathfrak{p}+\sigma\mathfrak{p}=\mathfrak{g}$ (or equivalently $\mathfrak{p}\cap\sigma\mathfrak{p}=\mathfrak{g}^f$).

PROPOSITION 2.1. *A polarization* $\mathfrak{p}$ *of* f *contains* $\mathfrak{g}^f$.

PROOF. We put $V=\mathfrak{g}/\mathfrak{g}^f$ (the quotient vector space over $\boldsymbol{C}$). Since

$$\mathfrak{g}^f=\{X\in\mathfrak{g}; f([X, Y])=0 \qquad \text{for every } Y\in\mathfrak{g}\},$$

a non-degenerate skew-symmetric bilinear form ω on V is well-defined by

$$\omega(\bar{X}, \bar{Y})=-f([X, Y]) \qquad \text{for every } X, Y\in\mathfrak{g},$$

where $\bar{X}=X+\mathfrak{g}^f$, $\bar{Y}=Y+\mathfrak{g}^f\in V$. We set $\mathfrak{q}=\mathfrak{p}+\mathfrak{g}^f$, and denote by $\bar{\mathfrak{q}}$ the subspace of V corresponding to $\mathfrak{q}$. By the condition i) of polarizations, we have

$$f([\mathfrak{q}, \mathfrak{q}])=\{0\}.$$

Hence $\omega(\bar{\mathfrak{q}}, \bar{\mathfrak{q}})=\{0\}$, and so, by the non-degeneracy of ω, we have

$$\dim\bar{\mathfrak{q}}\leqq\frac{1}{2}\dim V=\frac{1}{2}(\dim\mathfrak{g}-\dim\mathfrak{g}^f).$$

Therefore

$$\begin{aligned}\dim\mathfrak{q}&=\dim\bar{\mathfrak{q}}+\dim\mathfrak{g}^f\\&\leqq\frac{1}{2}(\dim\mathfrak{g}+\dim\mathfrak{g}^f)\\&=\dim\mathfrak{p},\end{aligned}$$

from the condition ii) of polarizations. This relation, combined with $\mathfrak{p} \subset \mathfrak{q}$, implies $\mathfrak{p} = \mathfrak{q}$. Thus we have

$$\mathfrak{g}^f \subset \mathfrak{p}.$$

Q.E.D.

It is easily seen that our (admissible) polarization in the above definition corresponds to an invariant (admissible) polarization of the homogeneous space G/G^f with respect to the canonical symplectic structure given by Kostant. Whereas, a weak polarization of f corresponds to an invariant polarization of the universal covering space of G/G^f, considered as a homogeneous space by the universal covering group of G. And a weak polarization $\mathfrak{p}$ of f becomes an invariant polarization of a suitable covering space of G/G^f. So, from the practical viewpoint of the unitary representation of G, it seems to be essential to study (admissible) w-polarizations of each element in $\mathfrak{g}_R^*$.

Throughout this paper we assume that G is a connected semisimple Lie group. In this case, the G-module $\mathfrak{g}_R^*$ is isomorphic to $\mathfrak{g}_R$ via the Killing form B. For an element X in $\mathfrak{g}_R$ and a complex subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, we call $\mathfrak{p}$ a polarization (resp. a w-polarization, etc.) of X if $\mathfrak{p}$ is a polarization (resp. a w-polarization) of f_X, where f_X is the element of $\mathfrak{g}_R^*$ corresponding to X by the above isomorphism.

The following theorem gives a characterization of a w-polarization:

THEOREM 2.2. *For $X \in \mathfrak{g}$ and a complex subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, the following conditions are equivalent:*

1) *$\mathfrak{p}$ is a w-polarization of X;*

2) *$\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$, and the space $[X, \mathfrak{p}]$ coincides with the nil-radical of $\mathfrak{p}$.*

Particularly in case that X is a nilpotent element of $\mathfrak{g}$, the above conditions are equivalent to

3) *X belongs to the orthogonal complement of $\mathfrak{p}$ with respect to the Killing form B of $\mathfrak{g}$, and $\mathfrak{p}$ satisfies the condition* ii) *of polarizations.*

PROOF. [1)$\Rightarrow$2)] Fix a compact real form $\mathfrak{k}$ of $\mathfrak{g}$, and denote by $\mathfrak{k} \vee \mathfrak{p}$ the vector subspace of $\mathfrak{g}$ generated by $\mathfrak{k}$ and $\mathfrak{p}$. First we shall prove that $\mathfrak{k} \vee \mathfrak{p} = \mathfrak{g}$.

We choose $Y, Z \in \mathfrak{k} \cap \mathfrak{p}$ such that $Y + \sqrt{-1}Z \in [X, \mathfrak{p}]$. Then by the condition $B([X, \mathfrak{p}], \mathfrak{p}) = \{0\}$, we have

$$B(Y + \sqrt{-1}\, Z,\ Y) = 0,$$

and

$$B(Y + \sqrt{-1}\, Z,\ Z) = 0.$$

Hence we have

$$B(Y,\ Y) + B(Z,\ Z) = 0.$$

Since B is strictly negative definite on $\mathfrak{k}$, we have $Y=Z=0$. Therefore

$$[(\mathfrak{k}\cap\mathfrak{p})\oplus\sqrt{-1}\ (\mathfrak{k}\cap\mathfrak{p})]\cap[X, \mathfrak{p}]=\{0\},$$

and $(\mathfrak{k}\cap\mathfrak{p})\oplus\sqrt{-1}\ (\mathfrak{k}\cap\mathfrak{p})$ and $[X, \mathfrak{p}]$ are mutually disjoint linear subspaces of the real vector space $\mathfrak{p}$.

Hence

$$\dim_R [(\mathfrak{k}\cap\mathfrak{p})\oplus\sqrt{-1}\ (\mathfrak{k}\cap\mathfrak{p})]+\dim_R [X, \mathfrak{p}]\leqq\dim_R \mathfrak{p}\cdots\cdots(1).$$

We set $n=\dim_C \mathfrak{g}$, $m=\dim_C \mathfrak{g}^X$ and $l=\dim_C \mathfrak{p}$. By the condition ii) of polarizations, we have

$$2l=m+n.$$

Since the kernel of the linear mapping φ of $\mathfrak{p}$ onto $[X, \mathfrak{p}]$ defined by $\varphi(Y)=[X, Y]$ coincides with $\mathfrak{p}\cap\mathfrak{g}^X=\mathfrak{g}^X$, we have

$$\dim_R [X, \mathfrak{p}]=\dim_R \mathfrak{p}-\dim_R \mathfrak{g}^X=2(l-m).$$

So the inequality (1) becomes

$$2\dim_R (\mathfrak{k}\cap\mathfrak{p})\leqq\dim_R \mathfrak{g}^X=2m.$$

Hence

$$\dim_R (\mathfrak{k}\cap\mathfrak{p})\leqq m.$$

Now we calculate the dimension of $\mathfrak{k}\vee\mathfrak{p}$:

$$\begin{aligned}\dim_R (\mathfrak{k}\vee\mathfrak{p})&=\dim_R \mathfrak{k}+\dim_R \mathfrak{p}-\dim_R (\mathfrak{k}\cap\mathfrak{p})\\&\geqq n+2l-m\\&=2n=\dim_R \mathfrak{g}.\end{aligned}$$

Thus we have

$$\mathfrak{k}\vee\mathfrak{p}=\mathfrak{g}.$$

Let $G^C=\mathrm{Int}\ \mathfrak{g}$ denote the group of all inner automorphisms of $\mathfrak{g}$, and P the analytic subgroup of G^C generated by $\mathfrak{p}$. We shall prove that P is a closed subgroup of G^C. It suffices to show that P is a connected component of the closure $\bar{P}$ of P in G^C, since P is connected. From the condition i) of polarizations, we have

$$B(X, Ad(g)Y)=B(X, Y) \qquad \text{for every} \quad g\in P \text{ and } Y\in\mathfrak{p}.$$

So we have

$$B(X, Ad(g)Y)=B(X, Y) \qquad \text{for every} \quad g\in \bar{P} \text{ and } Y\in\mathfrak{p}.$$

Therefore

$$t^{-1}\{B(X, Ad(\exp tZ)Y)-B(X, Y)\}=0$$

for every $Y\in\mathfrak{p}$, $Z\in\bar{\mathfrak{p}}$ and $t\in\boldsymbol{R}^*=\boldsymbol{R}-\{0\}$, where $\bar{\mathfrak{p}}$ is the Lie algebra of $\bar{P}$. From the above relation follows

$$B(X, [Z, Y])=0 \qquad \text{for every} \quad Y\in\mathfrak{p} \text{ and } Z\in\bar{\mathfrak{p}}.$$

So we have

$$B(X, [\bar{\mathfrak{p}}, \mathfrak{p}])=\{0\}.$$

The skew-symmetric bilinear form ω on $\mathfrak{g}/\mathfrak{g}^X$ defined by

$$\omega(\bar{u}, \bar{v})=-B(X, [u, v])$$

(where $u, v\in\mathfrak{g}$ and $\bar{u}=u+\mathfrak{g}^X\in\mathfrak{g}/\mathfrak{g}^X$, $\bar{v}=v+\mathfrak{g}^X\in\mathfrak{g}/\mathfrak{g}^X$) is non-degenerate, and so by the condition ii) of polarizations, $\mathfrak{p}/\mathfrak{g}^X$ is a maximal null-subspace of ω. Therefore $\bar{\mathfrak{p}}$ must coincide with $\mathfrak{p}$, and P is the connected component of $\bar{P}$ containing the unit. Thus P is a closed subgroup of G^C.

Next we shall prove that G^C/P is compact. Let $\mathfrak{k}$ be a compact real form of $\mathfrak{g}^C$ and K the analytic subgroup of G^C corresponding to $\mathfrak{k}$. Then K is compact, and acts transitively on G^C/P. Therefore G^C/P is compact.

Let $\mathfrak{n}$ be the orthogonal complement of $\mathfrak{p}$ with respect to B. Then, by conditions of w-polarizations, we have $\mathfrak{n}=[X, \mathfrak{p}]$ and $[\mathfrak{p}, \mathfrak{n}]\subset\mathfrak{n}$. The normalizer $P'=N_{G^C}(\mathfrak{n})$ of $\mathfrak{n}$ in G^C is a subgroup of G^C with Lie algebra $\mathfrak{p}$, and so includes P. Since P' is algebraic and G^C/P' is compact, P' is a parabolic subgroup of G^C. Thus we have proved that $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$.

[2)$\Rightarrow$1)] Let $\mathfrak{n}$ be the nil-radical of $\mathfrak{p}$.

i) Since B is G^C-invariant, we have

$$B(X, [\mathfrak{p}, \mathfrak{p}])=B([X, \mathfrak{p}], \mathfrak{p})=B(\mathfrak{n}, \mathfrak{p})=\{0\}.$$

ii) We set $n=\dim_C\mathfrak{g}$, $l=\dim_C\mathfrak{p}$ and $m=\dim_C\mathfrak{g}^X$. $\mathfrak{g}^X$ is included in $\mathfrak{p}$ (=the orthogonal complement of $\mathfrak{n}$), because

$$B(\mathfrak{n}, \mathfrak{g}^X)=B([X, \mathfrak{p}], \mathfrak{g}^X)=B(\mathfrak{p}, [X, \mathfrak{g}^X])=\{0\}.$$

Since the kernel of the linear mapping φ of $\mathfrak{p}$ onto $[X, \mathfrak{p}]$ defined by $\varphi(Y)=[X, Y]$ coincides with $\mathfrak{p}\cap\mathfrak{g}^X=\mathfrak{g}^X$, we have

$$\dim_C\mathfrak{p}=\dim_C[X, \mathfrak{p}]+\dim_C\mathfrak{g}^X.$$

Hence

$$\dim_C [X, \mathfrak{p}] = l - m.$$

On the other hand, we have

$$\dim_C \mathfrak{n} = \dim_C \mathfrak{g} - \dim_C \mathfrak{p} = n - l.$$

Thus, by $[X, \mathfrak{p}] = \mathfrak{n}$, we have $l - m = n - l$, and $\mathfrak{p}$ is a w-polarization of X.

[1)⇒3)] Since $\mathfrak{p}$ is parabolic, there exist a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathfrak{p}$, a positive root system Δ_+ of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and an additively closed subset Φ of Δ_+, such that

$$\mathfrak{p} = \mathfrak{h} + \sum_{\alpha \in \Delta_+ \cup (-\Phi)} \mathfrak{g}^\alpha,$$

where

$$-\Phi = \{-\alpha;\ \alpha \in \Phi\}$$

and

$$\mathfrak{g}^\alpha = \{X \in \mathfrak{g};\ (adH)X = \alpha(H)X \text{ for every } H \in \mathfrak{h}\}.$$

The element X can be expressed as

$$X = H + \sum_{\alpha \in \Delta_+ \cup (-\Phi)} c_\alpha X_\alpha,$$

where $H \in \mathfrak{h}$, $c_\alpha \in \boldsymbol{C}$ and $0 \neq X_\alpha \in \mathfrak{g}^\alpha$. From the condition i) of polarizations, we have

$$c_\alpha = 0 \quad \text{for} \quad \alpha \in \Phi \cup (-\Phi),$$

because

$$[\mathfrak{p}, \mathfrak{p}] = \sum_{\alpha \in \Phi} \boldsymbol{C} H_\alpha + \sum_{\alpha \in \Delta_+ \cup (-\Phi)} \mathfrak{g}^\alpha,$$

where $H_\alpha \in \mathfrak{h}$ is defined by

$$B(H_\alpha, H) = \alpha(H) \qquad \text{for every } H \in \mathfrak{h}.$$

So we have

$$X = H + \sum_{\alpha \in \Delta_+ - \Phi} c_\alpha X_\alpha,$$

and $H = 0$, since X is nilpotent.

Therefore

$$X = \sum_{\alpha \in \Delta_+ - \Phi} c_\alpha X_\alpha, \text{ and } B(X, \mathfrak{p}) = \{0\}.$$

[3)⇒1)] This is obvious from definition of w-polarizations. Q.E.D.

