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Energy of Functions on a Self-adjoint Harmonic Space I

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Dedicated to Professor Y. Tôki on his 60th birthday

Introduction.

One of the primary objectives of an axiomatic potential theory, or a theory of harmonic spaces, is a unified treatment of potential theoretic parts in the theories of various second order elliptic (and some parabolic) partial differential equations (see [2], [3]). Differential equations are considered on a space with differentiable structure and in the theory of such equations the notion of Dirichlet integrals or that of energy plays an important role. If the equation is, for example, given by $Lu\equiv\Delta u-Pu=0$ on a domain Ω in the Euclidean space R^n, then the Dirichlet integral of a function f is $D[f]=\int_\Omega \sum(\partial f/\partial x_i)^2 dx$ and the energy of f is $E[f]=D[f]+\int_\Omega f^2 P dx$. These values appear, for instance, in Green's formula, which is a basic tool in the theory of such an equation.

A harmonic space, however, is defined on a locally compact Hausdorff space on which we do not, in general, require any differentiable structure. Thus, on an abstract harmonic space, we cannot define the notion of Dirichlet integral or energy by means of differentiation of functions as above. Nevertheless, it is expected that the structure of harmonic space might yield a certain notion, which is reduced to the ordinary Dirichlet integral or energy in the special case where the structure is given by a differential equation on a differentiable manifold.

The purpose of the present and the subsequent papers is to introduce such a notion on a *self-adjoint* harmonic space. Here, a self-adjoint harmonic space means a harmonic space with a symmetric Green function $G(x, y)$. Recall that the canonical form of a self-adjoint second order partial differential equation is given by $Lu\equiv\Delta u-Pu=0$. Suppose this equation is considered on a domain $\Omega\subset R^n$ and suppose there is the corresponding Green function $G(x, y)$ such that $L_x G(x, y)=-\delta_y$ (δ_y: the Dirac measure). Then, for a positive measure μ on Ω such that $U^\mu(x)=\int G(x, y)d\mu(y)$ belongs to C^2, we have $LU^\mu=-\mu$. Thus, if a C^2-function f on Ω is expressed in the form

$$(*)\qquad f=u+U^\mu-U^\nu \qquad (u\text{: a solution of } Lu=0),$$

then $Lf=-\mu+\nu$. On the other hand, $L(f^2)=2f(Lf)+2\{\sum(\partial f/\partial x_i)^2+f^2P\}-f^2P$, so that

$$E[f]=-\int f(Lf)dx+\frac{1}{2}\int L(f^2)dx+\frac{1}{2}\int f^2Pdx.$$

Therefore, expressing $f^2=u'+U^{\mu'}-U^{\nu'}(Lu'=0)$ we have

$$E[f]=\int f(d\mu-d\nu)-\frac{1}{2}\int(d\mu'-d\nu')+\frac{1}{2}\int f^2Pdx,$$

which shows that $E[f]$ can be "calculated" without differentiation, once we have a Green function. This is the basic idea of our definition of energy.

In the present paper, we define the notion of energy for functions of the form (*) with bounded u, U^μ and U^ν (Chapter II) and then for general harmonic functions (Chapter III), on a self-adjont harmonic space. In the subsequent paper(s), we shall extend the definition to more general functions.

The most interesting feature of the present theory may be found in the fact that many results (e. g., Theorems in Chapters II and III), whose known proofs in the classical case essentially depend on differentiation, remain valid in our general case where the notion of differentiation loses its meaning.

Notation. Given a non-compact locally compact Hausdorff space Ω and a subset A of Ω, we use the following notation:

$\bar{A}$: the closure of A in Ω; ∂A: the boundary of A in Ω.

Ω^a: the one point (Alexandroff) compactification of Ω.

A^a: the closure of A in Ω^a; $\partial^a A$: the boundary of A in Ω^a.

ξ_a : the point at infinity, i.e., $\Omega^a-\Omega=\{\xi_a\}$.

Every function considered is extended real valued. The space of all finite continuous functions on A is denoted by $\boldsymbol{C}(A)$. By a measure on Ω, we mean a non-negative Radon measure on Ω. The support of a measure μ is denoted by $S(\mu)$. For a function f on Ω and a set A, the restriction of f to A is denoted by $f|A$; for a measure μ on Ω and a Borel set A, the restriction of μ to A is denoted by $\mu|A$.

CHAPTER I. Self-adjoint Harmonic Space

§ 1.1. Brelot's harmonic space.

Let Ω be a connected, locally connected, non-compact, locally compact Hausdorff space *with a countable base* and let $\mathfrak{H}=\{\mathscr{H}(\omega)\}_{\omega:\mathrm{open}\subset\Omega}$ be a structure of harmonic space on Ω satisfying Axioms 1, 2 and 3 of M.Brelot [2]. Functions in $\mathscr{H}(\omega)$ are called harmonic on ω. The notions of superharmonic functions on an open set ω and of potentials on Ω are defined with respect to this harmonic structure (see [2], [3], [5]). For a superharmonic function s on Ω, let

$$\sigma(s)=\Omega-\bigcup\{\omega\colon \text{open},\ s\,|\,\omega\in\mathscr{H}(\omega)\},$$

which is called the (harmonic) support of s.

In this paper, we shall assume the following additional axioms:

Axiom 4. The constant function 1 is superharmonic.

Axiom 5. There exists a positive potential on Ω.

Axiom 6. For each $y\in\Omega$, if p_1, p_2 are positive potentials on Ω such that $\sigma(p_1)=\sigma(p_2)=\{y\}$, then they are proportional (cf. [5]).

By Axiom 4, we have the following minimum principle ([2; Part IV, Theorem 3 (ii)]): Let ω be an open set in Ω and s be a superharmonic function on ω. If $\liminf_{x\to\xi,x\in\omega} s(x)\geqq 0$ for any $\xi\in\partial^a\omega$, then $s\geqq 0$ on ω.

Given an open set $\omega\subset\Omega$, the Dirichlet problem with respect to $\mathfrak{H}$ can be discussed by Perron-Brelot's method (see, e.g., [1], [2]): For an extended real valued function φ on $\partial^a\omega$, we set

$$\bar{\mathscr{S}}^{\omega}_{\varphi}=\left\{s;\ \begin{array}{l}\text{superharmonic, bounded below on } \omega,\\ \liminf_{x\to\xi,x\in\omega} s(x)\geqq\varphi(\xi) \text{ for all } \xi\in\partial^a\omega\end{array}\right\}\cup\{\infty\}$$

and $\underline{\mathscr{S}}^{\omega}_{\varphi}=-\bar{\mathscr{S}}^{\omega}_{-\varphi}$. We denote: $\bar{H}^{\omega}_{\varphi}\equiv\inf\ \bar{\mathscr{S}}^{\omega}_{\varphi}$ and $\underline{H}^{\omega}_{\varphi}\equiv\sup\ \underline{\mathscr{S}}^{\omega}_{\varphi}$.

LEMMA 1.1. *Any open set ω is resolutive, i.e., for any $\varphi\in\boldsymbol{C}(\partial^a\omega)$, $\bar{H}^{\omega}_{\varphi}=\underline{H}^{\omega}_{\varphi}$ and is harmonic on ω.*

This is a consequence of [1; Corollary 3 and Theorem 8].

We denote the function $\bar{H}^{\omega}_{\varphi}=\underline{H}^{\omega}_{\varphi}$ by H^{ω}_{φ} for resolutive φ. For each $x\in\omega$, there exists the harmonic measure μ^{ω}_{x} on $\partial^a\omega$ such that

$$\int\varphi\, d\mu^{\omega}_{x}=H^{\omega}_{\varphi}(x) \tag{1.1}$$

for all $\varphi\in\boldsymbol{C}(\partial^a\omega)$. By Axiom 4, $\mu^{\omega}_{x}(\partial^a\omega)\leqq 1$. Since ω^a is metrizable, a μ^{ω}_{x}-summable function φ on $\partial^a\omega$ is resolutive and satisfies (1.1) (cf. [2; Part IV, Proposition 21]).

If ω_0 is a domain in Ω, then $\mathfrak{H}\,|\,\omega_0=\{\mathscr{H}(\omega)\}_{\omega:\text{open}\subset\omega_0}$ is a harmonic structure on ω_0 and satisfies Axioms 1~6. Thus we have notions of potentials on ω_0, etc.

§ 1.2. Self-adjoint harmonic space.

DEFINITION. A harmonic structure $\mathfrak{H}$ on Ω satisfying Axioms 1, 2, 3, 5 and 6 is called *self-adjoint* if there exists a function $G(x, y)\colon \Omega\times\Omega\to(0, +\infty]$ such that

(i) $G(x, y)=G(y, x)$ for all $x, y\in\Omega$;

(ii) For each $y \in \Omega$, $G_y(x) \equiv G(x, y)$ is a potential on Ω and $\sigma(G_y)=\{y\}$.