PROPOSITION 2.3. *Any semisimple element in $\mathfrak{g}_R$ has an admissible polarization.*

PROOF. A semisimple element H of $\mathfrak{g}_R$ can be embedded in a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_R$. H can be decomposed into $H=H_1+H_2$, where all eigenvalues of $ad_{\mathfrak{g}}H_1$ (resp. $ad_{\mathfrak{g}}H_2$) are purely-imaginary (resp. real). Let $\mathfrak{h}$ be the complexification of $\mathfrak{h}_0$ and Δ the non-zero root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Now we set

$$\mathfrak{p}=\mathfrak{h}+\sum_{\alpha(H)=0}\mathfrak{g}^{\alpha}+\sum_{\alpha(H_2)>0}\mathfrak{g}^{\alpha}+\sum_{\substack{\alpha(H_2)=0\\ \sqrt{-1}\alpha(H_1)>0}}\mathfrak{g}^{\alpha},$$

where

$$\mathfrak{g}^{\alpha}=\{X\in\mathfrak{g};\ (adH)X=\alpha(H)X \qquad \text{for every } H\in\mathfrak{h}\}.$$

It follows from

$$\mathfrak{g}^{X}=\mathfrak{h}+\sum_{\alpha(H)=0}\mathfrak{g}^{\alpha}$$

and

$$[\mathfrak{p},\mathfrak{p}]=\sum_{\alpha(H)=0}CH_{\alpha}+\sum_{\alpha(H)=0}\mathfrak{g}^{\alpha}+\sum_{\alpha(H_2)>0}\mathfrak{g}^{\alpha}+\sum_{\substack{\alpha(H_2)=0\\ \sqrt{-1}\alpha(H_1)>0}}\mathfrak{g}^{\alpha},$$

that $\mathfrak{p}$ satisfies conditions i), ii) of polarizations. And

$$\mathfrak{p}+\sigma\mathfrak{p}=\mathfrak{h}+\sum_{\alpha(H_2)\geqq 0}\mathfrak{g}^{\alpha},$$

is a subalgebra of $\mathfrak{g}$. Since the centralizer $(G^C)^H$ of H in $G^C=\mathrm{Int}\mathfrak{g}$ is connected, G^H stabilizes $\mathfrak{p}$. Thus $\mathfrak{p}$ is an admissible polarization of H. Q.E.D.

LEMMA 2.4. *Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$, whose nil-radical is $\mathfrak{n}$. If $H\in\mathfrak{p}$ is a semisimple element, then $\mathfrak{p}\cap\mathfrak{g}^H$ is a paraboic subalgebra of $\mathfrak{g}^H$, whose nil-radical is $\mathfrak{n}\cap\mathfrak{g}^H$.*

PROOF. Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{p}$ containing H. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and of $\mathfrak{g}^H$ together. Now the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$ shows the lemma. Q.E.D.

Each element X in $\mathfrak{g}_R$ has the unique decomposition $X=H+e$ such that H is semisimple, e is nilpotent and $[H,e]=0$. The centralizer $\mathfrak{g}^H$ of H is reductive and e belongs to the semisimple part $[\mathfrak{g}^H,\mathfrak{g}^H]$ of $\mathfrak{g}^H$.

THEOREM 2.5. *Let $X=H+e$ be the decomposition of X as above. Then X has a w-polarization if and only if e has a w-polarization in the semisimple part of the centralizer $\mathfrak{g}^H$ of H in $\mathfrak{g}$.*

PROOF. First, suppose there exists a w-polarization $\mathfrak{p}$ of X in $\mathfrak{g}$. Then, by Theorem 2.2, $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ such that the nil-radical $\mathfrak{n}$ of $\mathfrak{p}$ coincides with $[X, \mathfrak{p}]$. Since $H \in \mathfrak{g}^X$, $\mathfrak{g}^H \subset \mathfrak{p}$ (Proposition 2.1), by Lemma 2.4, we see that $\mathfrak{p} \cap \mathfrak{g}^H$ is a parabolic subalgebra of $\mathfrak{g}^H$, whose nil-radical is $\mathfrak{n} \cap \mathfrak{g}^H$. Since $\mathfrak{g}^H = \mathfrak{c} + \mathfrak{g}'$, $\mathfrak{c} \subset \mathfrak{g}^X \subset \mathfrak{p}$, we have

$$\mathfrak{p} \cap \mathfrak{g}^H = \mathfrak{c} + (\mathfrak{p} \cap \mathfrak{g}'),$$

where $\mathfrak{c}$ (resp. $\mathfrak{g}'$) is the center (resp. semisimple part) of $\mathfrak{g}^H$. Thus $\mathfrak{p} \cap \mathfrak{g}'$ is a parabolic subalgebra of $\mathfrak{g}'$, whose nil-radical is $\mathfrak{n} \cap \mathfrak{g}^H$. We shall show that $\mathfrak{p} \cap \mathfrak{g}'$ is a w-polarization of e in $\mathfrak{g}'$. By the characterization theorem of polarizations (Theorem 2.2), it suffices to show that

$$[e, \mathfrak{p} \cap \mathfrak{g}'] = \mathfrak{n} \cap \mathfrak{g}^H = \mathfrak{n} \cap \mathfrak{g}'.$$

By the choice of $\mathfrak{p}$, we have $[X, \mathfrak{p}] = \mathfrak{n}$. Thus

$$\mathfrak{n} \cap \mathfrak{g}^H \supset [X, \mathfrak{p} \cap \mathfrak{g}^H] = [e, \mathfrak{p} \cap \mathfrak{g}^H] = [e, \mathfrak{p} \cap \mathfrak{g}'].$$

We set

$$f = ad(X) | \mathfrak{p} \quad \text{and} \quad h = ad(H) | \mathfrak{p},$$

then h is semisimple and $f \circ h = h \circ f$. We have

$$\mathfrak{p} = \operatorname{Ker}(h) \oplus h(\mathfrak{p}),$$

and

$$\begin{aligned} f(\mathfrak{p}) &= f(\operatorname{Ker}(h)) \oplus f(h(\mathfrak{p})) \\ &= f(\operatorname{Ker}(h)) \oplus h(f(\mathfrak{p})) \qquad \text{(direct sum)}. \end{aligned}$$

So we have

$$f(\operatorname{Ker}(h)) = \operatorname{Ker}(h) \cap f(\mathfrak{p}),$$

that is,

$$[X, \mathfrak{p} \cap \mathfrak{g}^H] = \mathfrak{g}^H \cap [X, \mathfrak{p}] = \mathfrak{n} \cap \mathfrak{g}^H.$$

On the other hand,

$$[e, \mathfrak{p} \cap \mathfrak{g}'] \subset \mathfrak{g}'.$$

Thus

$$\mathfrak{n} \cap \mathfrak{g}^H = [e, \mathfrak{p} \cap \mathfrak{g}'] \subset \mathfrak{g}'.$$

Hence

$$[e, \mathfrak{p} \cap \mathfrak{g}'] = \mathfrak{n} \cap \mathfrak{g}^H = \mathfrak{n} \cap \mathfrak{g}',$$

and $\mathfrak{p} \cap \mathfrak{g}'$ is a w-polarization of e in $\mathfrak{g}'$.

Next, suppose that e has a w-polarization $\mathfrak{p}'$ in $\mathfrak{g}'$. $\mathfrak{p}'$ is a parabolic subalgebra of $\mathfrak{g}'$, by Theorem 2.2. We denote its nil-radical by $\mathfrak{n}'$. We have

$$\mathfrak{g}^e \cap \mathfrak{g}' \subset \mathfrak{p}', \quad [e, \mathfrak{p}'] = \mathfrak{n}'.$$

For the semisimple element H in $\mathfrak{g}$, we choose an admissible polarization $\mathfrak{p}_s$ of H in $\mathfrak{g}$, whose nil-radical is denoted by $\mathfrak{n}_s$. (For the proof of this theorem, $\mathfrak{p}_s$ needs not to be admissible, but it is preferable for a later use to choose an admissible polarization.) Since H is semisimple, we have

$$\mathfrak{p}_s = \mathfrak{g}^H \oplus \mathfrak{n}_s \qquad \text{(direct sum as vector spaces)}.$$

Set

$$\mathfrak{p} = \mathfrak{c} \oplus \mathfrak{p}' \oplus \mathfrak{n}_s,$$
$$\mathfrak{n} = \mathfrak{n}' \oplus \mathfrak{n}_s.$$

We shall show that $\mathfrak{p}$ gives a w-polarization of $X = H + e$ in $\mathfrak{g}$.

0) $\mathfrak{p}$ is a subalgebra. In fact,

$$\mathfrak{p}' \oplus \mathfrak{c} \subset \mathfrak{g}^H, \quad \mathfrak{g}^H \subset \mathfrak{p}_s,$$

thus

$$[\mathfrak{p}' \oplus \mathfrak{c}, \mathfrak{n}_s] \subset \mathfrak{n}_s.$$

Hence

$$\mathfrak{p} \text{ is a subalgebra.}$$

i) Since $\mathfrak{c} \oplus \mathfrak{p}'$ contains a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$, whose nil-radical coincides with $\mathfrak{n}$. And

$$\begin{aligned} [X, \mathfrak{p}] &= [X, \mathfrak{c} + \mathfrak{p}' + \mathfrak{n}_s] = [X, \mathfrak{p}'] + [H + e, \mathfrak{n}_s] \\ &= [X, \mathfrak{p}'] + [e, \mathfrak{n}_s] \subset \mathfrak{n}' + \mathfrak{n}_s = \mathfrak{n}, \end{aligned}$$

since X belongs to the reductive part $\mathfrak{g}^H$ of $\mathfrak{p}'$. So we have $B([X, \mathfrak{p}], \mathfrak{p}) = \{0\}$.

ii) We have

$$\begin{aligned} 2\dim \mathfrak{p} &= 2(\dim \mathfrak{p}' + \dim \mathfrak{c} + \dim \mathfrak{n}_s) \\ &= (\dim \mathfrak{g}' + \dim(\mathfrak{g}' \cap \mathfrak{g}^e)) + 2\dim \mathfrak{c} + (\dim \mathfrak{g} - \dim \mathfrak{g}^H) \\ &= \dim \mathfrak{g} + \dim \mathfrak{g}^X, \end{aligned}$$

since $\mathfrak{g}^X = (\mathfrak{g}' \cap \mathfrak{g}^e) \oplus \mathfrak{c}$, $\mathfrak{g}^H = \mathfrak{g}' \oplus \mathfrak{c}$.

Q.E.D.

REMARK 2.1. It is easily seen from the proof of the above theorem that $\mathfrak{p} \cap \mathfrak{g}'$ is a polarization of e in $\mathfrak{g}'$ if and only if $\mathfrak{p}$ is a polarization of X, and that $\mathfrak{p} \cap \mathfrak{g}'$ is an admissible w-polarization in $\mathfrak{g}'$ if $\mathfrak{p}$ is an admissible w-polarization.

So we have

1) e has a polarization in $\mathfrak{g}'$ if and only if X has a polarization,

2) e has an admissible w-polarization in $\mathfrak{g}'$ if X has an admissible w-polarization.

So the problem to find a polarization for an arbitrary element in a real semisimple Lie algebra is reduced to the case where the element is nilpotent. The TDS plays an important role in finding out polarizations of a nilpotent element. Details of this method will be described in following sections.

PROPOSITION 2.6. *Decompose a real semisimple Lie algebra $\mathfrak{g}_R$ into the direct sum (as Lie algebras) of simple ideals:*

$$\mathfrak{g}_R=\mathfrak{g}_R^1+\cdots\cdots\cdots+\mathfrak{g}_R^m,$$

and let $\mathfrak{g}$ (resp. $\mathfrak{g}^i$) be the complexification of $\mathfrak{g}_R$ (resp. $\mathfrak{g}_R^i$). Then an element $X=\sum_{i=1}^m X^i \in \mathfrak{g}_R (X^i \in \mathfrak{g}_R^i)$ has a (admissible) (w-)polarization if and only if each X^i has a (admissible) (w-)polarization in $\mathfrak{g}^i$.

PROOF. We set $l_i=\operatorname{rank}(\mathfrak{g}^i)$ and $l=\sum_{i=1}^m l_i=\operatorname{rank}(\mathfrak{g})$.

First, suppose that X has a w-polarization $\mathfrak{p}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$, and $\mathfrak{h}^i$ the image of $\mathfrak{h}$ under the canonical projection π^i of $\mathfrak{g}$ onto $\mathfrak{g}^i$. Since π_i is a Lie algebra homomorphism, $\mathfrak{h}^i$ is an abelian subalgebra of $\mathfrak{g}^i$ and $ad_{\mathfrak{g}^i}(H)$ is semisimple for every $H \in \mathfrak{h}^i$. So $\mathfrak{h}^i$ is contained in a Cartan subalgebra of $\mathfrak{g}^i$, and we have

$$\dim \mathfrak{h}^i \leqq l_i.$$

So we have $l=\dim \mathfrak{h} \leqq \sum_{i=1}^m l_i=l$, since $\mathfrak{h} \subset \sum_{i=1}^m \mathfrak{h}^i$.

Hence $\mathfrak{h}^i$ is a Cartan subalgebra of $\mathfrak{g}^i$, and $\mathfrak{h}=\sum_{i=1}^m \mathfrak{h}^i$. Since $\mathfrak{p}$ is parabolic, we have (by using the root space decomposition)

$$\mathfrak{p}=\sum_{i=1}^m \pi^i(\mathfrak{p}) \quad \text{and} \qquad \pi^i(\mathfrak{p})=\mathfrak{p}\cap\mathfrak{g}^i.$$

We set $\mathfrak{p}^i=\pi^i(\mathfrak{p})$. We shall show that $\mathfrak{p}^i$ is a w-polarization of X^i.

By the condition i) ($B(X, [\mathfrak{p}, \mathfrak{p}])=\{0\}$) of polarizations and $\mathfrak{p}=\sum_{i=1}^m \mathfrak{p}^i$, we have

$$B(X^i, [\mathfrak{p}^i, \mathfrak{p}^i])=\{0\}.$$

From this relation and the non-degeneracy of ω in the proof of Proposition 2.1, we have

$$\dim \mathfrak{p}^i \leqq \frac{1}{2}(\dim \mathfrak{g}^i - \dim(\mathfrak{g}^i)^{X^i}).$$

Hence

$$\dim \mathfrak{p} \leqq \frac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{g}^X).$$

By the condition ii) of polarizations, the above inequality " $\leqq$ " is just " $=$ ". Thus $\mathfrak{p}^i$ satisfies the condition ii) of polarizations, and $\mathfrak{p}^i$ is a w-polarization of X^i in $\mathfrak{g}^i$.

Next, suppose that X^i has a w-polarization $\mathfrak{p}^i$ in $\mathfrak{g}^i$. Then it is easily seen that $\mathfrak{p} = \sum_{i=1}^{m} \mathfrak{p}^i$ is a w-polarization of X.

As to condition iii) or iv), the equivalence is easily checked. Q.E.D.

§3. Some properties of TDS

In this section we shall give a short description of some properties related to the TDS, which will become useful tools for the research of w-polarizations of a nilpotent element as is seen in following sections. Further detailed discussions of TDS are seen in Kostant ([8]) etc.

DEFINITION 3.1. For elements x, e and f in $\mathfrak{g}$,

(1) (x, e, f) is called *an S-triple* if

$$[x, e] = e, \ [x, f] = -f \qquad \text{and } [e, f] = x.$$

(2) In the above, x is called *the neutral element* and e (resp. f) is called *the nil-positive* (resp. *nil-negative*) *element* of the S-triple.

DEFINITION 3.2. An S-trinple (x, e, f) generates a complex subalgebra $\{x, e, f\}_C$ of $\mathfrak{g}$, isomorphic to $\mathfrak{sl}(2, \boldsymbol{C})$, which is called *a* TDS (*three-dimensional simple subalgebra*) *in* $\mathfrak{g}$ *associated to* (x, e, f).

Notations: 1) For $X \in \mathfrak{g}$, $\mathfrak{g}^X$ denotes the centralizer of X in $\mathfrak{g}$.

2) $G^C = \operatorname{Int} \mathfrak{g}$ is the group of all inner automorphisms of $\mathfrak{g}$, and $(G^C)^X$ is the centralizer of X in G^C.

3) We denote by $\frac{1}{2}\boldsymbol{Z}$ the set of all integers and half-integers.

Let e be a non-zero nilpotent elenent in $\mathfrak{g}$. Then e can be embedded in an S-triple (x, e, f) as a nil-positive element. We call x *a mono-semisimple element corresponding to* e. We remark here that a mono-semisimple element corresponding to e is never unique, and has the arbitrariness as follows: if x is a mono-semisimple element corresponding to e, then x added by an element in $\mathfrak{g}^e \cap [e, \mathfrak{g}]$ is also a mono-semisimple element corresponding to e. It is known (Kostant [8]) that an element x in $\mathfrak{g}$ is a mono-semisimple element corresponding to e, if and only if $x \in [e, \mathfrak{g}]$ and $[x, e]=e$. Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing x, and denote by Δ the non-zero root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Each root α in Δ determines the element H_α in $\mathfrak{h}$ by the relation $B(H_\alpha, H)=\alpha(H)$ for every $H \in \mathfrak{h}$. It is known from the representation theory of $\mathfrak{sl}(2, \boldsymbol{C})$ that the set of eigenvalues of $ad_{\mathfrak{g}}(x)$ forms a subset of $\frac{1}{2}\boldsymbol{Z}$. This concludes that the element x belongs to $\mathfrak{h}_R = \sum_{\alpha \in \Delta} \boldsymbol{R} H_\alpha$. For $j \in \frac{1}{2}\boldsymbol{Z}$, we set

$$\mathfrak{g}_j = \{X \in \mathfrak{g},\ ad(x)X = jX\}.$$

The space $\mathfrak{g}_j$ coincides with

$$\sum_{\substack{\alpha \in \Delta \\ \alpha(x)=j}} \mathfrak{g}^\alpha \qquad \text{if} \quad j \neq 0,$$

and with

$$\mathfrak{h} + \sum_{\substack{\alpha \in \Delta \\ \alpha(x)=0}} \mathfrak{g}^\alpha \qquad \text{if} \quad j=0,$$

where

$$\mathfrak{g}^\alpha = \{X \in \mathfrak{g};\ ad(H)X = \alpha(H)X \text{ for every } H \in \mathfrak{h}\}.$$

The following properties of $\mathfrak{g}_j$ are due to Kostant ([8]):

1) $\dim \mathfrak{g}_j = \dim \mathfrak{g}_{-j}$,

2) $\dim \mathfrak{g}^e = \dim \mathfrak{g}_0 + \dim \mathfrak{g}_{\frac{1}{2}}$,

3) $\mathfrak{g}^e \subset \sum_{j \geqq 0} \mathfrak{g}_j$,

4) $ad(e) \colon \mathfrak{g}_j \to \mathfrak{g}_{j+1}$ is injective if $j<0$, and $ad(e) \colon \mathfrak{g}_{j-1} \to \mathfrak{g}_j$ is surjective if $j>0$,

5) if $i+j \neq 0$, $\mathfrak{g}_i$ and $\mathfrak{g}_j$ are mutually orthogonal with respect to the Killing form B of $\mathfrak{g}$.

Let $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ be the fundamental root system for an arbitarily fixed lexicographic linear order in Δ. The set $\{\alpha_1, \cdots, \alpha_l\}$ forms a basis of $\mathfrak{h}^* = \mathrm{Hom}_C(\mathfrak{h}, \boldsymbol{C})$, and x has the following expression:

$$x = \sum_{i=1}^{l} \alpha_i(x) \varepsilon_i,$$

where $\{\varepsilon_1, \cdots, \varepsilon_l\}$ is the basis of $\mathfrak{h}$ dual to $\{\alpha_1, \cdots, \alpha_l\}$. We shall call $(\alpha_1(x), \cdots, \alpha_l(x))_\Pi$ *the characteristic of* x *with respect to* Π.

LEMMA 3.1. (Dynkin [4], Kostant [8]) *Choose such a linear order in Δ that the element x is contained in the positive Weyl chamber. Then the characteristic of x consists only of* 0, $\frac{1}{2}$ *and* 1.

PROOF. Each eigenspace $\mathfrak{g}_j$ of $ad_{\mathfrak{g}}(x)$ is a suitable sum of $\mathfrak{h}$ and $\mathfrak{g}^{\alpha}$'s ($\alpha \in \Delta$). It is enough to show that each root space $\mathfrak{g}^{\alpha_i}(1 \leqq i \leqq l)$ of a simple root appears only in $\mathfrak{g}_0$, $\mathfrak{g}_{\frac{1}{2}}$ and $\mathfrak{g}_1$. Since $\mathfrak{g}_j$ coincides with $[e, \mathfrak{g}_{j-1}]$ for $j \geqq \frac{1}{2}$, a root appearing in $\mathfrak{g}_j$ $\left(j \geqq \frac{3}{2}\right)$ is expressed as a sum of a root in $\mathfrak{g}_1$ and a root in $\mathfrak{g}_{j-1}$, both of which are positive roots because x belongs to the positive Weyl chamber. So a root which appears in $\mathfrak{g}_j$ can not be simple if $j \geqq \frac{3}{2}$. Thus every simple root belongs to $\mathfrak{g}_0$, $\mathfrak{g}_{\frac{1}{2}}$ and $\mathfrak{g}_1$. Q.E.D.

Next we define the subalgebra $\mathfrak{g}_e$ of $\mathfrak{g}$ by

$$\mathfrak{g}_e = \mathfrak{g}^e \cap \sum_{j>0} \mathfrak{g}_j = \mathfrak{g}^e \cap [e, \mathfrak{g}],$$

and we set $(G^C)_e$ the analytic subgroup of G^C generated by $\mathfrak{g}_e$. Then it follows from well-known facts about linear nilpotent Lie algebras that $(G^C)_e$ is closed, connected and simply connected, and that $\exp|\mathfrak{g}_e$ is a diffeomorphism of $\mathfrak{g}_e$ onto $(G^C)_e$. Further, Kostant ([8] Theorem 3.6) has proved that the following mappings are bijections:

$$\begin{array}{ccl}
gx & \in & x + \mathfrak{g}_e = \{x + X;\ X \in \mathfrak{g}_e\} \\
\uparrow & & \quad\uparrow \\
g & \in & (G^C)_e \\
\downarrow & & \quad\downarrow \\
(gx, e, gf) & \in & \text{the set of all } S\text{-triples containing } e \text{ as the nil-positive element.}
\end{array}$$

Using this bijection, we have

LEMMA 3.2. 1) *$(G^C)_e$ is a normal subgroup of $(G^C)^e$.*
2) *$(G^C)^e$ is the semi-direct product of $(G^C)_e$ and $(G^C)^e \cap (G^C)^x$.*

PROOF. Subgroups $(G^C)^e$, $(G^C)_e$ and $(G^C)^e \cap (G^C)^x$ are closed subgroups of G^C. We shall prove that i) each element $g \in (G^C)^e$ has the unique decomposition $g = g'g''$ where $g' \in (G^C)_e$ and $g'' \in (G^C)^e \cap (G^C)^x$, and that ii) $(G^C)^e \cap (G^C)^x$ normalizes $(G^C)_e$. Since (gx, e, gf) is an S-triple containing e as the nil-positive element, there exists $g' \in (G^C)_e$ satisfying $gx = g'x$ owing to the existence of the above bijection of $(G^C)_e$ onto $x + \mathfrak{g}_e$. By putting $g'' = g'^{-1}g$, we have $g'' \in (G^C)^e \cap (G^C)^x$ and $g = g'g''$. Thus an element $g \in (G^C)^e$ has a decomposition $g = g'g''$, where $g' \in (G^C)_e$ and $g'' \in (G^C)^e \cap (G^C)^x$. It is enough for the uniqueness of this decomposition to show that $[(G^C)^e \cap (G^C)^x] \cap (G^C)_e$ contains only the unit. An element g_0 in $[(G^C)^e \cap (G^C)^x] \cap (G^C)_e$ satisfies $gg_0x = gx$, which fact combined with the bijectiveness of the mapping $y \to yx$ of $(G^C)_e$ to $x + \mathfrak{g}_e$ proves $g = gg_0$. So $g_0 = 1$, and the uniqueness of the decomposition is proved. Next we shall prove ii). Each $g \in (G^C)^e \cap (G^C)^x$ stabilizes $\mathfrak{g}^e \cap \mathfrak{g}_j$ for every $j \in \frac{1}{2}\boldsymbol{Z}$, so the space $\mathfrak{g}_e$ is stable under the adjoint action of every $g \in (G^C)^e \cap (G^C)^x$. Thus we have proved ii) because of $(G^C)_e = \exp \mathfrak{g}_e$.

Q.E.D.

LEMMA 3.3. *If e is a (non-zero) nilpotent element in $\mathfrak{g}_R$, an S-triple (x, e, f) can be chosen in $\mathfrak{g}_R$.*

PROOF. Let (x', e, f') be an S-triple containing e as the nil-positive element. We set

$$x = \frac{1}{2}(x' + \sigma x') = \text{the real part of } x',$$

$$f = \frac{1}{2}(f' + \sigma f') = \text{the real part of } f'.$$

Then (x, e, f) is also an S-triple. Q.E.D.

§ 4. G-conjugate classess in $\mathfrak{g}_R$

Let θ be a Cartan involution of $\mathfrak{g}_R$, and $\mathfrak{g}_R = \mathfrak{k}_0 + \mathfrak{p}_0$ be the Cartan decomposition of $\mathfrak{g}_R$ associated to θ, where $\mathfrak{k}_0$ is a maximal compactly imbedded subalgebra of $\mathfrak{g}_R$. Let $\mathfrak{h}_0^1, \cdots, \mathfrak{h}_0^k$ be representatives of the G-conjugate classes of Cartan subalgebras of $\mathfrak{g}_R$. They can be chosen θ-stable and such that $\mathfrak{h}_-^i \subset \mathfrak{h}_-^1$, $\mathfrak{h}_+^i \subset \mathfrak{h}_+^k$ and $\dim \mathfrak{h}_+^i \leqq \dim \mathfrak{h}_+^{i+1}$ for every i, where $\mathfrak{h}_-^i = \mathfrak{h}_0^i \cap \mathfrak{k}_0$ (the toroidal part of $\mathfrak{h}_0^i$) and $\mathfrak{h}_+^i = \mathfrak{h}_0^i \cap \mathfrak{p}_0$ (the vector part of $\mathfrak{h}_0^i$). We set $\mathfrak{h}^i = (\mathfrak{h}_0^i)^C$ and $\mathfrak{h}_R^i = \sqrt{-1}\,\mathfrak{h}_-^i + \mathfrak{h}_+^i$. The non-zero root system Δ^i of $(\mathfrak{g}, \mathfrak{h}^i)$ admits the direct sum decomposition

$$\Delta^i = \Sigma_{\mathfrak{k}}^i \cup \Sigma_{\mathfrak{p}}^i \cup \Lambda^i,$$

where

$$\Sigma_{\mathfrak{k}}^i = \{\alpha \in \Delta^i ; \mathfrak{g}^\alpha \subset \mathfrak{k}\},$$

$$\Sigma_{\mathfrak{p}}^i = \{\alpha \in \Delta^i ; \mathfrak{g}^\alpha \subset \mathfrak{p}\}$$

and

$$\Lambda^i = \{\alpha \in \Delta^i ; \alpha | \mathfrak{h}_+^i \neq 0\}.$$

The set $\Sigma^i=\Sigma^i_{\mathfrak{k}}\cup\Sigma^i_{\mathfrak{p}}$ coincides with the set of all purely-imaginary roots in Δ^i (i.e., roots which vanish on $\mathfrak{h}^i_+$). A lexicographic order in $\mathfrak{h}^i_R$ compatible to $\mathfrak{h}^i_+$ induces a linear order in Δ^i and determines positive subsystems Δ^i_+ and Λ^i_+.