If $\mathfrak{H}$ is self-adjoint, then $(\Omega, \mathfrak{H})$ is called a self-adjoint harmonic space.

By Axiom 6, we can easily show:

PROPOSITION 1.1. *If $\mathfrak{H}$ is self-adjoint, then the function $G(x, y): \Omega \times \Omega \to (0, +\infty]$ satisfying* (i) *and* (ii) *above is uniquely determined up to a multiplicative constant.*

We call $G(x, y)$ a Green function for $\mathfrak{H}$. By [5; Proposition 18.1], we see that $G(x, y)$ is lower semi-continuous on $\Omega \times \Omega$.

REMARK. R.-M. Hervé showed that, under Axioms 1,2,3,5 and 6, there is a function $p(x, y): \Omega \times \Omega \to (0, +\infty]$ such that $x \to p(x, y)$ is a potential with support $\{y\}$ for each $y \in \Omega$ and $y \to p(x, y)$ is continuous on $\Omega - \{x\}$ for each $x \in \Omega$ ([5; Théorème 18.1]). The above definition simply means that $\mathfrak{H}$ is self-adjoint if we can choose $p(x, y)$ to be symmetric.

Hereafter, we assume that $(\Omega, \mathfrak{H})$ is a self-adjoint harmonic space satisfying Axioms 1~6 and $G(x, y)$ is a fixed Green function. For $y \in \Omega$, we shall often use the notation G_y: $G_y(x) \equiv G(x,y)$.

LEMMA 1.2. *For any $y \in \Omega$ and any open set ω containing y,*

$$\sup_{x \in \Omega-\omega} G(x,y) < +\infty.$$

PROOF. Let ω_0 be a relatively compact open set such that $y \in \omega_0 \subset \bar{\omega}_0 \subset \omega$. Then $\alpha \equiv \sup_{x \in \partial\omega_0} G(x, y) < +\infty$, since $y \to G(x, y)$ is finite continuous on $\partial\omega_0$. By [5; Lemme 3.1], we have $\alpha \geqq G(x, y)$ for all $x \in \Omega - \omega_0 \supset \Omega - \omega$.

LEMMA 1.3. *For each $y \in \Omega$, there is a non-negative superharmonic function s_y^* on Ω such that whenever $x_n \to \xi_a$ and* $\liminf_{n\to\infty} G(x_n, y) > 0$, *we have*

$$\lim_{n\to\infty} s_y^*(x_n) = +\infty.$$

This is a special case of [1; Lemma 1].

LEMMA 1.4. *Let ω be an open set in Ω. Then*

(i) $$\int_{\partial\omega} G(\xi, y)\, d\mu_x^\omega(\xi) \leqq G(x, y) \qquad \textit{for all } x \in \omega,\ y \in \Omega;$$

(ii) $$\int_{\partial\omega} G(\xi, y)\, d\mu_x^\omega(\xi) = \int_{\partial\omega} G(x, \xi)\, d\mu_y^\omega(\xi) \qquad \textit{for all } x, y \in \omega.$$

PROOF. For each $y \in \Omega$, let $\varphi_y(\xi) = G_y(\xi)$ if $\xi \in \partial\omega$, $=0$ if $\xi = \xi_a$.
(i) Since G_y is a positive superharmonic function on Ω, $G_y|\omega \in \mathscr{S}_{\varphi_y}^\omega$ for

any $y \in \Omega$. It follows that φ_y is μ_x^ω-summable and $G_y(x) \geqq H_{\varphi_y}^\omega(x) = \int_{\partial\omega} G(\xi, y) d\mu_x^\omega(\xi)$ for any $x \in \omega$ and $y \in \Omega$.

(ii) Let $x \in \omega$ be fixed. Since $y \to G(\xi, y)$ is harmonic on ω for each $\xi \in \partial\omega$, $w(y) \equiv \int_{\partial\omega} G(\xi, y) d\mu_x^\omega(\xi)$ is harmonic on ω. By (i) above, Lemma 1.2 and the minimum principle, we see that w is bounded on ω. Then, it follows from (i) that $w - \varepsilon(s_x^* | \omega) \in \mathscr{S}_{\varphi_x}^\omega$ for any $\varepsilon > 0$, where s_x^* is the superharmonic function given in the above lemma. Hence, $w \leqq H_{\varphi_x}$, i.e., $\int_{\partial\omega} G(\xi, y) d\mu_x^\omega(\xi) \leqq \int_{\partial\omega} G(x, \xi) d\mu_y^\omega(\xi)$ for any $x, y \in \omega$. By symmetry, we obtain the equality.

Proposition 1.2. *For any domain ω in Ω, $\mathfrak{H}|\omega$ is also self-adjoint and there exists a Green function $G^\omega(x, y)$ for $\mathfrak{H}|\omega$ such that*

$$G^\omega(x, y) = G(x, y) + h_y(x) \qquad (x, y \in \omega) \tag{1.2}$$

with $h_y \in \mathscr{H}(\omega)$. In fact h_y is given by

$$h_y(x) = \int_{\partial\omega} G(\xi, y) d\mu_x^\omega(\xi). \tag{1.3}$$

Proof. Let $G^\omega(x, y)$ be defined by (1.2) and (1.3). By (ii) of the above lemma, we have $h_y(x) = h_x(y)$, so that $G^\omega(x, y) = G^\omega(y, x)$ for $x, y \in \omega$. If we fix $y \in \omega$, then, by (i) of the above lemma, $h_y \leqq G_y$ on ω. On the other hand, if u is harmonic on ω and $u \leqq G_y$ on ω, then $u - \varepsilon(s_y^* | \omega) \in \mathscr{S}_{\varphi_y}^\omega$ for any $\varepsilon > 0$ as in the proof of (ii) of the above lemma. Thus, $u \leqq H_{\varphi_y}^\omega = h_y$, which shows that h_y is the greatest harmonic minorant of G_y on ω. Hence, $G_y^\omega \equiv G_y - h_y$ is a potential on ω. Obviously, G_y^ω is harmonic on $\omega - \{y\}$. Therefore, $G^\omega(x, y)$ is a Green function for $\mathfrak{H}|\omega$ and $\mathfrak{H}|\omega$ is self-adjoint.

§ 1.3. Potentials on a self-adjoint harmonic space.

For a measure μ on Ω, let

$$U^\mu(x) \equiv \int_\Omega G(x, y) d\mu(y).$$

Then, U^μ is a potential on Ω unless it is constantly infinite ([5; Théorème 18.3]). Since $G(x, y)$ is symmetric, we have $\int U^\mu d\nu = \int U^\nu d\mu$ for any measures μ, ν.

Lemma 1.5. *If $\mu(\Omega) < +\infty$, then U^μ is a potential.*

This is proved using Lemma 1.2 by a standard method in the classical theory (cf. [5; Corollaire de la Proposition 17.1]).

Now, by [5; Théorème 18.2, 2)], we know

LEMMA 1.6. *Any potential on Ω is expressed as U^{μ} by a uniquely determined measure μ.*

Note that the uniqueness follows from Axiom 6. By this lemma we see that any superharmonic function s having a harmonic minorant on Ω can be uniquely expressed as $s=u+U^{\mu}$ with $u \in \mathscr{H}(\Omega)$ and a measure μ on Ω. This measure μ is called the associated measure of s.

The associated measure of the constant function 1 will be denoted by π; thus $1=h_1+U^{\pi}$ with $h_1 \in \mathscr{H}(\Omega)$. Obviously, $U^{\pi} \leqq 1$. Note that $\pi=0$ if and only if $1 \in \mathscr{H}(\Omega)$.

LEMMA 1.7. *If $\mu(\Omega)<+\infty$, then $\int_{\Omega} U^{\mu} d\pi<+\infty$.*

PROOF. $\int_{\Omega} U^{\mu} d\pi=\int_{\Omega} U^{\pi} d\mu \leqq \mu(\Omega)<+\infty$.

For a domain ω and a measure μ on ω, we use the notation

$$U_{\omega}^{\mu}(x) \equiv \int_{\omega} G^{\omega}(x, y) d\mu(y) \qquad (x \in \omega).$$

In case μ is a measure on Ω, we shall write U_{ω}^{μ} instead of $U_{\omega}^{\mu|\omega}$.