We put

$$\mathfrak{n}^i_0=\Big(\sum_{\alpha\in\Lambda^i_+}\mathfrak{g}^\alpha\Big)\cap\mathfrak{g}_R.$$

Notations: 1) Hereafter we write sometimes $\mathfrak{a}_-$, $\mathfrak{a}_+$, $\mathfrak{a}_0$, and $\mathfrak{a}$ instead of $\mathfrak{h}^k_-$, $\mathfrak{h}^k_+$, $\mathfrak{h}^k_0$ and $\mathfrak{h}^k$. That is, in this paper $\mathfrak{a}_0$ denotes the θ-stable Cartan subalgebra of $\mathfrak{g}_R$ with maximal vector part. A θ-stable Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_R$ is called *standard relative to* $\mathfrak{a}_0$ if the vector part of $\mathfrak{h}_0$ is a subspace of $\mathfrak{a}_+$.

2) For a θ-stable Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_R$ notations $\mathfrak{h}_+$, $\mathfrak{h}_-$ $\Sigma_{\mathfrak{k}}$, $\Sigma_{\mathfrak{p}}$, Λ, $\mathfrak{n}_0$ and so on are used to express ones defined in the same way as above.

PROPOSITION 4.1. *Let e be a (non-zero) nilpotent element in $\mathfrak{g}_R$.*

Then

1) *a mono-semisimple element corresponding to e can be chosen G-conjugate to an element in $\mathfrak{a}_+$, and*

2) *the element e is G-conjugate to an element in $\mathfrak{n}^k_0$.*

PROOF. 1) By Lemma 3.3, a mono-semisimple element x corresponding to e exists in $\mathfrak{g}_R$. Since x is a semisimple element of $\mathfrak{g}_R$, x can be imbedded into a standard (relative to $\mathfrak{a}$) θ-stable Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_R$ under the adjoint action of G, i.e., there exists $g\in G$ such that $gx\in\mathfrak{h}_0$. Since the eigenvalues of $ad_{\mathfrak{g}}(gx)$ are all real numbers, gx belongs to the vector part of $\mathfrak{h}_0$, which is a subspace of $\mathfrak{a}_+$. Thus we have $gx\in\mathfrak{a}_+$.

2) Replacing x and e by gx and ge, we can assume that the mono-semisimple element x belongs to $\mathfrak{a}_+$. And by the action of an element k in K (k is chosen in the normalizer of $\mathfrak{a}_+$ in K), x is transferred into the closure of the positive Weyl chamber in $\mathfrak{a}_+$. Since ke belongs to the 1-eigenspace of $ad_{\mathfrak{g}}(kx)$, we have

$$ke\in\Big(\sum_{\substack{\alpha\in\Delta^k\\ \alpha(kx)=1}}\mathfrak{g}^\alpha\Big)\cap\mathfrak{g}_R\subset\Big(\sum_{\alpha\in\Lambda^k_+}\mathfrak{g}^\alpha\Big)\cap\mathfrak{g}_R=\mathfrak{n}^k_0.$$

Q.E.D.

For the sake of simplicity, an S-triple (x, e, f) in $\mathfrak{g}$ is called standard with respect to the Iwasawa decomposition $\mathfrak{g}_R=\mathfrak{k}_0+\mathfrak{a}_++\mathfrak{n}_0$, if $x\in\mathfrak{a}_+$ and $e\in\mathfrak{n}_0$. Then, from Proposition 4.1, we have the following:

COROLLARY 4.2. *Every S-triple in $\mathfrak{g}_R$ is G-conjugate to a standard S-triple.*

LEMMA 4.3. *For a θ-stable Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_R$, the conditions* 1) *and* 2) *are equivalent*:

1) $\mathfrak{h}_0$ *has a maximal vector part,*
2) $\Sigma_{\mathfrak{p}}$ *is empty.*

PROOF. First we note that

$$\text{i)}\quad Z_{\mathfrak{g}}(\mathfrak{h}_+)=\mathfrak{h}+\sum_{\alpha\in\Sigma}\mathfrak{g}^{\alpha}$$

where $Z_{\mathfrak{g}}(\mathfrak{h}_+)$ denotes the centralizer of $\mathfrak{h}_+$ in $\mathfrak{g}$ and

$$\text{ii)}\quad \sum_{\alpha\in\Sigma}\mathfrak{g}^{\alpha}\quad\text{is}\quad \theta\text{-stable}.$$

These facts are seen easily from the definition of Σ. Now the condition 1) is equivalent to

$$Z_{\mathfrak{g}}(\mathfrak{h}_+)\cap\mathfrak{p}=\mathfrak{h}_+^C,$$

which is equivalent to

$$(\sum_{\alpha\in\Sigma}\mathfrak{g}^{\alpha})\cap\mathfrak{p}=\{0\}$$

because of i), and this is equivalent to

$$\sum_{\alpha\in\Sigma}\mathfrak{g}^{\alpha}\subset\mathfrak{k}$$

because of ii), and this is equivalent to

$$\Sigma_{\mathfrak{p}}=\phi(\text{empty set}).$$

Thus the statement of Lemma 4.3 has been proved. Q.E.D.

LEMMA 4.4. *A semisimple element H_0 can be imbedded by G-action into such a θ-stable Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_R$ that*

$$\alpha(H_0')\neq 0\quad \textit{for every } \alpha\in\Sigma_{\mathfrak{p}}$$

where H_0' is an element in $\mathfrak{h}_0$ G-conjugate to H_0.

PROOF. There exist an element g in G and an integer i $(1\leqq i\leqq k)$ such that $Ad(g)H_0$ belongs to $\mathfrak{h}_0^i$. Among the above g's and i's, we choose i as large as possible (in other words, dim $\mathfrak{h}_+^i$ is as large as possible under the condition that $\mathfrak{h}_0^i$ contains an element G-conjugate to H_0), and hereafter we fix i and g as such. We put

$$H_0'=Ad(g)H_0 \qquad \text{and } \mathfrak{h}_0=\mathfrak{h}_0^i,$$

and we shall prove that

$$\alpha(H_0')\neq 0\quad\text{for every}\quad \alpha\in\Sigma_{\mathfrak{p}}.$$

Now suppose that $\alpha(H_0')=0$ for some $\alpha \in \Sigma_{\mathfrak{p}}$ and fix a root $\alpha \in \Sigma_{\mathfrak{p}}$ as such. Since $\sigma\alpha=-\alpha(\alpha \in \Sigma_{\mathfrak{p}} \subset \Sigma)$, we can choose non-zero vectors $e_{\pm\alpha}$ in $\mathfrak{g}^{\pm\alpha}$ so that $e_\alpha+e_{-\alpha}$ may be σ-stable (i.e., $e_\alpha+e_{-\alpha} \in \mathfrak{g}_R$). We also note that $e_\alpha+e_{-\alpha} \in \mathfrak{p}_0$ since $\alpha \in \Sigma_{\mathfrak{p}}$. We set

$\mathfrak{h}_-'=$orthogonal complement of H_α in $\mathfrak{h}_-$ with respect to the Killing form B,

$$\mathfrak{h}_+'=\mathfrak{h}_+ + \boldsymbol{R}(e_\alpha+e_{-\alpha}),$$

and

$$\mathfrak{h}_0'=\mathfrak{h}_-'+\mathfrak{h}_+'.$$

The space $\mathfrak{h}_0'$ is a maximal abelian subalgebra of $\mathfrak{g}_R$, and so a (θ-stable) Cartan subalgebra of $\mathfrak{g}_R$, the dimension of whose vector part is equal to $\dim \mathfrak{h}_+ +1$. Moreover, H_0' belongs to $\mathfrak{h}_0'$ by the assumption that $\alpha(H_0')=0$. This contradicts the choice of i (the maximality of the vector part), and so we have

$$\alpha(H_0') \neq 0 \quad \text{for every} \quad \alpha \in \Sigma_{\mathfrak{p}}.$$

Thus the proof is accomplished. Q.E.D.

Proposition 4.5. *An element X in $\mathfrak{g}_R$ is G-conjugate to an element in $\mathfrak{h}_0^i+\mathfrak{n}_0^i$ for some i.*

Proof. X has a unique decomposition $X=H+e$ where H (resp. e) is a semisimple (resp. nilpotent) element in $\mathfrak{g}_R$ and $[H, e]=0$. The element H is transferred into some $\mathfrak{h}_0^i$ by the adjoint action of G. We choose i as large as possible. Hereafter, for the sake of simplicity, we assume that H itself is contained in $\mathfrak{h}_0^i$. Then, due to Lemma 4.4, we have

$$\alpha(H) \neq 0 \quad \text{for every} \quad \alpha \in \Sigma_{\mathfrak{p}}^i.$$

So if we set

$$\Delta'=\{\alpha \in \Delta^i;\ \alpha(H)=0\},$$

Δ' is included in $\Sigma_{\mathfrak{k}}^i \cup \Lambda^i$. The centralizer $\mathfrak{g}^H$ of H in $\mathfrak{g}$ is expressed as

$$\mathfrak{g}^H=\mathfrak{h}^i+\sum_{\alpha\in\Delta'}\mathfrak{g}^\alpha,$$

which is a reductive Lie algebra with the center

$$\mathfrak{z}=\sum_{\substack{\alpha\in\Delta^i \\ \alpha|\mathfrak{h}'=0}} \boldsymbol{C}H_\alpha$$

and the semisimple part

$$\mathfrak{h}'+\sum_{\alpha\in\Delta'}\mathfrak{g}^\alpha,$$

where

$$\mathfrak{h}'=\sum_{\alpha\in\Delta'}\boldsymbol{C}H_\alpha.$$

Let $\mathfrak{l}_0$ denote the semisimple part of $(\mathfrak{g}_R)^H$, and L the analytic subgroup of G generated by $\mathfrak{l}_0$.

Since

$$\mathfrak{l}_0=\mathfrak{h}_0'+(\sum_{\alpha\in\Delta'}\mathfrak{g}^{\alpha})\cap\mathfrak{g}_R \text{ (where } \mathfrak{h}_0'=\mathfrak{h}'\cap\mathfrak{g}_R)$$

and

$$\Delta'\cap\Sigma_{\mathfrak{p}}=\phi,$$

$\mathfrak{h}_0'$ is a Cartan subalgebra of $\mathfrak{l}_0$ with maximal vector part (Lemma 4.3). And since e belongs to the semisimple part $\mathfrak{l}_0$ of $(\mathfrak{g}_R)^H$, e is transferred into $(\sum_{\substack{\alpha\in\Delta_+^i\\ \alpha(H)=0}}\mathfrak{g}^{\alpha})\cap\mathfrak{g}_R$ by the adjoint action of an element g in L. Thus we have proved that

$$Ad(g)X\in\mathfrak{h}_0^i+\mathfrak{n}_0^i.$$

Q.E.D.

§5. The TDS and w-polarizations

In this section we make an investigation into w-polarizations of nilpotent elements using the TDS.

PROPOSITION 5.1. *Let x be a mono-semisimple element corresponding to a (non-zero) nilpotent element e in $\mathfrak{g}_R$. And assume that the characteristic of x consists only of integers. Then the subalgebra $\sum_{j\geqq 0}\mathfrak{g}_j$ of $\mathfrak{g}$ is a w-polarization of e.*

PROOF. We set $\mathfrak{p}=\sum_{j\geqq 0}\mathfrak{g}_j$. We shall prove that $\mathfrak{p}$ satisfies the conditions i) and ii) of a w-polarization.

i) The orthogonality of e and $[\mathfrak{p},\mathfrak{p}]$ is true because $e\in\mathfrak{g}_1$ and $B(\mathfrak{g}_i,\mathfrak{g}_j)=\{0\}$ for $i+j\neq 0$.

ii) The calculation of the dimension of $\mathfrak{p}$:

$$\dim\mathfrak{g}-\dim\mathfrak{p}=\dim\sum_{j<0}\mathfrak{g}_j,$$

and

$$\dim\mathfrak{p}-\dim\mathfrak{g}^e=\dim\mathfrak{p}-\dim\mathfrak{g}_0=\dim\sum_{j>0}\mathfrak{g}_j.$$

Thus we have

$$\dim\mathfrak{g}-\dim\mathfrak{p}=\dim\mathfrak{p}-\dim\mathfrak{g}^e$$

since $\dim \mathfrak{g}_j = \dim \mathfrak{g}_{-j}$.

Q.E.D.

PROPOSITION 5.2. *Let x be a mono-semisimple element corresponding to a (non-zero) nilpotent element e in $\mathfrak{g}_R$. Assume that the characteristic of x contains half-integers and that there exists such a subspace V of $\mathfrak{g}_{-\frac{1}{2}}$ that 1) V is an abelian subalgebra of $\mathfrak{g}$, 2) V is stable under the adjoint action of each element in $\mathfrak{g}_0$ and 3) the dimension of V is a half of $\dim \mathfrak{g}_{-\frac{1}{2}}$. Then the subspace $\sum_{j\geqq 0} \mathfrak{g}_j + V$ is a w-polarization of e.*

PROOF. We set $\mathfrak{p} = \sum_{j\geqq 0} \mathfrak{g}_j + V$. By the assumption 1) and 2) on V, $\mathfrak{p}$ is a subalgebra of $\mathfrak{g}$. And the same discussion as in the proof of Proposition 5.1 shows that $\mathfrak{p}$ satisfies conditions i) of a w-polarization. So we need only to calculate the dimension of $\mathfrak{p}$:

$$\begin{aligned}
\dim \mathfrak{g} - \dim \mathfrak{p} &= \frac{1}{2}\dim \mathfrak{g}_{-\frac{1}{2}} + \dim \sum_{j\leqq -1} \mathfrak{g}_j \\
&= \frac{1}{2}\dim \mathfrak{g}_{-\frac{1}{2}} + \dim \sum_{j\geqq 1} \mathfrak{g}_j,
\end{aligned}$$

$$\begin{aligned}
\dim \mathfrak{p} - \dim \mathfrak{g}^e &= \dim \mathfrak{p} - (\dim \mathfrak{g}_0 + \dim \mathfrak{g}_{\frac{1}{2}}) \\
&= (\dim \sum_{j\geqq 0} \mathfrak{g}_j + \frac{1}{2}\dim \mathfrak{g}_{-\frac{1}{2}}) \\
&\quad - (\dim \mathfrak{g}_0 + \dim \mathfrak{g}_{\frac{1}{2}}) \\
&= \dim \sum_{j\geqq 1} \mathfrak{g}_j + \frac{1}{2}\dim \mathfrak{g}_{-\frac{1}{2}}.
\end{aligned}$$

Thus we have

$$\dim \mathfrak{g} - \dim \mathfrak{p} = \dim \mathfrak{p} - \dim \mathfrak{g}^e,$$

and $\mathfrak{p}$ is a w-polarization of e. Q.E.D.