LEMMA 1.8. *Let μ, ν be measures on Ω such that U^{μ}, U^{ν} are potentials and let ω be an open set in Ω. If $U^{\mu}|\omega=U^{\nu}|\omega+u$ with $u \in \mathscr{H}(\omega)$, then $\mu|\omega=\nu|\omega$.*

PROOF. By considering each component of ω, we may assume that ω is a domain. Then U_{ω}^{μ} and U_{ω}^{ν} are potentials on ω and, by Proposition 2.1, we see that $U^{\mu}=U_{\omega}^{\mu}+u_1$ and $U^{\nu}=U_{\omega}^{\nu}+v_1$ on ω with u_1, $v_1 \in \mathscr{H}(\omega)$. Hence, by the assumption of the lemma, $U_{\omega}^{\mu}=U_{\omega}^{\nu}+h$ with $h \in \mathscr{H}(\omega)$. It follows that $U_{\omega}^{\mu}=U_{\omega}^{\nu}$. Then, applying Lemma 1.6 on ω, we have $\mu|\omega=\nu|\omega$.

By this lemma, we see that, given any superharmonic function s on Ω, there corresponds a unique measure μ on Ω such that $s|\omega=U_{\omega}^{\mu}+u_{\omega}$ with $u_{\omega} \in \mathscr{H}(\omega)$ for any relatively compact domain ω. This measure μ is again called the associated measure of s. In this case, we have $\sigma(s)=S(\mu)$; in particular, $\sigma(U^{\mu})=S(\mu)$. Also, note that $1=H_1^{\omega}+U_{\omega}^{\pi}$ on ω for any domain ω.

LEMMA 1.9. *For any relatively compact open set ω in Ω, $p_{\omega} \equiv \inf\{s;$ non-negative superharmonic on Ω, $s \geqq 1$ on $\omega\}$ is a potential on Ω. The associated measure λ_{ω} of p_{ω} has the following properties:*

(i) *$U^{\lambda_{\omega}} \leqq 1$ on Ω and $U^{\lambda_{\omega}}=1$ on ω;*

(ii) *$S(\lambda_{\omega}) \subset \bar{\omega}$, and hence $\lambda_{\omega}(\Omega)<+\infty$;*

(iii) $\lambda_\omega | \omega = \pi | \omega$.

PROOF. By [2; Part IV, Proposition 10 and a result in p.124], we see that p_ω is a potential. Then, property (i) is obvious. By [2; Part IV, Theorem 8], p_ω is harmonic on $\Omega - \bar{\omega}$. Hence, $S(\lambda_\omega) \subset \bar{\omega}$. Since $U^{\lambda_\omega} | \omega = 1 = U^\pi | \omega + h_1 | \omega$, we obtain (iii) by Lemma 1.8.

LEMMA 1.10. *$U^\mu \leqq U^\nu$ implies $\mu(\Omega) \leqq \nu(\Omega)$.*

PROOF. For any relatively compact open set ω in Ω, $\mu(\omega) \leqq \int_\Omega U^{\lambda_\omega} d\mu = \int_\Omega U^\mu d\lambda_\omega \leqq \int_\Omega U^\nu d\lambda_\omega = \int_\Omega U^{\lambda_\omega} d\nu \leqq \nu(\Omega)$. Hence $\mu(\Omega) \leqq \nu(\Omega)$.

CHAPTER II. Energy of Bounded Functions

§2.1. The spaces $\boldsymbol{H}_{BE}$ and $\boldsymbol{B}_E$.

LEMMA 2.1. *If $u \in \mathscr{H}(\Omega)$ and $\alpha \geqq 1$, then $-|u|^\alpha$ is superharmonic on Ω.*

PROOF. $|u|^\alpha$ is a continuous function and for any regular domain ω and $x \in \omega$,

$$|u|^\alpha(x) = \left|\int u \, d\mu_x^\omega\right|^\alpha \leqq \left(\int |u|^\alpha d\mu_x^\omega\right)\left(\int d\mu_x^\omega\right)^{1-(1/\alpha)} \leqq \int |u|^\alpha d\mu_x^\omega.$$

Hence, $-|u|^\alpha$ is superharmonic.

The associated measure of $-u^2$ will be denoted by μ_u for $u \in \mathscr{H}(\Omega)$. If u is bounded, i.e., $|u| \leqq M$, then we have $u^2 = h - U^{\mu_u}$ with $h \in \mathscr{H}(\Omega)$ and $0 \leqq U^{\mu_u} \leqq h \leqq M^2$. We consider the class

$$\boldsymbol{H}_{BE} = \boldsymbol{H}_{BE}(\Omega) \equiv \left\{u \in \mathscr{H}(\Omega);\ \text{bounded},\ \mu_u(\Omega) < +\infty \text{ and } \int_\Omega u^2 d\pi < +\infty\right\}.$$

From $(u+v)^2 + (u-v)^2 = 2(u^2+v^2)$, it follows that $\mu_{u+v} + \mu_{u-v} = 2(\mu_u + \mu_v)$ for $u, v \in \mathscr{H}(\Omega)$. Therefore, we see that $\boldsymbol{H}_{BE}$ is a linear space. Let

$$\boldsymbol{M}_B = \boldsymbol{M}_B(\Omega) \equiv \{\mu;\ \text{measure on } \Omega \text{ such that } U^\mu \text{ is bounded and } \mu(\Omega) < +\infty\},$$

$$\boldsymbol{B}_E = \boldsymbol{B}_E(\Omega) \equiv \{u + U^\mu - U^\nu;\ u \in \boldsymbol{H}_{BE},\ \mu, \nu \in \boldsymbol{M}_B\}.$$

$\boldsymbol{B}_E$ is a linear space of bounded functions on Ω and $\boldsymbol{H}_{BE}$ is a linear subspace of $\boldsymbol{B}_E$. We shall define the notion of energy for functions in $\boldsymbol{B}_E$. For this we need some preparations.

For $u, v \in \mathscr{H}(\Omega)$, let $u \vee v$ (resp. $u \wedge v$) be the least harmonic majorant of $\max(u, v)$ (resp. the greatest harmonic minorant of $\min(u, v)$) on Ω whenever it exists. If u, v are bounded, then $u \vee v$ and $u \wedge v$ exist.

LEMMA 2.2 *If* $u \in \boldsymbol{H}_{BE}$, *then* $u \vee 0$, $u \wedge 0 \in \boldsymbol{H}_{BE}$.

PROOF. Since $u \vee 0 = \{u \vee (-u) + u\}/2$ and $u \wedge 0 = \{u - u \vee (-u)\}/2$, it is enough to show that $u \vee (-u) \in \boldsymbol{H}_{BE}$. Let $u^2 = h - U^{\mu_u}$. Then $h^{1/2} \geqq |u|$. For any regular domain ω and $x \in \omega$, $\int h^{1/2} d\mu_x^\omega \leqq \left(\int h \, d\mu_x^\omega\right)^{1/2} = h(x)^{1/2}$. Hence $h^{1/2}$ is superharmonic on Ω. It follows that $h^{1/2} \geqq u \vee (-u) \geqq |u|$, i.e., $h \geqq [u \vee (-u)]^2 \geqq u^2$. Obviously, $u \vee (-u)$ is bounded. $h - [u \vee (-u)]^2$ is non-negative superharmonic and majorized by $h - u^2 = U^{\mu_u}$. Hence, it is a potential, so that $[u \vee (-u)]^2 = h - U^\nu$, where $\nu = \mu_{u \vee (-u)}$. Since $U^\nu \leqq U^{\mu_u}$, Lemma 1.10 implies that $\nu(\Omega) \leqq \mu_u(\Omega) < +\infty$. Finally, since $\int_\Omega u^2 d\pi < +\infty$ and $\int_\Omega U^{\mu_u} d\pi < +\infty$ (Lemma 1.7), $\int_\Omega h \, d\pi < +\infty$, and hence, $\int_\Omega [u \vee (-u)]^2 d\pi < +\infty$. Therefore $u \vee (-u) \in \boldsymbol{H}_{BE}$.

LEMMA 2.3 *If* f, $g \in \boldsymbol{B}_E$, *then* fg *is expressed as* $fg = u + U^\mu - U^\nu$ *with* $u \in \mathscr{H}(\Omega)$ *and* μ, $\nu \in \boldsymbol{M}_B$.

PROOF. Let $\boldsymbol{B} = \{u + U^\mu - U^\nu;\ u \in \mathscr{H}(\Omega),\ \mu, \nu \in \boldsymbol{M}_B\}$. Then $\boldsymbol{B}$ is a linear space of real valued functions on Ω. Since $fg = \{(f+g)^2 - f^2 - g^2\}/2$, it is enough to prove that $f \in \boldsymbol{B}_E$ implies $f^2 \in \boldsymbol{B}$. If $f = u + U^\mu - U^\nu$ with $u \in \boldsymbol{H}_{BE}$ and μ, $\nu \in \boldsymbol{M}_B$, then $f = f_1 - f_2$ with $f_1 = u \vee 0 + U^\mu$ and $f_2 = (-u) \vee 0 + U^\nu$. By the above lemma, $f_1, f_2 \in \boldsymbol{B}_E$. Since $f^2 = 2(f_1^2 + f_2^2) - (f_1 + f_2)^2$, we may assume that $f = u + U^\mu$ with $u \geqq 0$ ($u \in \boldsymbol{H}_{BE}$, $\mu \in \boldsymbol{M}_B$).

Let $M = \sup_{x \in \Omega} f(x)$. Then $0 \leqq M < +\infty$. Let $s = M^2 - (M - f)^2$. Then $f^2 = 2Mf - s$. Obviously, $2Mf \in \boldsymbol{B}$. We shall show that $s \in \boldsymbol{B}$. First we remark that $0 \leqq s \leqq M^2$. Since $M - f$ is non-negative upper semi-continuous, s is lower semi-continuous. For any regular domain ω and $x \in \omega$,

$$\begin{aligned}\int s \, d\mu_x^\omega &= 2M \int f \, d\mu_x^\omega - \int f^2 d\mu_x^\omega \\ &\leqq 2M \int f \, d\mu_x^\omega - \left(\int f \, d\mu_x^\omega\right)^2 \\ &= M^2 - \left(M - \int f \, d\mu_x^\omega\right)^2 \leqq M^2 - [M - f(x)]^2 = s(x),\end{aligned}$$

where we used the superharmonicity of f for the last inequality. Therefore, s is non-negative superharmonic on Ω. Let $s = k + U^{\mu'}$ with $k \in \mathscr{H}(\Omega)$. Since s is bounded, $U^{\mu'}$ is bounded. Let $u^2 = h - U^{\mu_u}$ with $h \in \mathscr{H}(\Omega)$. Then, from

$$(u + U^\mu)^2 = f^2 = 2Mf - s = 2M(u + U^\mu) - k - U^{\mu'}$$

it follows that

$$h - U^{\mu_u} = u^2 = -2u U^\mu - (U^\mu)^2 + 2Mu + 2MU^\mu - k - U^{\mu'},$$

or

$$h-2Mu+k=U^{\mu_u}+2MU^{\mu}-[2uU^{\mu}+(U^{\mu})^2+U^{\mu'}].$$

Thus, noting that $U^{\mu}\leqq M$, we have

$$-(3MU^{\mu}+U^{\mu'})\leqq h-2Mu+k\leqq U^{\mu_u}+2MU^{\mu}.$$

It follows that $h-2Mu+k=0$, i.e., $k=2Mu-h$. Hence,

$$U^{\mu'}=s-k=2M(u+U^{\mu})-(u+U^{\mu})^2-2Mu+h$$

$$\leqq 2MU^{\mu}-u^2+h=2MU^{\mu}+U^{\mu_u}.$$

Hence, by Lemma 1.10, we have $\mu'(\Omega)\leqq 2M\mu(\Omega)+\mu_u(\Omega)<+\infty$, since $\mu\in\boldsymbol{M}_B$ and $u\in\boldsymbol{H}_{BE}$. Therefore, $\mu'\in\boldsymbol{M}_B$, and hence $s\in\boldsymbol{B}$.

LEMMA 2.4. *If* $f\in\boldsymbol{B}_E$, *then* $\int_\Omega f^2 d\pi<+\infty$.

PROOF. As in the proof of the previous lemma, we may assume that $f=u+U^{\mu}$ with $u\in\boldsymbol{H}_{BE}$, $u\geqq 0$ and $\mu\in\boldsymbol{M}_B$. Let $f\leqq M$. Then $f^2\leqq u^2+2fU^{\mu}\leqq u^2+2MU^{\mu}$. By definition, $\int_\Omega u^2 d\pi<+\infty$. By Lemma 1.7, $\int_\Omega U^{\mu}d\pi<+\infty$. Hence $\int_\Omega f^2 d\pi<+\infty$.

§ 2.2. Definition of energy for functions in $\boldsymbol{B}_E$.

If a function f is expressed as $f=u+U^{\mu}-U^{\nu}$ with $u\in\mathscr{H}(\Omega)$ and μ, ν being measures such that $\mu(\Omega)$, $\nu(\Omega)$ are finite, then the signed measure $\mu-\nu$ is determined by the function f (Lemma 1.6). Thus this signed measure is denoted by σ_f. In this case, $|\sigma_f|(\Omega)\leqq\mu(\Omega)+\nu(\Omega)<+\infty$, so that $\sigma_f(\Omega)(=\mu(\Omega)-\nu(\Omega))$ is well-defined. Obviously, the mapping $f\to\sigma_f$ is linear.

DEFINITION. For f, $g\in\boldsymbol{B}_E$, we define

$$E[f,\ g]=\frac{1}{2}\left\{\int_\Omega f d\sigma_g+\int_\Omega g d\sigma_f-\sigma_{fg}(\Omega)+\int_\Omega fg\ d\pi\right\}, \tag{2.1}$$

which is called the *mutual energy* of f and g. The *energy* of $f\in\boldsymbol{B}_E$ is defined by

$$E[f]=E[f,f]. \tag{2.2}$$

$E[f,\ g]$ for f, $g\in\boldsymbol{B}_E$ is well-defined by virtue of Lemmas 2.3 and 2.4. The mapping $(f,\ g)\to E[f,\ g]$ is obviously a symmetric bilinear form on $\boldsymbol{B}_E\times\boldsymbol{B}_E$.

The above definition is based on the observation made in the introduction.

In fact, if Ω is a domain in a Euclidean space R^n ($n \geqq 3$), then the solutions of $\Delta u = Pu$ ($P \geqq 0$; locally Hölder continuous) form a self-adjoint harmonic structure on Ω which satisfies Axioms $1 \sim 6$ (cf. [5; Chap. VII]). In this case, if f, g are bounded C^2-functions with finite Dirichlet integrals on Ω such that $\int_\Omega f^2 P\, dx$ and $\int_\Omega g^2 P\, dx$ are finite, then the right hand side of (2.1) is equal to the ordinary mutual energy

$$\int_\Omega \sum \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i}\, dx + \int_\Omega fgPdx.$$

Remark that the measure π in this case is Pdx.

PROPOSITION 2.1. *If $u \in \boldsymbol{H}_{BE}$, then*

$$E[u] = \frac{1}{2}\left\{\mu_u(\Omega) + \int_\Omega u^2 d\pi\right\},$$

so that $E[u] \geqq 0$.

PROOF. Since $\sigma_u = 0$ and $\sigma_{u^2} = -\mu_u$, this proposition immediately follows from (2.1) and (2.2).

THEOREM 2.1. *If $\mu \in \boldsymbol{M}_B$, then $U^\mu \in \boldsymbol{B}_E$ and*

$$E[U^\mu] = \int_\Omega U^\mu d\mu.$$

PROOF. Obviously, $U^\mu \in \boldsymbol{B}_E$ for $\mu \in \boldsymbol{M}_B$ and $\sigma_{U^\mu} = \mu$. Hence it is enough to prove that

$$\sigma_{(U^\mu)^2}(\Omega) = \int_\Omega (U^\mu)^2 d\pi. \tag{2.3}$$

For any $\alpha > 0$, $\min(U^\mu/\alpha, 1)$ is a potential. Let μ_α be its associated measure. Then $0 \leqq U^{\mu_\alpha} \leqq 1$ and $U^{\mu_\alpha} \uparrow 1$ as $\alpha \downarrow 0$. By Lemma 2.3, $(U^\mu)^2 = U^{\mu_1} - U^{\mu_2}$ with $\mu_1, \mu_2 \in \boldsymbol{M}_B$. Since $\sigma_{(U^\mu)^2} = \mu_1 - \mu_2$,

$$\begin{aligned}
\sigma_{(U^\mu)^2}(\Omega) &= \lim_{\alpha \to 0} \int_\Omega U^{\mu_\alpha} d\sigma_{(U^\mu)^2} \\
&= \lim_{\alpha \to 0} \left\{ \int_\Omega U^{\mu_\alpha} d\mu_1 - \int_\Omega U^{\mu_\alpha} d\mu_2 \right\} \\
&= \lim_{\alpha \to 0} \int_\Omega (U^{\mu_1} - U^{\mu_2}) d\mu_\alpha = \lim_{\alpha \to 0} \int_\Omega (U^\mu)^2 d\mu_\alpha.
\end{aligned}$$

Let $\omega_\alpha = \{x \in \Omega;\, U^\mu(x) > \alpha\}$. Then ω_α is an open set and $U^{\mu_\alpha} = 1$ on ω_α. Hence, by Lemma 1.8, $\mu_\alpha | \omega_\alpha = \pi | \omega_\alpha$. Therefore,

$$\int_\Omega (U^\mu)^2 d\mu_\alpha = \int_{\Omega - \omega_\alpha} (U^\mu)^2 d\mu_\alpha + \int_{\omega_\alpha} (U^\mu)^2 d\pi.$$

Since $\omega_\alpha \uparrow \Omega$ as $\alpha \downarrow 0$, $\int_{\omega_\alpha} (U^\mu)^2 d\pi \to \int_\Omega (U^\mu)^2 d\pi$ as $\alpha \to 0$. On the other hand, since $U^\mu \leqq \alpha$ on $\Omega - \omega_\alpha$,

$$0 \leqq \int_{\Omega-\omega_\alpha} (U^\mu)^2 d\mu_\alpha \leqq \alpha \int_{\Omega-\omega_\alpha} U^\mu d\mu_\alpha$$

$$\leqq \alpha \int_\Omega U^\mu d\mu_\alpha = \alpha \int_\Omega U^{\mu_\alpha} d\mu \leqq \alpha\mu(\Omega) \to 0 \ (\alpha \to 0).$$

Therefore, $\lim_{\alpha\to 0} \int_{\Omega-\omega_\alpha} (U^\mu)^2 d\mu_\alpha = 0$, and hence we obtain (2.3).