PROPOSITION 5.3. *A w-polarization $\mathfrak{p}$ of a nilpotent element e in $\mathfrak{g}_R$, if it exists, contains mono-semisimple elements corresponding to e.*

PROOF. Let x be a mono-semisimple element corresponding to e, and $\mathfrak{n}$ the nil-radical of $\mathfrak{p}$. By Theorem 2.2, we have

$$\begin{aligned}
B(x, \mathfrak{n}) &= B(x, [e, \mathfrak{p}]) \\
&= B([x, e], \mathfrak{p}) = B(e, \mathfrak{p}) = \{0\}.
\end{aligned}$$

Thus we have

$$x \in \mathfrak{n}^{\perp} = \mathfrak{p}.$$

Q.E.D.

The following is an easy consequence of the above proposition:

COROLLARY 5.4. *Let $\mathfrak{p}$ be a* w*-polarization of a nilpotent element e in $\mathfrak{g}_R$, x a mono-semisimple element corresponding to e, and $\mathfrak{g}_j$ the j-eigenspace of $ad_{\mathfrak{g}}(x)$. Then we have a direct decomposition (as vector spaces)*

$$\mathfrak{p} = \sum_{j \in \frac{1}{2} Z} (\mathfrak{g}_j \cap \mathfrak{p}).$$

PROPOSITION 5.5. *Let e be a nilpotent element in $\mathfrak{g}_R$ written in the following form: $e = \sum_{\alpha \in \Phi} e_\alpha$, where Φ is a subset of a fundamental root system $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ of $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$, and e_α is a non-zero element in the root space $\mathfrak{g}^\alpha$. Then any w-polarization of e, if it exists, contains $\mathfrak{h}$.*

PROOF. First we shall show that there exists a mono-semisimple element in $\mathfrak{h}$ corresponding to e. Choose the element $e_{-\alpha}$ in $\mathfrak{g}^{-\alpha} (\alpha \in \Phi)$ so that $B(e_\alpha, e_{-\alpha}) = 1$, and we put

$$x = \sum_{\alpha \in \Phi} c_\alpha H_\alpha \qquad (c_\alpha \in \boldsymbol{C}).$$

The element x belongs to $[e, \mathfrak{g}]$ because $x = [e, \sum_{\alpha \in \Phi} c_\alpha e_{-\alpha}]$. So if x satisfies the relation $[x, e] = e$, x is a mono-semisimple element corresponding to e as is discussed in §3. Since $[x, e] = \sum_{\alpha, \beta \in \Phi} (\alpha, \beta) c_\beta e_\beta$, the relation $[x, e] = e$ is equivalent to a system of linear equations:

$$\sum_{\beta \in \Phi} (\alpha, \beta) c_\beta = 1 \qquad \text{for every } \alpha \in \Phi,$$

where (α, β) denotes the inner product of H_α and H_β by the Killing form B. For the sake of simplicity, by an appropriate rearrangement of $\alpha_1, \cdots, \alpha_l$, we assume that $\Phi = \{\alpha_1, \cdots, \alpha_k\}$, and write c_i instead of $c_{\alpha_i} (1 \leqq i \leqq k)$. Then the above linear equations are rewritten as

$$\sum_{j=1}^{k} (\alpha_i, \alpha_j) c_j = 1 \qquad \text{for } i = 1, \cdots, k.$$

Since the $k \times k-$ matrix $((\alpha_i, \alpha_j))_{1 \leqq i, j \leqq k}$ is strictly positive definite, the above system of linear equations has a unique solution, and $c_1, \cdots, c_k$ are real numbers. With such $c_1, \cdots, c_k$, we have $[x, e] = e$, which, combined with the fact that $x \in [e, \mathfrak{g}]$, assures us that x is a mono-semisimple element corresponding to e.

Next we shall prove that $\mathfrak{p}\cap\sum_{\alpha\in\Phi}\mathfrak{g}^{-\alpha}=\{0\}$. We put

$$\Psi=\Pi-\Phi=\{\alpha_{k+1},\cdots,\alpha_l\},$$

$$\Delta'=\{\alpha\in\Delta;\ \alpha(x)=1\}$$

and

$$\Delta''=\Delta'-\Phi.$$

Using the dual basis $\{\varepsilon_1,\cdots,\varepsilon_l\}$ of $\{\alpha_1,\cdots,\alpha_l\}$, x has the expansion:

$$x=\sum_{i=1}^{k}\varepsilon_i+\sum_{i=k+1}^{l}c_i\varepsilon_i.$$

By Corollary 5.4, we have $\mathfrak{p}=\sum_{j\in\frac{1}{2}\boldsymbol{Z}}(\mathfrak{p}\cap\mathfrak{g}_j)$. We shall make here some investigation into $\mathfrak{p}\cap\mathfrak{g}_{-1}$. The space $\mathfrak{h}_\Psi=\sum_{i=k+1}^{l}\boldsymbol{C}\varepsilon_i$ is a subspace of $\mathfrak{g}^e\cap\mathfrak{g}_0$, and it is obvious from the definition of Ψ and Δ'' that

$$\{\text{weights of } \mathfrak{h}_\Psi \quad \text{on} \quad \sum_{\alpha\in\Delta''}\mathfrak{g}^{-\alpha}\}\not\ni 0,$$

$$\{\text{weights of } \mathfrak{h}_\Psi \quad \text{on} \quad \sum_{\alpha\in\Phi}\mathfrak{g}^{-\alpha}\}=\{0\}$$

and

$$\mathfrak{g}_{-1}=\sum_{\alpha\in\Delta''}\mathfrak{g}^{-\alpha}+\sum_{\alpha\in\Phi}\mathfrak{g}^{-\alpha}\quad\text{(direct sum as vector spaces).}$$

This implies, with $\mathfrak{h}_\Psi\subset\mathfrak{g}^e\subset\mathfrak{p}$, that the vector space $\mathfrak{p}\cap\mathfrak{g}_{-1}$ is the direct sum of subspaces $(\mathfrak{p}\cap\sum_{\alpha\in\Delta''}\mathfrak{g}^{-\alpha})$ and $(\mathfrak{p}\cap\sum_{\alpha\in\Phi}\mathfrak{g}^{-\alpha})$. Assume that $\mathfrak{p}\cap\sum_{\alpha\in\Phi}\mathfrak{g}^{-\alpha}\neq\{0\}$. Then one can find a non-zero element $X=\sum_{i=1}^{k}r_ie_{-\alpha_i}$ in $\mathfrak{p}\cap\sum_{\alpha\in\Phi}\mathfrak{g}^{-\alpha}$. We set

$$H=\sum_{i=1}^{k}r_iH_{\alpha_i}(=[e,X]\in\mathfrak{p}).$$

The element $(adH)X$ is expressed as

$$(adH)X=-\sum_{i,j=1}^{k}r_ir_j(\alpha_i,\alpha_j)e_{-\alpha_j},$$

which is not zero because $k\times k-$ matrix $((\alpha_i,\alpha_j))_{1\leq i,j\leq k}$ is strictly positive definite. Choosing a suitable linear combination of $X,(adH)X,(adH)^2X,\cdots,(adH)^{k-1}X$ (these element are contained in $\mathfrak{p}$), there exists an integer $s(1\leqq s\leqq k)$ and a subsequence $(q_1,\cdots,q_s)$ of $(1,\cdots,k)$ satisfying that

1) $r_{q_i}\neq 0$ for every $1\leqq i\leqq s$,

2) $\alpha_{q_1}(H)=\cdots=\alpha_{q_s}(H)\neq 0$,

and

3) $\sum_{i=1}^{s} r_{q_i} e_{-\alpha_i} \in \mathfrak{p}$

we put $X' = \sum_{i=1}^{s} r_{q_i} e_{-\alpha_i}$ and $H' = [e, X']$. Repeating the above process, we finally obtain elements X and H in $\mathfrak{p}$ with the following properties:

i) $X = \sum_{i=1}^{t} r_{p_i} e_{-\alpha_{p_i}}$ is a non-zero element in $\mathfrak{p}$, where $(p_1, \cdots, p_t)$ is a subsequence of $(1, \cdots, k)$,

ii) $H = [e, X] = \sum_{i=1}^{t} r_{p_i} H_{p_i}$,

and

iii) $\beta_{p_1}(H) = \cdots = \beta_{p_t}(H) \neq 0$

The condition iii) can be expressed as

$$\sum_{i=1}^{t} r_{p_i} (\alpha_{p_i}, \alpha_{p_j}) = r \qquad (\text{for } j = 1, \cdots, t),$$

where $r = \beta_{p_1}(H)$. We can assume that r is a positive number by replacing X, if necessary, by a scalar multiple of X. We set

$$r_{p_i}^{*} = \frac{1}{2} r_{p_i} (\alpha_{p_i}, \alpha_{p_i})$$

and

$$\alpha_{p_i}^{*} = 2\alpha_{p_i} (\alpha_{p_i}, \alpha_{p_i})^{-1}$$

for $i = 1, \cdots, t$. Then the above condition iii) becomes

$$\sum_{i=1}^{t} r_{p_i}^{*} (\alpha_{p_i}^{*}, \alpha_{p_j}) = r \qquad (\text{for } j = 1, \cdots, t).$$

In the matrix forms, this becomes

$$(r_{p_1}^{*}, \cdots, r_{p_t}^{*}) A = (r, \cdots, r),$$

where

$$A = \begin{pmatrix} (\alpha_{p_1}^{*}, \alpha_{p_1}) \cdots\cdots (\alpha_{p_1}^{*}, \alpha_{p_t}) \\ \vdots \qquad\qquad \vdots \\ (\alpha_{p_t}^{*}, \alpha_{p_1}) \cdots\cdots (\alpha_{p_t}^{*}, \alpha_{p_t}) \end{pmatrix}.$$

So we have

$$(r_{p_1}^{*}, \cdots, r_{p_t}^{*}) = (r, \cdots, r) A^{-1}.$$

Since r is positive by the assumption on r and all matrix elements of A^{-1} (the inverse of a Cartan matrix) are non-negative real numbers, every $r^*_{p_i}(1 \leqq i \leqq t)$ is a positive number. So $\sum_{i=1}^{t} r_{p_i}$ is also a positive number. On the other hand, by Theorem 2.2, $B(e, \mathfrak{p})=\{0\}$ is valid. So $B(e, X)=0$, i.e., $\sum_{i=1}^{t} r_{p_i}=0$, which contradicts the above. This contradiction has arisen from the assumption that $\mathfrak{p} \cap \sum_{\alpha \in \Phi} \mathfrak{g}^{-\alpha}$ contains a non-zero element. Thus $\mathfrak{p} \cap \sum_{\alpha \in \Phi} \mathfrak{g}^{-\alpha}=\{0\}$, and we have

$$\mathfrak{p} \cap \mathfrak{g}_{-1}=\mathfrak{p} \cap \sum_{\alpha \in \Delta''} \mathfrak{g}^{-\alpha}.$$

Since e is in $\sum_{\alpha \in \Phi} \mathfrak{g}^{\alpha}$ and the set Φ is disjoint with Δ'', each vector in the space $[e, \sum_{\alpha \in \Delta''} \mathfrak{g}^{-\alpha}]$ has no $\mathfrak{h}$-component, i.e., $[e, \sum_{\alpha \in \Delta''} \mathfrak{g}^{-\alpha}]$ is a subspace of $\sum_{\alpha \in \Delta} \mathfrak{g}^{\alpha}$. So we have $[e, \mathfrak{p} \cap \mathfrak{g}_{-1}]=[e, \mathfrak{p} \cap \sum_{\alpha \in \Delta''} \mathfrak{g}^{-\alpha}] \subset [e, \sum_{\alpha \in \Delta''} \mathfrak{g}^{-\alpha}] \subset \sum_{\substack{\alpha \in \Delta \\ \alpha(x)=0}} \mathfrak{g}^{\alpha}$.

Denoting by the suffix "$\perp$" (resp. "$\perp\!\!\!\perp$") placed on the right-hand shoulder the orthogonal complement in $\mathfrak{g}$ (resp. $\mathfrak{g}_0$) with respect to B, we have

$$\begin{aligned}\mathfrak{h} &= \Big(\sum_{\substack{\alpha \in \Delta \\ \alpha(x)=0}} \mathfrak{g}^{\alpha}\Big)^{\perp\!\!\!\perp} \subset [e, \mathfrak{p} \cap \mathfrak{g}_{-1}]^{\perp\!\!\!\perp} \\ &= ([e, \mathfrak{p}] \cap \mathfrak{g}_0)^{\perp\!\!\!\perp} = [e, \mathfrak{p}]^{\perp} \cap \mathfrak{g}_0 = \mathfrak{p} \cap \mathfrak{g}_0.\end{aligned}$$

Thus $\mathfrak{h} \subset \mathfrak{p}$ is proved. Q.E.D.

DEFINITION. A nilpotent element e in $\mathfrak{g}$ is called *principal nilpotent* if $\dim \mathfrak{g}^e =$ rank of $\mathfrak{g}$.

It is known from the theory of Lie algebras that $\mathfrak{g}_R$ contains a principal nilpotent element in $\mathfrak{g}$ if and only if there exist no purely-imaginary roots in the root system of $\mathfrak{g}$ with respect to the complexification $\mathfrak{h}$ of a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_R$ with maximal vector part.

COROLLARY 5.6. *Assume that $\mathfrak{g}_R$ contains a principal nilpotent element e of $\mathfrak{g}$. Then e has a unique* w-*polarization $\mathfrak{p}$. Moreover $\mathfrak{p}$ is a real polarization of e and a Borel subalgebra of $\mathfrak{g}$.*

PROOF. Due to Kostant ([8]), we can choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a fundamental root system $\Pi=\{\alpha_1, \cdots, \alpha_l\}$ so that e may be expressed as $e=\sum_{i=1}^{l} e_{\alpha_i}$. As is shown in the proof of Proposition 5.5, $x=\sum_{i=1}^{l} \varepsilon_i$ is a mono-semisimple element corresponding to e. So, by Proposition 5.1, $\sum_{j \geqq 0} \mathfrak{g}_j$ is a w-polarization of e, and the existence of w-polarizations of e is thus proved.