COROLLARY. *If $f_i = U^{\mu_i} - U^{\nu_i}$, $i=1, 2$, with $\mu_1, \mu_2, \nu_1, \nu_2 \in \boldsymbol{M}_B$, then $f_1, f_2 \in \boldsymbol{B}_E$ and*

$$E[f_1, f_2] = \int_\Omega f_1 d\sigma_{f_2} = \int_\Omega f_2 d\sigma_{f_1} \qquad (\sigma_{f_i} = \mu_i - \nu_i,\ i=1, 2).$$

§ 2.3. Orthogonality.

In this section, we shall prove

THEOREM 2.2. *If $u \in \boldsymbol{H}_{BE}$ and $\mu \in \boldsymbol{M}_B$, then*

$$E[u, U^\mu] = 0.$$

The proof of this theorem will be given by a series of lemmas. For each $y \in \Omega$ and $\alpha > 0$, let $\omega_{\alpha,y} = \{x \in \Omega;\ G(x, y) > \alpha\}$. We see that $\omega_{\alpha,y}$ is a domain whenever $\alpha < G(y, y)$ (cf. [4; n°17]).

LEMMA 2.5. $\pi(\omega_{\alpha,y}) \leqq \dfrac{1}{\alpha}$ *and* $\lim\limits_{\alpha\to 0} \alpha\pi(\omega_{\alpha,y}) = 0.$

PROOF.

$$\pi(\omega_{\alpha,y}) \leqq \frac{1}{\alpha} \int_{\omega_{\alpha,y}} G(x, y) d\pi(y) \leqq \frac{1}{\alpha} U^\pi(y) \leqq \frac{1}{\alpha}.$$

Suppose $\varepsilon_0 \equiv \limsup_{\alpha\to 0} \alpha\pi(\omega_{\alpha,y}) > 0$. Fix $\alpha_1 > 0$. We choose $\{\alpha_n\}$ by induction as follows: Suppose $\alpha_1, \cdots, \alpha_n$ are already chosen. Then we can find $\alpha_{n+1} > 0$ such that $\alpha_{n+1} < \alpha_n$, $\alpha_{n+1}\pi(\omega_{\alpha_n,y}) < \varepsilon_0/3$ and $\alpha_{n+1}\pi(\omega_{\alpha_{n+1},y}) > 2\varepsilon_0/3$. Let $\omega_n \equiv \omega_{\alpha_n,y}$ for simplicity. Then

$$1 \geqq U^\pi(y) \geqq \sum_{n=1}^\infty \int_{\omega_{n+1}-\omega_n} G(x, y) d\pi(x)$$

$$\geqq \sum_{n=1}^\infty \alpha_{n+1}\{\pi(\omega_{n+1}) - \pi(\omega_n)\} \geqq \sum_{n=1}^\infty \frac{\varepsilon_0}{3} = +\infty,$$

a contradiction. Hence we have the lemma.

LEMMA 2.6. $\mu_x^{\omega_{\alpha,y}}(\{\xi_a\})=0$ *for any* $x \in \omega_{\alpha,y}$.

PROOF. Let $\omega\equiv\omega_{\alpha,y}$ for fixed y and α. Let φ be the characteristic function of $\{\xi_a\}$ on $\partial^a\omega$. For the superharmonic function s_y^* in Lemma 1.3, we see that $\varepsilon(s_y^*|\omega) \in \bar{\mathscr{S}}_\varphi^\omega$ for any $\varepsilon>0$. Hence $0\leqq H_\varphi^\omega\leqq\varepsilon s_y^*$ on ω for any $\varepsilon>0$, so that $H_\varphi^\omega=0$, i.e., $\mu_x^\omega(\{\xi_a\})=0$ for all $x \in \omega$.

Now we consider the mapping $\varphi \in \boldsymbol{C}(\partial^a\omega_{\alpha,y}) \rightarrow \int_{\omega_{\alpha,y}} H_\varphi^{\omega_{\alpha,y}} d\pi$. By Lemma 2.5, the integral exists and is finite for each φ. It is easy to see that this mapping is a non-negative linear functional on $\boldsymbol{C}(\partial^a\omega_{\alpha,y})$. Hence there is a measure $\nu_{\alpha,y}$ on $\partial^a\omega_{\alpha,y}$ such that

$$\int_{\partial^a\omega_{\alpha,y}} \varphi\, d\nu_{\alpha,y}=\int_{\omega_{\alpha,y}} H_\varphi^{\omega_{\alpha,y}}\, d\pi \tag{2.4}$$

for all $\varphi \in \boldsymbol{C}(\partial^a\omega_{\alpha,y})$. We can show that (2.4) holds for any bounded $\mu_x^{\omega_{\alpha,y}}$-measurable function φ on $\partial^a\omega_{\alpha,y}$.

LEMMA 2.7. *If u is a bounded harmonic function on Ω, then*

$$\int_{\partial\omega_{\alpha,y}} u\, d\nu_{\alpha,y}=\int_{\omega_{\alpha,y}} u\, d\pi.$$

PROOF. Let $\omega\equiv\omega_{\alpha,y}$ and let $\varphi(\xi)=u(\xi)$ if $\xi \in \partial\omega$ and $=0$ if $\xi=\xi_a$. Then φ is a bounded μ_x^ω-measurble function on $\partial^a\omega$. Hence, by (2.4),

$$\int_{\partial\omega} u\, d\nu_{\alpha,y}=\int_{\partial^a\omega} \varphi\, d\nu_{\alpha,y}=\int_\omega H_\varphi^\omega\, d\pi.$$

Now, by Lemma 2.6, we see that $H_\varphi^\omega=u$, since u is harmonic. Hence we have the lemma.

For each $y \in \Omega$ and $\alpha>0$, let $\mu_{\alpha,y}\equiv\mu_y^{\omega_{\alpha,y}}$. By Lemma 2.6, $\mu_{\alpha,y}(\{\xi_a\})=0$ and $\nu_{\alpha,y}(\{\xi_a\})=0$. Therefore, $\mu_{\alpha,y}$ and $\nu_{\alpha,y}$ can be regarded as measures on Ω.

LEMMA 2.8. *Let*

$$w_{\alpha,y}=\frac{1}{\alpha} U^{\mu_{\alpha,y}}-U^{\nu_{\alpha,y}}+U^{\pi|\omega_{\alpha,y}}.$$

Then, $0\leqq w_{\alpha,y}\leqq 2$ *on* Ω *and* $w_{\alpha,y}=1$ *on* $\omega_{\alpha,y}$.