Now let $\mathfrak{p}$ be any w-polarization of e. By Proposition 5.5, $\mathfrak{p}$ includes the

Cartan subalgebra $\mathfrak{h}$, which coincides with $\mathfrak{g}_0$ by the regularity of x. Considering also the fact that $e \in \mathfrak{p}$ and $[e, \mathfrak{g}_j]=\mathfrak{g}_{j+1}$ $(j \geqq 0)$, we have $\sum_{j \geqq 0} \mathfrak{g}_j \subset \mathfrak{p}$. By the condition ii) (as to the dimension) of w-polarizations, $\mathfrak{p}$ must coincides with $\sum_{j \geqq 0} \mathfrak{g}_j$. So the uniqueness of w-polarizations is proved.

For the w-polarization $\mathfrak{p}$ of e, subalgebras $\sigma\mathfrak{p}$ and $Ad(g)\mathfrak{p}$ $(g \in (G^C)^e)$ are also w-polarizations of e, which coincide with $\mathfrak{p}$ by the uniqueness. Thus $\mathfrak{p}$ is a real polarization.

Since $x=\sum_{i=1}^{l} \varepsilon_i$, we have $\sum_{j \geqq 0} \mathfrak{g}_j = \mathfrak{h} + \sum_{\alpha>0} \mathfrak{g}^{\alpha}$. So $\mathfrak{p}$ is a Borel subalgebra of $\mathfrak{g}$. Q.E.D.

§6. Some examples

From the point of view of w-polarizations, simple Lie algebras of type (A) have a distinct property from those of (B) (C) (D) (E) (F) or (G). In case of type (A), every element has a w-polarization, while in other cases, the existence of w-polarizations does not necessarily hold. The number of w-polarizations or admissible polarizations is somewhat complicated (Examples 6.2–6.4), and it seems to be a difficult problem to find out or to control all the w-polarizations or admissible w-polarizations for an arbitrarily given element in $\mathfrak{g}_R$.

PROPOSITION 6.1. *Assume that $\mathfrak{g}_R$ is a real semisimple Lie algebra of type (A) (i.e., all the simple factor of $\mathfrak{g}_R$ are simple Lie algebras of type (A)). Then every (non-zero) nilpotent element e in $\mathfrak{g}_R$ has a w-polarization.*

PROOF. By Proposition 2.6, we can assume that $\mathfrak{g}_R$ is simple. Let x be a mono-semisimple element corresponding to e, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ containing x. Let $\Pi=\{\alpha_1,\cdots,\alpha_l\}$ be a simple root system, arranged according to the Dynkin diagram. (So any positive root $\alpha \in \Delta_+$ can be written $\alpha=\alpha_i+\cdots+\alpha_j$.) The element x may be assumed contained in the closure of the positive Weyl chamber. Let $(a_1,\cdots,a_l)$ be the characteristic of x with respect to Π:

$$x=\sum_{i=1}^{l} a_i \varepsilon_i$$

where $\{\varepsilon_1,\cdots,\varepsilon_l\}$ is the basis of $\mathfrak{h}$ dual to $\alpha_1,\cdots,\alpha_l$. We put

$$m=\text{the number of } 1 \text{ in } (a_1,\cdots,a_l),$$

and $(t_1,\cdots,t_m)$ be the subsequence of $(a_1,\cdots,a_l)$ such that

$$a_{t_j}=1 \qquad (1 \leqq j \leqq m).$$

For the convenience of notations, we set

$$t_0=0 \quad \text{and} \quad t_{m+1}=l+1,$$

and

$$C_j=(a_i)_{t_j<i<t_{j+1}} \qquad \text{for } j=0,\cdots,m.$$

Let p_j be the multiplicity of $\frac{1}{2}$ in C_j, and $(s_1,\cdots,s_{p_j})$ the subsequence of C_j such that $a_{s_i}=\frac{1}{2}$ $(1\leqq i\leqq p_j)$. For every $j(0\leqq j\leqq m)$, we set

$$\mu_j^+=\sum_{i=\text{even}}\varepsilon_{s_i},$$

$$\mu_j^-=\sum_{i=\text{odd}}\varepsilon_{s_i},$$

$$A_j^0=\left\{\alpha\in\Delta;\ \alpha(x)=\frac{1}{2},\ \alpha(\mu_j^+)=\frac{1}{2}\right\},$$

$$A_j^1=\left\{\alpha\in\Delta;\ \alpha(x)=\frac{1}{2},\ \alpha(\mu_j^-)=\frac{1}{2}\right\}.$$

The following statements are obvious from the fact that $e\in\mathfrak{g}_1$ and $ad(e)$ is a linear isomorphism of $\mathfrak{g}_{-\frac{1}{2}}$ onto $\mathfrak{g}_{\frac{1}{2}}$:

1) $p_j=0$ or $p_j\geqq 2$,

2) The cardinal number of the set A_j^0 is equal to that of A_j^1.

We set

$$A^i=\bigcup_{j=0}^m A_j^i,$$

$$\mathfrak{g}^i_{-\frac{1}{2}}=\sum_{\alpha\in A^i}\mathfrak{g}^{-\alpha} \qquad (i=1,2).$$

Then it is easily seen from our way of construction that $\mathfrak{g}^i{}_{-\frac{1}{2}}$ $(i=1,2)$ is an abelian subalgebra of $\mathfrak{g}$, that the space $\mathfrak{g}_{-\frac{1}{2}}$ is the direct sum of $\mathfrak{g}^1{}_{-\frac{1}{2}}$ and $\mathfrak{g}^2{}_{-\frac{1}{2}}$, and that $\mathfrak{g}^i{}_{-\frac{1}{2}}$ $(i=1,2)$ is stable under the adjoint action of $\mathfrak{g}_0$. So, by Proposition 5.2, $\mathfrak{p}^i=\sum_{j\geqq 0}\mathfrak{g}_j+\mathfrak{g}^i{}_{-\frac{1}{2}}$ $(i=1,2)$ is a w-polarization of e.

Q.E.D.

COROLLARY 6.2. *Every element in a semisimple Lie algebra of type* (A) *has a* w-*polarization.*

PROOF. Each element X in $\mathfrak{g}_R$ has the unique decomposition $X=H+e$, where H (resp. e) is a semisimple (resp. nilpotent) element in $\mathfrak{g}_R$ and $[H, e]=0$. Since the semisimple part of $\mathfrak{g}^H$ is also semisimple of type (A), X has a w-polarization by Theorem 2.5 and Proposition 6.1. Q.E.D.

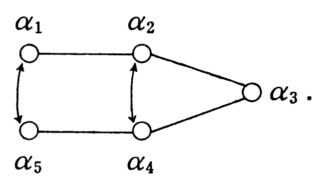
COROLLARY 6.3. *Every nilpotent element in a semisimple Lie algebra of type* (AI) *and* (AII) *has an admissible w-polarization.*

PROOF. The Satake diagram of a simple Lie algebra of type (AI) or (AII) does not contain arrows. When $\mathfrak{g}_R$ is of type (AI) or (AII), A_j^i ($j=0,\cdots, m$ and $i=0, 1$) is σ-stable. So every nilpotent element in $\mathfrak{g}_R$ has a real w-polarization. Q.E.D.

The following example shows that a nilpotent element in a simple Lie algebra of type (AIII) has not necessarily an admissible w-polarization.

Example 6.1. $\mathfrak{g}_R=\mathfrak{su}(3, 3)$

The Satake diagram of $\mathfrak{g}_R$ is



We put

$$e=e_{\alpha_1}+e_{\alpha_5},$$

$$x=\varepsilon_1+\varepsilon_5-\frac{1}{2}(\varepsilon_2+\varepsilon_4).$$

Then x is a mono-semisimple element corresponding to e, and the characteristic of x is $(1, -\frac{1}{2}, 0, -\frac{1}{2}, 1)$. For the sake of simplicity, we express a root $\alpha=\sum_{i=1}^{5} a_i\alpha_i$ by

$$(a_1,\cdots, a_5) \qquad \text{if } \alpha \text{ is positive, and}$$

$$-(-a_1,\cdots, -a_5) \quad \text{if } \alpha \text{ is negative.}$$

We set

$$\Delta_1 \;=\{(10000), (00001), (11111), -(01110)\}.$$

$$\Delta^1_{\frac{1}{2}}=\{(11000), (11100), -(00010), -(00110)\},$$

$$\Delta^2_{\frac{1}{2}}=\{-(01000), -(01100), (00011), (00111)\},$$

$$\Delta_{\frac{1}{2}} \quad = \Delta^1_{\frac{1}{2}} \cup \Delta^2_{\frac{1}{2}},$$

$$\Delta_0 \quad = \{\pm(00100), \pm(11110), \pm(01111)\}.$$

Then

$$\mathfrak{g}_i = \{0\} \qquad \text{if} \quad |i| \geqq \frac{3}{2}$$

$$\mathfrak{g}_i = \sum_{\alpha \in \Delta_i} \mathfrak{g}^{\alpha} \qquad \text{if} \quad i = \frac{1}{2} \text{ or } 1,$$

$$\mathfrak{g}_0 = \mathfrak{h} + \sum_{\alpha \in \Delta_0} \mathfrak{g}^{\alpha},$$

$$\mathfrak{g}_i = \sum_{\alpha \in \Delta_{-i}} \mathfrak{g}^{-\alpha} \qquad \text{if} \quad i = -\frac{1}{2}, -1.$$

Let $\mathfrak{p}$ be a w-polarization of e. From $\mathfrak{h} \subset \mathfrak{p}$ (Proposition 5.5) and $\mathfrak{g}^e \subset \mathfrak{p}$, we have

$$\mathfrak{p} \supset \sum_{j \geqq 0} \mathfrak{g}_j .$$

And by the proof of Proposition 5.5, we have

$$\mathfrak{p} \cap (\mathfrak{g}^{-\alpha_1} + \mathfrak{g}^{-\alpha_5}) = \{0\}.$$

So, if we suppose $\mathfrak{p} \cap \mathfrak{g}_{-1} \neq \{0\}$, $\mathfrak{p}$ includes $\mathfrak{g}^{-(11111)}$ or $\mathfrak{g}^{(01110)}$. If $\mathfrak{g}^{-(11111)} \subset \mathfrak{p}$, then $\mathfrak{g}^{-\alpha_1} = [\mathfrak{g}^{(01111)}, \mathfrak{g}^{-(11111)}] \subset \mathfrak{p}$, which contradicts the fact that $\mathfrak{p} \cap \mathfrak{g}^{-\alpha_1} = \{0\}$. If $\mathfrak{g}^{(01110)} \subset \mathfrak{p}$, then

$$\mathfrak{g}^{-\alpha_1} = [\mathfrak{g}^{-(11110)}, \mathfrak{g}^{(01110)}] \subset \mathfrak{p},$$

which is also a contradiction. Hence $\mathfrak{g}_{-1} \cap \mathfrak{p} = \{0\}$, and by the condition ii) of polarizations, we have

$$\dim(\mathfrak{p} \cap \mathfrak{g}_{-\frac{1}{2}}) = \frac{1}{2} \dim \mathfrak{g}_{-\frac{1}{2}} .$$

We set

$$V^i = \sum_{\alpha \in \Delta^i_{\frac{1}{2}}} \mathfrak{g}^{-\alpha} \qquad (i=1, 2).$$

Then $V^i (i=1, 2)$ is an abelian subalgebra of $\mathfrak{g}$ and $ad(\mathfrak{g}_0)$-irreducible, and $\dim V^i = \frac{1}{2} \dim \mathfrak{g}_{-\frac{1}{2}}$. So, by Proposition 5.2,

$$\mathfrak{p}_i = \sum_{j \geqq 0} \mathfrak{g}_j + V^i \qquad (i=1, 2)$$

is a w-polarization of e, and $\mathfrak{p}$ must coincide with $\mathfrak{p}_1$ or $\mathfrak{p}_2$. Since $\sigma\alpha_i=\alpha_{5-i}$ $(1\leqq i\leqq 5)$, we have $\sigma\mathfrak{p}_1=\mathfrak{p}_2$, so we have

$$\mathfrak{p}_1+\sigma\mathfrak{p}_1=\mathfrak{p}_2+\sigma\mathfrak{p}_2=\sum_{j\geqq-\frac{1}{2}}\mathfrak{g}_j,$$

which is not a subalgebra. Thus a w-polarization of e cannot be admissible.

In the following examples, we fix a θ-stable Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_R$ with maximal vector part, where θ is the Cartan involution associated to a fixed Cartan decomposition $\mathfrak{g}_R=\mathfrak{k}_0+\mathfrak{p}_0$. In the non-zero root system Δ of $\mathfrak{g}$ with respect to the complexification $\mathfrak{h}$ of $\mathfrak{h}_0$, is introduced a lexicographic linear order compatible to the vector part $\mathfrak{h}_+$ of $\mathfrak{h}_0$. Let Δ_+ be the set of all positive roots and $\Pi=\{\alpha_1,\cdots,\alpha_l\}$ the fundamental system of Δ_+. We set

$$\Lambda_+=\{\alpha\in\Delta_+;\ \alpha|\mathfrak{h}_+\neq 0\},$$

and

$$\mathfrak{g}^\alpha=\{X\in\mathfrak{g};\ ad(H)X=\alpha(H)X\quad\text{for every}\quad H\in\mathfrak{h}\},$$

for $\alpha\in\Delta$. We choose $e_\alpha\in\mathfrak{g}^\alpha$ (for $\alpha\in\Delta$) such that

$$B(e_\alpha,e_{-\alpha})=1,\quad\text{and}\quad\sigma e_\alpha=e_{\sigma\alpha},$$

where $\sigma\alpha\in\Delta$ is defined by

$$(\sigma\alpha)(H)=\overline{\alpha(\sigma H)}\quad\text{for every}\quad H\in\mathfrak{h}.$$

Let $\{\varepsilon_1,\cdots,\varepsilon_l\}$ be the basis of $\mathfrak{h}$ dual to $\{\alpha_1,\cdots,\alpha_l\}$.