PROOF. Fix y and α and let $\mu=\mu_{\alpha,y}$, $\nu=\nu_{\alpha,y}$, $\omega=\omega_{\alpha,y}$ and $w=w_{\alpha,y}$. By Lemma 1.4, $U^\mu(x)\leqq G(x, y)$ for all $x \in \Omega$ and $U^\mu(x)=\int_\Omega G(\xi, y)\, d\mu_x^\omega(\xi)=\alpha\int_{\partial\omega} d\mu_x^\omega$ for $x \in \omega$. Hence $0\leqq U^\mu\leqq\alpha$ on Ω and, by Lemma 2.6 we see that $U^\mu=\alpha H_1^\omega$ on ω. For any $x \in \Omega$, let $\varphi_x(\xi)=G(x, \xi)$ if $\xi \in \partial\omega$ and $=0$ if $\xi=\xi_a$. Then φ_x

is non-negative lower semi-continuous on $\partial^a\omega$. Choose $\varphi_n \in \boldsymbol{C}(\partial^a\omega)$ such that $\varphi_n \geqq 0$ and $\varphi_n \uparrow \varphi_x$ on $\partial^a\omega$. Then

$$U^\nu(x)=\int_\Omega G(x, \xi)d\nu(\xi)=\int_{\partial\omega}\varphi_x\, d\nu$$

$$=\lim_{n\to\infty}\int_{\partial\omega}\varphi_n\, d\nu$$

$$=\lim_{n\to\infty}\int_\omega H^\omega_{\varphi_n}\, d\pi=\lim_{n\to\infty}\int_\omega\left(\int\varphi_n d\mu_z^\omega\right)d\pi(z).$$

Since $\varphi_n \leqq \varphi_x$, $\int\varphi_n d\mu_z^\omega \leqq G(x, z)$ (cf. Lemma 1.4, (i)). Hence $U^\nu(x)\leqq\int_\omega G(x, z)\, d\pi(z)=U^{\pi|\omega}(x)$. Thus, $0\leqq U^{\pi|\omega}-U^\nu\leqq U^\pi\leqq 1$ on Ω. It follows that $0\leqq w\leqq 2$. Furthermore, for each $x\in\omega$, φ_x is a bounded function on $\partial^a\omega$, and hence $U^\nu(x)=\int_\omega H^\omega_{\varphi_x}d\pi$. By Proposition 1.2, $H^\omega_{\varphi_x}(z)=G(x, z)-G^\omega(x, z)$. Hence, $U^\nu=U^{\pi|\omega}-U^\pi_\omega$ on ω. On the other hand, since $U^\pi_\omega=1-H^\omega_1$, we have $U^{\pi|\omega}-U^\nu=1-H^\omega_1$ on ω. Thus $w=(1/\alpha)U^\mu-U^\nu+U^{\pi|\omega}=H^\omega_1+(1-H^\omega_1)=1$ on ω.

Lemma 2.9. *If μ is a measure with $\mu(\Omega)<+\infty$, then*

$$\mu(\Omega)=\lim_{\alpha\to 0}\left\{\frac{1}{\alpha}\int_\Omega U^\mu d\mu_{\alpha,y}-\int_\Omega U^\mu d\nu_{\alpha,y}\right\}+\int_\Omega U^\mu d\pi$$

for any $y\in\Omega$.

Proof. In the notation of the previous lemma, we have $\mu(\Omega)=\lim_{\alpha\to 0}\int_\Omega w_{\alpha,y}d\mu$, since $\{w_{\alpha,y}\}_\alpha$ is uniformly bounded, $w_{\alpha,y}=1$ on $\omega_{\alpha,y}$ and $\omega_{\alpha,y}\uparrow\Omega$ as $\alpha\downarrow 0$. Now

$$\int_\Omega w_{\alpha,y}d\mu=\frac{1}{\alpha}\int_\Omega U^{\mu_{\alpha,y}}d\mu-\int_\Omega U^{\nu_{\alpha,y}}d\mu+\int_\Omega U^{\pi|\omega_{\alpha,y}}d\mu$$

$$=\frac{1}{\alpha}\int_\Omega U^\mu d\mu_{\alpha,y}-\int_\Omega U^\mu d\nu_{\alpha,y}+\int_{\omega_{\alpha,y}}U^\mu d\pi.$$

The last integral tends to $\int_\Omega U^\mu d\pi$ as $\alpha\to 0$. Hence we obtain the lemma.

Lemma 2.10. *Let $\{\Omega_n\}$ be an exhaustion of Ω, i.e., a sequence of relatively compact open sets in Ω such that $\overline{\Omega_n}\subset\Omega_{n+1}$, $n=1, 2,\cdots$, and $\bigcup_{n=1}^\infty\Omega_n=\Omega$. Then, for any $u\in\boldsymbol{H}_{BE}$ and $y\in\Omega$,*

$$u(y)=\lim_{n\to\infty}\int_{\Omega-\Omega_n}G(x, y)u(x)d\lambda_{\Omega_n}(x). \tag{2.5}$$

Proof. Fix $y\in\Omega$ and choose m such that $y\in\Omega_m$. Let $\beta=\sup_{x\in\Omega-\Omega_m}G(x, y)$. Then, $0<\beta<+\infty$ (Lemma 1.2). Put $p=\min(G_y, \beta)$. Since $p\leqq G_y$

and p is bounded, $p \in \boldsymbol{B}_E$. Since u is bounded, $|up| \leqq Mp$ for some M. Then, it follows from Lemma 2.3 that $up = U^{\mu_1} - U^{\mu_2}$ with $\mu_1, \mu_2 \in \boldsymbol{M}_B$. For simplicity, let $\lambda_n \equiv \lambda_{\Omega_n}$, $\mu_\alpha \equiv \mu_{\alpha,y}$ and $\nu_\alpha \equiv \nu_{\alpha,y}$. First we see that

$$\mu_1(\Omega) - \mu_2(\Omega) = \lim_{n\to\infty} \left\{ \int_\Omega U^{\lambda_n} d\mu_1 - \int_\Omega U^{\lambda_n} d\mu_2 \right\}$$

$$= \lim_{n\to\infty} \int_\Omega (U^{\mu_1} - U^{\mu_2}) d\lambda_n = \lim_{n\to\infty} \int_\Omega up\ d\lambda_n.$$

Since $\lambda_n | \Omega_n = \pi | \Omega_n$ and up is π-summable (Lemma 1.7), we have

$$\lim_{n\to\infty} \int_\Omega up\ d\lambda_n = \lim_{n\to\infty} \int_{\Omega - \Omega_n} up\ d\lambda_n + \int_\Omega up\ d\pi.$$

On the other hand, by the previous lemma,

$$\mu_1(\Omega) - \mu_2(\Omega) = \lim_{\alpha\to 0} \left\{ \frac{1}{\alpha} \int_\Omega up\ d\mu_\alpha - \int_\Omega up\ d\nu_\alpha \right\} + \int_\Omega up\ d\pi.$$

Hence,

$$\lim_{n\to\infty} \int_{\Omega - \Omega_n} up\ d\lambda_n = \lim_{\alpha\to 0} \left\{ \frac{1}{\alpha} \int_\Omega up\ d\mu_\alpha - \int_\Omega up\ d\nu_\alpha \right\}. \tag{2.6}$$

For $0 < \alpha < \beta$, $p = G_y = \alpha$ on $\partial\omega_{\alpha,y}$. Since $S(\mu_\alpha)$, $S(\nu_\alpha) \subset \partial\omega_{\alpha,y}$, the right hand side of (2.6) is equal to $\lim_{\alpha\to 0} \left\{ \int_\Omega u\ d\mu_\alpha - \alpha \int_\omega u\ d\nu_\alpha \right\}$. By Lemmas 2.6 and 2.7, we have $\int_\Omega u\ d\mu_\alpha = u(y)$ and $\int_\Omega u\ d\nu_\alpha = \int_{\omega_{\alpha,y}} u\ d\pi$. Since u is bounded, $\lim_{\alpha\to 0} \alpha \int_{\omega_{\alpha,y}} u\ d\pi = 0$ by Lemma 2.5. Therefore, by (2.6)

$$\lim_{n\to\infty} \int_{\Omega - \Omega_n} up\ d\lambda_n = u(y),$$

which is the required formula (2.5), since $p = G_y$ on $\Omega - \Omega_n$ for $n \geqq m$.

Finally, we prove:

LEMMA 2.11. *Let $\{\Omega_n\}$ be an exhaustion of Ω. For any $u \in \boldsymbol{H}_{BE}$ and $\mu \in \boldsymbol{M}_B$,*

$$\lim_{n\to\infty} \int_\Omega u U^\mu d\lambda_{\Omega_n} = \int_\Omega u\ d\mu + \int_\Omega u U^\mu d\pi.$$

PROOF. For simplicity, let $\lambda_n \equiv \lambda_{\Omega_n}$. Since $\lambda_n | \Omega_n = \pi | \Omega_n$ and uU^μ is π-summable (Lemma 1.7),

$$\lim_{n\to\infty} \int_\Omega u U^\mu d\lambda_n = \lim_{n\to\infty} \int_{\Omega - \Omega_n} u U^\mu d\lambda_n + \int_\Omega u U^\mu d\pi.$$

Hence, it is enough to show that

$$\int_{\Omega} u\, d\mu = \lim_{n\to\infty} \int_{\Omega-\Omega_n} u U^{\mu} d\lambda_n. \tag{2.7}$$

Now, since uU^{μ} is bounded and $\lambda_n(\Omega) < +\infty$, Fubini's theorem can be applied, so that

$$\begin{aligned}\int_{\Omega-\Omega_n} u U^{\mu} d\lambda_n &= \int_{\Omega-\Omega_n} u(x)\left\{\int_{\Omega} G(x, y)\, d\mu(y)\right\} d\lambda_n(x) \\ &= \int_{\Omega}\left\{\int_{\Omega-\Omega_n} G(x, y) u(x) d\lambda_n(x)\right\} d\mu(y).\end{aligned}$$

Since u is bounded and U^{λ_n} is μ-summable (note that $\mu(\Omega) < +\infty$), Lebesgue's convergence theorem implies

$$\lim_{n\to\infty}\int_{\Omega-\Omega_n} u U^{\mu} d\lambda_n = \int_{\Omega}\left\{\lim_{n\to\infty}\int_{\Omega-\Omega_n} G(x, y) u(x) d\lambda_n(x)\right\} d\mu(y),$$

the right hand side of which is equal to $\int_{\Omega} u\, d\mu$ by (2.5) of the previous lemma. Hence we obtain (2.7).

Proof of Theorem 2.2. Since $|uU^{\mu}| \leqq MU^{\mu}$ for some $M>0$, $uU^{\mu} = U^{\mu_1} - U^{\mu_2}$ with $\mu_1, \mu_2 \in \boldsymbol{M}_B$ by Lemma 2.3. Hence, by the above lemma,

$$\sigma_{uU^{\mu}}(\Omega) = \int_{\Omega} u\, d\mu + \int_{\Omega} u U^{\mu} d\pi.$$

Since $\sigma_u = 0$ and $\sigma_{U^{\mu}} = \mu$,

$$E[u, U^{\mu}] = \frac{1}{2}\left\{\int_{\Omega} u\, d\mu - \sigma_{uU^{\mu}}(\Omega) + \int_{\Omega} u U^{\mu} d\pi\right\} = 0.$$

§2.4. An estimate.

As an application of Lemma 2.11, we shall prove

Proposition 2.2. *For any $u \in \mathscr{H}(\Omega)$ and any domain $\omega \subset \Omega$*

$$\int_{\omega} u^2 d\pi \leqq \mu_u(\omega). \tag{2.8}$$

We need one more lemma to prove this proposition.

Lemma 2.12. *For any $u \in \mathscr{H}(\Omega)$ and any relatively compact domain $\omega \subset \Omega$, $u^2 | \omega \in \boldsymbol{B}_E(\omega)$.*

Proof. Since ω is relatively compact, $u|\omega$ is bounded, and hence $u^2|\omega$ is bounded. By Lemma 2.1, $-u^2$ and $-u^4$ are superharmonic. The associated measure of $-u^2$ is μ_u by definition. Let ν by the associated measure of $-u^4$. Since μ_u and ν are measures on Ω, $\mu_u(\omega)<+\infty$ and $\nu(\omega)<+\infty$. Let $u^2|\omega=h-U_\omega^{\mu_u}$ and $u^4|\omega=k-U_\omega^\nu$ with $h, k \in \mathscr{H}(\omega)$. Since $u^2|\omega$ is bounded, $U_\omega^{\mu_u}$ is bounded, so that $\mu_u|\omega \in \boldsymbol{M}_B(\omega)$. From $u^2|\omega \leqq M$ for some M and

$$k-h^2=u^4|\omega+U_\omega^\nu-(u^2|\omega+U_\omega^{\mu_u})^2$$

it follows that

$$-3MU_\omega^{\mu_u} \leqq k-h^2 \leqq U_\omega^\nu \qquad \text{on } \omega.$$

Since $k-h^2$ is superharmonic on ω, the above inequalities imply that $k-h^2$ is a potential on ω. Let $k-h^2=U_\omega^{\mu'}$. Then $\mu'=\mu_h$. Since $U_\omega^{\mu'} \leqq U_\omega^\nu$, Lemma 1.10 implies $\mu_h(\omega)=\mu'(\omega)\leqq \nu(\omega)<+\infty$. Since $\pi(\omega)<+\infty$ and h is bounded, $\int_\omega h^2 d\pi<+\infty$. Therefore, $h \in \boldsymbol{H}_{BE}(\omega)$, and hence $u^2|\omega \in \boldsymbol{B}_E(\omega)$.

PROOF of Proposition 2.2. First we suppose $\pi(\Omega)<+\infty$ and $u^2 \in \boldsymbol{B}_E$, and prove (2.8) for $\omega=\Omega$. It is trivial if $u=0$; thus let $u\neq 0$. Let $u^2=h-U^{\mu_u}$ with $h \in \mathscr{H}(\Omega)$. Then $h>0$ on Ω, and hence u^2/h is a finite continuous function on Ω. For any regular domain ω and for any $x \in \omega$,

$$u(x)^2=\left(\int u \, d\mu_x^\omega\right)^2 \leqq \left(\int \frac{u^2}{h} \, d\mu_x^\omega\right)\left(\int h \, d\mu_x^\omega\right)=h(x)\int \frac{u^2}{h} d\mu_x^\omega.$$

Therefore $-u^2/h$ is superharmonic on Ω. Since $u^2/h \leqq 1$ on Ω, we have $u^2/h \leqq 1-U^\pi$, i.e., $u^2 \leqq h-hU^\pi$. It then follows that $U^{\mu_u} \geqq hU^\pi$. Thus, for an exhaustion $\{\Omega_n\}$ of Ω, letting $\lambda_n=\lambda_{\Omega_n}$, we have

$$\mu_u(\Omega)=\lim_{n\to\infty}\int_\Omega U^{\lambda_n} d\mu_u=\lim_{n\to\infty}\int_\Omega U^{\mu_u} d\lambda_n \geqq \lim_{n\to\infty}\int_\Omega hU^\pi d\lambda_n. \tag{2.9}$$

By our assumption, $\pi \in \boldsymbol{M}_B$ and $h \in \boldsymbol{H}_{BE}$. Hence, by Lemma 2.11, the last term of (2.9) is equal to $\int_\Omega h \, d\pi+\int_\Omega hU^\pi d\pi$. Therefore

$$\mu_u(\Omega) \geqq \int_\Omega h \, d\pi \geqq \int_\Omega u^2 d\pi.$$

Next, let ω be any domain. For any relatively compact domain ω' contained in ω, $\pi(\omega')<+\infty$ and, by the previous lemma, $u^2|\omega' \in \boldsymbol{B}_E(\omega')$. Therefore, applying the above result to $\mathfrak{H}|\omega'$, we have $\mu_u(\omega') \geqq \int_{\omega'} u^2 d\pi$. Hence

$$\mu_u(\omega)=\sup_{\omega'} \mu_u(\omega') \geqq \sup_{\omega'}\int_{\omega'} u^2 d\pi=\int_\omega u^2 d\pi,$$

where the suprema are taken over all relatively compact domains ω' contained in ω.

REMARK. Proposition 2.2 implies that (2.8) holds for any Borel set ω.

CHAPTER III. Energy-finite Harmonic Functions

§ 3.1. Energy of harmonic functions.

If $u \in \mathscr{H}(\Omega)$, then there corresponds the measure μ_u, which is the associated measure of $-u^2$. Thus we can define the value

$$E[u]=E_\Omega[u]\equiv\frac{1}{2}\left\{\mu_u(\Omega)+\int_\Omega u^2 d\pi\right\},$$

which is in $[0, +\infty]$. By Proposition 2.1, this value coincides with $E[u]$ defined in the previous chapter for $u \in \boldsymbol{H}_{BE}$. Thus, it is also called the *energy* of u on Ω. If $E[u]<+\infty$, then we call u an *energy-finite harmonic function.* Let

$$\boldsymbol{H}_E=\boldsymbol{H}_E(\Omega)\equiv\{u \in \boldsymbol{H}(\Omega);\ E[u]<+\infty\}.$$

Obviously, $\boldsymbol{H}_{BE}=\{u \in \boldsymbol{H}_E;\ \text{bounded}\}$. By virtue of Proposition 2.2, we see that $\mu_u(\Omega)/2\leqq E[u]\leqq\mu_u(\Omega)$ for any $u \in \mathscr{H}(\Omega)$, and hence $\boldsymbol{H}_E=\{u \in \mathscr{H}(\Omega);\ \mu_u(\Omega)<+\infty\}$.

We define $\|u\|$ for $u \in \boldsymbol{H}_E$ as follows: In case $1 \in \mathscr{H}(\Omega)$,

$$\|u\|=\{E[u]+|u(x_0)|^2\}^{1/2}$$

for a fixed $x_0 \in \Omega$; in case $1 \notin \mathscr{H}(\Omega)$,

$$\|u\|=E[u]^{1/2}.$$

PROPOSITION 3.1. $\|u\|=0$ *if and only if* $u=0$. *In case* $1 \in \mathscr{H}(\Omega)$, $E[u]=0$ *if and only if* $u=$ *const.*

PROOF. If $1 \in \mathscr{H}(\Omega)$, then $\pi=0$ and $\mu_c=0$ for any constant c. Hence $E[c]=0$. Now suppose $E[u]=0$ $(u \in \boldsymbol{H}_E)$. Then $\mu_u=0$, so that $u^2 \in \mathscr{H}(\Omega)$. Hence $[u-u(x_0)]^2=u^2-2u(x_0)u+u(x_0)^2$ is superharmonic on Ω. Since it is non-negative and vanishes at x_0, it must vanish identically, i.e., $u\equiv u(x_0)$. In case $1 \in \mathscr{H}(\Omega)$ and $\|u\|=0$, $u(x_0)=0$ by the definition of $\|u\|$; in case $1 \notin \mathscr{H}(\Omega)$, $u(x_0)$ must be zero since no non-zero constant is harmonic.