Notation: A root $\alpha=\sum_{i=1} a_i\alpha_i$ is denoted by $(a_1,\cdots,a_l)$ if $\alpha>0$, and by $-(-a_1,\cdots,-a_l)$ if $\alpha<0$.

Example 6.2. $\mathfrak{g}_R=\mathfrak{o}(3,2)$ (the normal form of type (B_2))

The Satake diagram of $\mathfrak{g}_R$ is $\overset{\alpha_1}{\bigcirc}\!\!\Longrightarrow\!\!\overset{\alpha_2}{\bigcirc}$, and each root vector is contained in $\mathfrak{g}_R$. The characteristics of mono-semisimple elements corresponding to non-zero nilpotent elements in $\mathfrak{g}_R$ are as follow:

$$\left(0,\frac{1}{2}\right),\ (1,0)\quad\text{and}\quad(1,1).$$

There are three G^c-conjugate classes of non-zero nilpotent elements in $\mathfrak{g}_R$.

1) $\left(0,\frac{1}{2}\right)$ $\left(\text{i.e., } x=\frac{1}{2}\varepsilon_2\right)$

In this case $e=e_{\alpha_1+2\alpha_2}$ is a nil-positive element corresponding to x. This is because x satisfies $x\in[e,\mathfrak{g}^{-(1,2)}]\subset[e,\mathfrak{g}]$ and $[x,e]=e$. We set

$$\Delta_{\frac{1}{2}}=\{\alpha_2,\ \alpha_1+\alpha_2\},$$

and

$$\Delta_0=\{\alpha_1,\ -\alpha_1\}.$$

Then we have $\mathfrak{g}_{\frac{1}{2}}=\sum_{\alpha\in\Delta_{\frac{1}{2}}}\mathfrak{g}^{\alpha},\ \mathfrak{g}_0=\mathfrak{h}+\sum_{\alpha\in\Delta_0}\mathfrak{g}^{\alpha},$

and

$$\dim\mathfrak{g}^{e}=\dim\mathfrak{g}_0+\dim\mathfrak{g}_{\frac{1}{2}}=4+2=6.$$

Since $\dim\mathfrak{g}=10$, the dimension of a w-polarization of e, if it exists, must be equal to 8. The dimension of a parabolic subalgebra of $\mathfrak{g}$ is, however, equal to 6,7 or 10, and so by Theorem 2.2, e has not a w-polarization.

2) (1, 0) (i.e., $x=\varepsilon_1$)

In this case, $e=e_{\alpha_1+\alpha_2}$ is a nil-positive element corresponding to x. We have

$$\mathfrak{g}_1=\mathfrak{g}^{\alpha_1}+\mathfrak{g}^{\alpha_1+\alpha_2}+\mathfrak{g}^{\alpha_1+2\alpha_2},$$

$$\mathfrak{g}_0=\mathfrak{h}+\mathfrak{g}^{\alpha_2}+\mathfrak{g}^{-\alpha_2},$$

$$\mathfrak{g}_{-1}=\mathfrak{g}^{-\alpha_1}+\mathfrak{g}^{-(\alpha_1+\alpha_2)}+\mathfrak{g}^{-(\alpha_1+2\alpha_2)}.$$

Since a w-polarization of e contains $\mathfrak{h}$ by Proposition 5.5, w-polarizations of e are exhausted by the following three:

$$\mathfrak{p}_1=\sum_{j\geqq 0}\mathfrak{g}_j,$$

$$\mathfrak{p}_2=\mathfrak{h}+\mathfrak{g}_1+\mathfrak{g}^{\alpha_2}+\mathfrak{g}^{-\alpha_1},$$

$$\mathfrak{p}_3=\mathfrak{h}+\mathfrak{g}_1+\mathfrak{g}^{-\alpha_2}+\mathfrak{g}^{-(\alpha_1+2\alpha_2)}.$$

$\mathfrak{p}_2$ is conjugate to $\mathfrak{p}_3$ under $\mathrm{Aut}(\mathfrak{g})$, while $\mathfrak{p}_1$ is conjugate to none of them.

3) (1, 1) (i.e., $x=\varepsilon_1+\varepsilon_2$)

In this case, $e=e_{\alpha_1}+e_{\alpha_2}$ is a nil-positive element corresponding to x. Since e is a principal nilpotent element of $\mathfrak{g}$, e has the unique w-polarization and it is an admissible polarization of e (Corollary 5.6).

Example 6.3. The normal form of type (G_2).

The Satake diagram of $\mathfrak{g}_R$ is $\overset{\alpha_1}{\bigcirc}\!\Rrightarrow\!\overset{\alpha_2}{\bigcirc}$.

The characteristics of semisimple elements in the closure of the positive Weyl chamber of $\mathfrak{h}$ which can be mono-semisimple elements of nilpotent elements in $\mathfrak{g}_R$ are as follow:

$$\left(\frac{1}{2}, 0\right),\ \left(0, \frac{1}{2}\right),\ (1, 0) \text{ and } (1, 1)$$

(see Dynkin [4] p. 176).

1) $x=\left(\frac{1}{2}, 0\right)$.

$e=e_{(2,3)}$ is a nilpositive element corresponding to x. We set

$$\Delta_{\frac{1}{2}}=\{(1, 0), (1, 1), (1, 2), (1, 3)\},$$

$$\Delta_0 \ =\{\pm(0, 1)\}.$$

Then

$$\mathfrak{g}_{\frac{1}{2}}=\sum_{\alpha\in\Delta_{\frac{1}{2}}}\mathfrak{g}^{\alpha},\ \mathfrak{g}_0=\mathfrak{h}+\sum_{\alpha\in\Delta_0}\mathfrak{g}^{\alpha},$$

and

$$\dim \mathfrak{g}^e=\dim \mathfrak{g}_0+\dim \mathfrak{g}_{\frac{1}{2}}=4+4=8.$$

Since $\dim \mathfrak{g}=14$, the dimension of a w-polarization, if it exists, should be equal to 11. The dimension of a parabolic subalgebra of $\mathfrak{g}$ is, however, equal to 8, 9 or 14, and so by Theorem 2.2, e has no w-polarizations.

2) $x=\left(0, \frac{1}{2}\right)$

In this case $e=e_{(1,2)}$ is a nilpositive element corresponding to x. We set

$$\Delta_j=\{\alpha\in\Delta;\ \alpha(x)=j\} \qquad \text{for } j\in\frac{1}{2}\boldsymbol{Z}.$$

That is

$$\Delta_{\frac{3}{2}}=\{(1, 3), (2, 3)\},$$

$$\Delta_1 \ =\{(1, 2)\},$$

$$\Delta_{\frac{1}{2}}=\{(0, 1), (1,1)\},$$

$$\Delta_0 \ =\{\pm(1, 0)\},$$

and

$$\Delta_j=\phi \qquad \text{if} \quad |j|\geqq 2.$$

Since $\mathfrak{g}^e=\boldsymbol{C}\varepsilon_1+\mathfrak{g}^{(1,0)}+\mathfrak{g}^{-(1,0)}+\mathfrak{g}_1+\mathfrak{g}_{\frac{3}{2}}$, we have

$$\mathfrak{p}\supset\mathfrak{g}_0+\mathfrak{g}_1+\mathfrak{g}_{\frac{3}{2}}$$

by Proposition 2.1 and Proposition 5.5. $\mathfrak{p}$ cannot contain f, because $f\in\mathfrak{p}$ and $\mathfrak{g}^e\subset\mathfrak{p}$ implies $\mathfrak{p}=\mathfrak{g}$. So $\mathfrak{p}\cap\mathfrak{g}_{-1}=\{0\}$. Thus, by the condition ii) of polarizations, $\mathfrak{p}\cap\mathfrak{g}_{-\frac{1}{2}}\neq\{0\}$. Since $\mathfrak{g}_{-\frac{1}{2}}$ is $ad(\mathfrak{g}_0)$-irreducible, we have $\mathfrak{p}\supset\mathfrak{g}_{-\frac{1}{2}}$. So we have

$$\mathfrak{g}_{-1}=[\mathfrak{g}_{-\frac{1}{2}},\mathfrak{g}_{-\frac{1}{2}}]\subset[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{p},$$

which contradicts the fact that $f\notin\mathfrak{p}$. Thus we have proved that e has no w-polarizations. (But for the above discussion, it may be shown only by calculating the dimension of $\mathfrak{g}^e(\dim\mathfrak{g}^e=8)$ that e has no w-polarizations.)

3) $x=(1, 0)$

$e=e_{(1,1)}+e_{(1,2)}$ is a nilpositive element in $\mathfrak{g}_R$ corresponding to x.

We set

$$\Delta_2=\{(2, 3)\},$$
$$\Delta_1=\{(1, 0), (1, 1), (1, 2), (1, 3)\},$$
$$\Delta_0=\{\pm(0, 1)\}.$$

Then we have

$$\mathfrak{g}_0=\mathfrak{h}+\sum_{\alpha\in\Delta_0}\mathfrak{g}^\alpha,$$

and

$$\mathfrak{g}_i=\sum_{\alpha\in\Delta_i}\mathfrak{g}^\alpha,\ \mathfrak{g}_{-i}=\sum_{\alpha\in\Delta_i}\mathfrak{g}^{-\alpha} \qquad \text{if } i\geqq 1.$$

W-polarizations of e containing $\mathfrak{h}$ are exhausted by the following three:

$$\mathfrak{p}_1=\sum_{j\geqq 0}\mathfrak{g}_j,$$
$$\mathfrak{p}_2=\mathfrak{h}+\sum_{j\geqq 1}\mathfrak{g}_j+\mathfrak{g}^{(0,1)}+\mathfrak{g}^{-(1,0)},$$
$$\mathfrak{p}_3=\mathfrak{h}+\sum_{j\geqq 1}\mathfrak{g}_j+\mathfrak{g}^{-(0,1)}+\mathfrak{g}^{-(1,3)}.$$

It is easily seen that $\mathfrak{p}_3$ is G^c-conjugate to $\mathfrak{p}_2$, and not to $\mathfrak{p}_1$. We cannot tell whether there exists a w-polarization of e other than the above three or not.

4) $x=(1, 1)$

A nilpotent element e in $\mathfrak{g}_R$ corresponding to x is a principal nilpotent element of $\mathfrak{g}$, and so e has the unique w-polarization, according to Corollary 5.6.

Example 6.4. (F II)

The Satake diagram of $\mathfrak{g}_R$ is $\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet\!\!\Longrightarrow\!\!\bullet\!\!-\!\!\!-\!\!\!-\!\!\circ$.

The characteristics of elements in the closure of the positive Weyl chamber of $\mathfrak{h}$ which can be mono-semisimple elements corresponding to nilpotent elements in $\mathfrak{g}_R$ are $\left(0, 0, 0, \frac{1}{2}\right)$ and $(0, 0, 0, 1)$.

1) $x=\left(0, 0, 0, \frac{1}{2}\right)$

$e=e_{(1232)}$ is a nil-positive element corresponding to x. We have

$$\begin{aligned}\dim \mathfrak{g}^e &= \dim \mathfrak{g}_0 + \dim \mathfrak{g}_{\frac{1}{2}} \\ &= 22+8=30.\end{aligned}$$

Since $\dim \mathfrak{g}=52$, the dimension of a w-polarization, if it exists, should be equal to 41. The dimension of a parabolic subalgebra of $\mathfrak{g}$ is, however, equal to 28, 29, 30, 31, 32, 37 or 52, and so by Theorem 2.2, e has no w-polarizations.

2) $x=(0, 0, 0, 1)$

In this case, $e=e_{(0001)}+e_{(1231)}$ is a nilpositive element corresponding to e. Since the characteristic of x consists only of integers, e has a w-polarization (Proposition 5.1). Let $\mathfrak{p}$ be any w-polarization of e. The same discussion as in the proof of Proposition 5.5 is valid in this case, and we have $\mathfrak{h}\subset\mathfrak{p}$. By a simple calculation of roots using this fact and $\mathfrak{g}^e\cap\mathfrak{g}_0\subset\mathfrak{p}$ (Proposition 2.1), one can see that $\mathfrak{g}_0\subset\mathfrak{p}$, and so $\sum_{j\geq 0}\mathfrak{g}_j\subset\mathfrak{p}$. Now, by the condition ii) of polarization, $\mathfrak{p}$ must coincide with $\sum_{j\geq 0}\mathfrak{g}_j$. Thus e has the unique w-polarization, and it is an admissible polarization of e.

Note: A non-compact real form of type (F_4) is $(F\text{I})$ (=the normal form) or $(F\text{II})$. And so Example 6.4 implies that in a non-compact real form of type (F_4) there exists a nilpotent element with no w-polarizations.

Note: By a similar discussion, we can see that there exists an element in $\mathfrak{g}_R$ with no w-polarizations if a certain simple factor of $\mathfrak{g}_R$ is the normal form of type (B) (C) (D) (E) (F) or (G).

§7. P-orbits in $\mathfrak{n}$

In this section, the notion of polarizations is slightly modified. For an element X in $\mathfrak{g}$ and a subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, we consider the following conditions:

i) $B(X, [\mathfrak{p}, \mathfrak{p}])=\{0\}$;
ii) $\dim \mathfrak{p}-\dim \mathfrak{g}^X=\dim \mathfrak{g}-\dim \mathfrak{p}$:
iii) $\mathfrak{p}$ is $Ad((G^C)^X)$-stable;
iv) $\mathfrak{p}+\sigma\mathfrak{p}$ is a complex subalgebra of $\mathfrak{g}$.

Definition 7.1. Let $X \in \mathfrak{g}$. A complex subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is called

1) *a* w-*polarization of* X if $\mathfrak{p}$ satisfies i) and ii),
2) *a polarization of* X if $\mathfrak{p}$ satisfies i)—iii),
3) *an admissible* w-*polarization of* X if $\mathfrak{p}$ satisfies i), ii) and iv).

The arguments and propositions in the previous sections are all valid for such definitions.

In §6, we gave examples of nilpotent elements with no polarizations. Now there arises the problem "is every parabolic subalgebra of $\mathfrak{g}$ obtained as a w-polarization of a nilpotent element?"

Proposition 7.1. *Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$, whose nil-radical is $\mathfrak{n}$.*

1) *If e is an element in $\mathfrak{n}$ such that $[e, \mathfrak{p}]=\mathfrak{n}$, the P-orbit through e is an open subset of $\mathfrak{n}$.*

2) *The set of e in $\mathfrak{n}$ such that $[e, \mathfrak{p}]=\mathfrak{n}$ is, if not empty, an open, dense and connected subset of $\mathfrak{n}$.*

Proof. 1) By Theorem 2.2, $\mathfrak{p}$ is a w-polarization of e, and so $\mathfrak{g}^e\subset\mathfrak{p}$. Define a mapping φ of P to n by $\varphi(p)=Ad(p)e$, where P is the parabolic subgroup of G^C with Lie algebra $\mathfrak{p}$. Then

$$\begin{aligned}\dim(\mathrm{Im}\varphi)&=\dim \mathfrak{p}-\dim \mathfrak{g}^e\\&=\dim \mathfrak{g}-\dim \mathfrak{p}=\dim \mathfrak{n}.\end{aligned}$$

Thus the image of φ is open in $\mathfrak{n}$.

2) For each $e \in \mathfrak{n}$, define a linear mapping A_e of $\mathfrak{p}$ to $\mathfrak{n}$ by $A_e(X)=[e, X]$ (for every $X \in \mathfrak{p}$). Then we have

$$\begin{aligned}&\{e \in \mathfrak{n};\ [e, \mathfrak{p}]=\mathfrak{n}\}\\&=\{e \in \mathfrak{n};\ \text{the rank of } A_e \text{ is } \dim \mathfrak{n}\}.\end{aligned}$$

Thus the condition "$[e, \mathfrak{p}]=\mathfrak{n}$" is expressed by using a certain polynomial function f on $\mathfrak{n}$ as "the value of f at e does not vanish," and the set of e satisfying the above condition is (if not empty) open, dense and connected in $\mathfrak{n}$. Q.E.D.

COROLLARY 7.2. *Let P be a parabolic subgroup of G^C with Lie algebra $\mathfrak{p}$ whose nil-radical is $\mathfrak{n}$. Assume that $\mathfrak{p}$ is a* w*-polarization of e and e' simultaneously ($e, e' \in \mathfrak{n}$). Then e and e' are P-conjugate.*

PROOF. We set

$$\mathfrak{n}'=\{X \in \mathfrak{n};\ [X, \mathfrak{p}]=\mathfrak{n}\}.$$

Since $\mathfrak{n}'$ is an open, dense, connected subset of $\mathfrak{n}$ (Proposition 7.1), $\mathfrak{n}'$ is a single P-orbit. And, by Theorem 2.2, we have $e, e' \in \mathfrak{n}'$. Q.E.D.

PROPOSITION 7.3. *Let e be a nilpotent element in $\mathfrak{g}$, $\mathfrak{p}_i (i=1,2)$ a* w*-polarization of e, and $P_i (i=1, 2)$ the parabolic subgroup of G^C corresponding to $\mathfrak{p}_i$. Assume that $(G^C)^e$ is contained in P_1 and that $\mathfrak{p}_1$ is G^C-conjugate to $\mathfrak{p}_2$. Then $\mathfrak{p}_1$ coincides with $\mathfrak{p}_2$.*

PROOF. By the assumtion, there exists an element g in G^C such that $Ad(g)\mathfrak{p}_1=\mathfrak{p}_2$. Since $\mathfrak{p}_2$ is a w-polarization of e, $Ad(g^{-1})\mathfrak{p}_2=\mathfrak{p}_1$ is a w-polarization of $Ad(g^{-1})e$. Thus $\mathfrak{p}_1$ is a w-polarization of e and $Ad(g^{-1})e$, and so e is P_1-conjugate to $Ad(g^{-1})e$ (Corollary 7.2). Choose an element $p \in P_1$ such that $Ad(g^{-1})e=Ad(p)e$. Then we have

$$gp \in (G^C)^e \subset P_1.$$

Hence $$g \in P_1, \quad \text{and} \quad \mathfrak{p}_2=Ad(g)\mathfrak{p}_1=\mathfrak{p}_1$$

Q.E.D.

The following is an immediate consequence from Proposition 2.1 and Proposition 7.3:

COROLLARY 7.4. *Let e be a nilpotent element in $\mathfrak{g}$, such that $(G^C)^e$ is connected. Then any two* w*-polarizations of e are not G^C-conjugate.*

Note: $(G^C)^e$ is not necessarily connected. For example, in Example 6.3.3), $e=e_{(11)}+e_{(12)}$ has at least three w-polarizations $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_3$. Among them, $\mathfrak{p}_3$ is G^C-conjugate to $\mathfrak{p}_2$ and not to $\mathfrak{p}_1$. So, by Corollary 7.4, $(G^C)^e$ is not connected.

PROPOSITION 7.5. *If e is a principal nilpotent element of $\mathfrak{g}$, $(G^C)^e$ is a connected subgroup of G^C.*

PROOF. Let x be a mono-semisimple element corresponding to e. The isotropy subgroup $H=(G^C)^x$ is a Cartan subgroup of G^C with Lie algebra $\mathfrak{h}=\mathfrak{g}^x$.

It suffices to show that $H\cap(G^C)^e$ contains only the unit (Lemma 3.2). Choose the linear order in the non-zero root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ such that x belongs to the positive Weyl chamber, and let $\Pi=\{\alpha_1,\cdots,\alpha_l\}$ (resp. Δ_+) be the fundamental (resp. positive) root system. We can assume that

$$e=\sum_{i=1}^{l} e_{\alpha_i},$$

from the G^C-conjugacy of principal nilpotent elements of $\mathfrak{g}$ (Kostant [8]). For any $g\in H\cap(G^C)^e$, we have

$$Ad(g)e_{\alpha_i}=e_{\alpha_i} \qquad \text{for } 1\leqq i\leqq l,$$

and so $Ad(g)$ is the identity mapping on $\mathfrak{h}+\sum_{i=1}^{l}\mathfrak{g}^{\alpha_i}$.

Therefore $Ad(g)$ is the identity on $\mathfrak{h}+\sum_{\alpha\in\Delta_+}\mathfrak{g}^{\alpha}$. Since the Killing form B induces a non-degenerate, $Ad(g)$-invariant pairing between $\sum_{\alpha\in\Delta_+}\mathfrak{g}^{\alpha}$ and $\sum_{\alpha\in\Delta_+}\mathfrak{g}^{-\alpha}$, $Ad(g)$ is the identity also on $\sum_{\alpha\in\Delta_+}\mathfrak{g}^{-\alpha}$. Hence g is the unit of G^C.

Q.E.D.

Note: In the above proposition, "$G^C=\text{Int}\,\mathfrak{g}$" is essential.

PROPOSITION 7.6. *A Borel-subalgebra* $\mathfrak{b}$ *of* $\mathfrak{g}$ *is a* w*-polarization of a principal nilpotent element contained in the nil-radical* $\mathfrak{n}$ *of* $\mathfrak{b}$.

PROOF. Let $e\in\mathfrak{n}$ be a principal nilpotent element. Then we have

$$\dim\mathfrak{b}=\frac{1}{2}\Big(\dim\mathfrak{g}+\text{rank}(\mathfrak{g})\Big)=\frac{1}{2}\Big(\dim\mathfrak{g}+\dim\mathfrak{g}^e\Big),$$

$$B(e,[\mathfrak{b},\mathfrak{b}])=B(e,\mathfrak{n})=\{0\}.$$

Thus $\mathfrak{b}$ is a w-polarization of e. Q.E.D.

PROPOSITION 7.7. *Let* $\mathfrak{g}$ *be a (complex) semisimple Lie algebra of type* (A); *then every parabolic subalgebra is a* w*-polarization of a nilpotent element in* $\mathfrak{g}$.

PROOF. It suffices to show the proposition in the case where $\mathfrak{g}$ is simple (Proposition 2.6). Again it suffices to show the proposition for the Lie algebra $\mathfrak{g}=\text{End}(V)$ where V is an n-dimensional complex vector space. In this case, for a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, we can choose a decreasing sequence

$$V=V_0\supset V_1\supset\cdots\cdots\supset V_m=\{0\}$$

of subspaces of V such that

$$\mathfrak{p}=\{x\in\text{End}(V);\ x(V_i)\subset V_i,\qquad i=1,\cdots,m-1\}.$$

Furthermore we have

$$\mathfrak{n}=\{x \in \mathrm{End}(V);\ x(V_i)\subset V_{i+1}, \qquad i=1,\cdots, m-1\},$$

$$P=\{g \in \mathrm{Aut}(V);\ g(V_i)\subset V_i, \qquad i=1,\cdots, m-1\}.$$

For an element $v \in V$, we set

$$\gamma(v)=\max\{i;\ v \in V_i\}.$$

Then an element g in $\mathrm{Aut}(V)$ belongs to P if and only if $\gamma(g(v))=\gamma(v)$ for all $v \in V$.

Consider the adjoint action of P on $\mathfrak{n}$. To show the existence of an element e in $\mathfrak{n}$ satisfying $[e, \mathfrak{p}]=\mathfrak{n}$, it is sufficient to show that there exists an element e in $\mathfrak{n}$ such that the P-orbit of e in $\mathfrak{n}$ is open, or equivalently that there exists a non-empty open set O in $\mathfrak{n}$ such that any two elements in O are conjugate to each other by an element in P.

Let $$a_i=\dim(V_{i-1}/V_i) \qquad (i=1,\cdots, m).$$

Then $\sum_{i=1}^{m} a_i=n.$ Reorder the a_i's so that

$$b_1\leqq b_2\leqq\cdots\cdots\leqq b_m,$$

and set

$$\alpha_1=b_1,$$

$$\alpha_i=b_i-b_{i-1} \qquad (i=2,\cdots, m).$$

From the definition, we have

$$\sum_{i=1}^{m}(m+1-i)\alpha_i=n.$$

Consider the set

$$O=\{x \in \mathfrak{n};\ \mathrm{rank}(x^{m-1})=\alpha_1,\cdots, \mathrm{rank}(x)=\alpha_{m-1}\}.$$

One can show that

$$O=\{x \in \mathfrak{n};\ \mathrm{rank}(x^{m-1})\geqq\alpha_1,\cdots, \mathrm{rank}(x)\geqq\alpha_{m-1}\},$$

and hence O is an open set in $\mathfrak{n}$. We shall show that O satisfies the required property.

In general, for a nilpotent mapping f of V such that $f^m=0$, we can choose a sequence $\{U_i\}$ of subspaces of V such that i) $\dim U_i=\dim f(U_i)$, $f^{m-i}(U_i)=\{0\}$ for each i, and ii) V decomposes into the direct sum of $\{f^j(U_i)\}_{i=1,\dots, m;\, j=0,1\dots,m-1}$

(the ordered set $\{\dim U_i\}$ is an invariance of f).

Suppose $x \in O$. Then, we have

$$\dim U_i = \alpha_i.$$

An inductive argument would show that, for an element v in $x^j(u_i)$, $\gamma(v)$ is determined only by (j, i) and is independent of x. Suppose $x, x' \in O$. Choose $\{U_i\}$, $\{U_i'\}$ as above for x and x' respectively. Then the set of isomorphisms $U_i \longrightarrow U_i'$ induces an automorphism g of V such that $xg = gx'$. Furthermore $g \in P$ since

$$\gamma(g(v)) = \gamma(v) \qquad \text{for every} \quad v \in V.$$

An inductive argument shows that, for a non-zero vector v in $x^j(U_i)$ $(i=1,\cdots, m; j=0,\cdots, m-i)$, $\gamma(v)$ is determined only by i and j, and is independent of x. Thus g preserves γ, and hence g belongs to P.

Q.E.D.

Note: We have a conjecture that, for any parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, there exists a nilpotent element in $\mathfrak{n}$ with a w-polarization $\mathfrak{p}$, or equivalently, that any $(P, \mathfrak{n})$ is a pre-homogeneous vector space. This conjecture is correct if 1) P is a Borel subgroup of G^C (Proposition 7.6) or 2) $\mathfrak{g}$ is a semisimple Lie algebra of type (A) (Proposition 7.7), and further, from case-wise discussions, we can assert that the conjecture is true when $\mathfrak{g}$ is a simple Lie algebra of type (E_6) (F_4) (G_2) or a simple Lie algebra of lower ranks of type (B) (C) (D).

References

[1] L. Auslander and B. Kostant: Quantization and representations of solvable Lie groups, Bull. Amer. Math. Soc., **73** (1967), 692–695.

[2] ————: Polarization and unitary representations of solvable Lie groups, Inventiones Math., **14** (1971), 255–354.

[3] R. Bott: Homogeneous vector bundles, Ann. of Math., **66** (1957), 203–248.

[4] E. B. Dynkin: Semisimple subalgebras of semisimple Lie algebras, Amer. Math. Soc. Trans., **6** (1957), 111–244.

[5] S. Helgason: Differential geometry and symmetric spaces, Academic Press (1962).

[6] A. A. Kirillov: Unitary representations of nilpotent Lie groups, Russian Math. Surveys, **17** (1962), 53–104.

[7] ————: Construction of irreducible unitary representations of Lie groups, Vestnik Mosc. Univ., **2** (1970), 41–51.

[8] B. Kostant: The principal three dimensional subgroup and the Betti numbers of a complex simple Lie group, Amer. J. Math., **81** (1959), 973–1032.

[9] ————: Orbits, symplectic structures and representation theory, Proc. U.S.-Japan Sem. on Diff. Geom. Kyoto, (1965), 71.

[10] ————: Quantization and unitary representations, Lectures in Modern Analysis and Applications III. Lecture Notes (Springer-Verlag), **170** (1970), 87–208.

[11] K. Nomizu: Fundamentals of linear algebra, Mc. Graw-Hill, 1966.

[12] M. Sugiura: Conjugate classes of Cartan subalgebras in real semisimple Lie algebras, J. Math. Soc. Japan, **11** (1959), 374–434.

[13] J. A. Wolf: The action of a real semisimple group on a complex flag manifold, I: Oribit structure and holomorphic arc components, Bull. Amer. Math. Soc., **75** (1969), 1121–1237.

[14] ————: The action of a real semisimple group on a complex flag manifold, II: Unitary representations on partially holomorphic cohomology spaces (to appear).

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