LEMMA 3.1. *If* $u \in \boldsymbol{H}_E$, *then* u^2 *has a harmonic majorant on* Ω, *so that* $u^2=h-U^{\mu_u}$ *with* $h \in \mathscr{H}(\Omega)$.

PROOF. Since $\mu_u(\Omega)<+\infty$, U^{μ_u} is a potential (Lemma 1.5). It then follows that $u^2+U^{\mu_u}$ is harmonic.

LEMMA 3.2. *$\boldsymbol{H}_E$ is a linear subspace of $\mathscr{H}(\Omega)$ and if $u, v \in \boldsymbol{H}_E$, then uv is expressed as $uv = h + U^{\mu} - U^{\nu}$ with $h \in \mathscr{H}(\Omega)$, $\mu(\Omega) < +\infty$ and $\nu(\Omega) < +\infty$.*

PROOF. From $(u+v)^2 + (u-v)^2 = 2(u^2+v^2)$, it follows that $\mu_{u+v} + \mu_{u-v} = 2(\mu_u + \mu_v)$ (cf. §2.1). Hence $\mu_{u+v}(\Omega) + \mu_{u-v}(\Omega) = 2(\mu_u(\Omega) + \mu_v(\Omega)) < +\infty$. Therefore, $u+v \in \boldsymbol{H}_E$. It is obvious that $\alpha u \in \boldsymbol{H}_E$ for any $u \in \boldsymbol{H}_E$ and a real number α. Thus $\boldsymbol{H}_E$ is a linear subspace of $\mathscr{H}(\Omega)$. Since $uv = \{(u+v)^2 - u^2 - v^2\}/2$, Lemma 3.1 implies that $v = h + U^{\mu} - U^{\nu}$ with $h \in \mathscr{H}(\Omega)$, $\nu = \mu_{u+v}/2$ and $\mu = (\mu_u + \mu_v)/2$.

By the above lemma, σ_{uv} is defined for any $u, v \in \boldsymbol{H}_E$ and $\sigma_{uv}(\Omega)$ is a finite value. We define

$$E[u, v] = \frac{1}{2}\left\{-\sigma_{uv}(\Omega) + \int_{\Omega} uv \, d\pi\right\}$$

for $u, v \in \boldsymbol{H}_E$ and call it the mutual energy of u and v. It is easy to see that $E[u, v]$ is a symmetric bilinear form on $\boldsymbol{H}_E \times \boldsymbol{H}_E$ and $E[u, u] = E[u]$. Hence, together with Proposition 3.1, we have

PROPOSITION 3.2. *$\boldsymbol{H}_E$ is a pre-Hilbert space with respect to the inner product*

$$(u, v) = \begin{cases} E[u, v] + u(x_0)v(x_0), & \text{if } 1 \in \mathscr{H}(\Omega) \\ E[u, v] \quad , & \text{if } 1 \notin \mathscr{H}(\Omega), \end{cases}$$

for which $(u, u) = \|u\|^2$.

We remark here that, in case $1 \in \mathscr{H}(\Omega)$, Proposition 3.1 implies that $E[u, c] = 0$, and hence $E[u+c] = E[u]$, for any constant c and $u \in \boldsymbol{H}_E$.

§ 3.2. Lattice structure of $\boldsymbol{H}_E$.

LEMMA 3.3. *If $u \in \boldsymbol{H}_E$, then $u \vee (-u)$ exists and belongs to $\boldsymbol{H}_E$. Furthermore,*

$$E[u \vee (-u)] \leqq E[u].$$

PROOF. Let $u^2 = h - U^{\mu_u}$ with $h \in \mathscr{H}(\Omega)$ (Lemma 3.1). Then, as in the proof of Lemma 2.2, we see that $h^{1/2}$ is superharmonic on Ω majorizing $|u|$. Hence, $u \vee (-u)$ exists and $h \geqq [u \vee (-u)]^2 \geqq u^2$. Then $[u \vee (-u)]^2 = h - U^{\nu}$, where $\nu = \mu_{u \vee (-u)}$. Thus, $U^{\nu} \leqq U^{\mu_u}$, and by Lemma 1.10, $\nu(\Omega) \leqq \mu_u(\Omega) < +\infty$. Let $\{\Omega_n\}$ be an exhaustion of Ω and let $\lambda_n \equiv \lambda_{\Omega_n}$. Since $\lambda_n | \Omega_n = \pi | \Omega_n$ and U^{ν}, U^{μ_u} are π-summable (Lemma 1.7), we have

$$\mu_u(\Omega) - \nu(\Omega) = \lim_{n\to\infty}\left\{\int_{\Omega} U^{\lambda_n} d\mu_u - \int_{\Omega} U^{\lambda_n} d\nu\right\}$$

$$= \lim_{n\to\infty}\int_{\Omega} (U^{\mu_u} - U^{\nu}) d\lambda_n$$

$$=\lim_{n\to\infty}\int_{\Omega-\Omega_n}(U^{\mu_u}-U^\nu)\,d\lambda_n+\int_\Omega(U^{\mu_u}-U^\nu)\,d\pi$$

$$\geqq\int_\Omega(U^{\mu_u}-U^\nu)\,d\pi=\int_\Omega\{[u\vee(-u)]^2-u^2\}\,d\pi.$$

Hence, $\int_\Omega[u\vee(-u)]^2d\pi<+\infty$ and

$$\mu_{u\vee(-u)}(\Omega)+\int_\Omega[u\vee(-u)]^2d\pi\leqq\mu_u(\Omega)+\int_\Omega u^2d\pi,$$

i.e., $u\vee(-u)\in \boldsymbol{H}_E$ and $E[u\vee(-u)]\leqq E[u]$.

REMARK. The above proof also shows that $\mu_{u\vee(-u)}(\Omega)\leqq\mu_u(\Omega)$.

THEOREM 3.1. *$\boldsymbol{H}_E$ is a vector lattice with respect to the operations $\vee$ and $\wedge$. For any $u, v\in\boldsymbol{H}_E$,*

$$E[u\vee v]+E[u\wedge v]\leqq E[u]+E[v].$$

PROOF. Since $u=u\vee 0+u\wedge 0$ and $u\vee(-u)=u\vee 0-u\wedge 0$, the above lemma implies that $u\vee 0$, $u\wedge 0\in\boldsymbol{H}_E$ for any $u\in\boldsymbol{H}_E$ and

$$E[u\vee 0]+E[u\wedge 0]\leqq E[u].$$

For any $u, v\in\boldsymbol{H}_E$, $u\vee v=v+[(u-v)\vee 0]$ and $u\wedge v=v+[(u-v)\wedge 0]$ exist and belong to $\boldsymbol{H}_E$. Furthermore,

$$\begin{aligned}&E[u\vee v]+E[u\wedge v]\\&=2E[v]+2E[v,(u-v)\vee 0]+2E[v,(u-v)\wedge 0]+E[(u-v)\vee 0]+E[(u-v)\wedge 0]\\&\leqq 2E[v]+2E[v,u-v]+E[u-v]=E[u]+E[v].\end{aligned}$$

The following lemma will be used in the next section:

LEMMA 3.4. *If $u\in\boldsymbol{H}_E$ and ω is a non-empty relatively compact open set in Ω, then*

$$\inf_{x\in\bar{\omega}}\min\{(u\vee 0)(x),[(-u)\vee 0](x)\}\leqq\left\{\frac{\mu_u(\Omega)}{4\lambda_\omega(\Omega)}\right\}^{1/2}. \tag{3.1}$$

PROOF. Let ν be the associated measure of the superharmonic function $\min(u, 0)$. Then,

$$\min(u,0)=u\wedge 0+U^\nu, \tag{3.2}$$

and since $(u\vee 0)+(u\wedge 0)=u=\max(u,0)+\min(u,0)$, we have

$$\max(u,0)=u\vee 0-U^\nu. \tag{3.3}$$

Hence, $|u|=u\vee(-u)-2U^{\nu}$. Let $u^2=h-U^{\mu_u}$ with $h\in\mathscr{H}(\Omega)$. We have shown (see the proof of Lemma 3.3) that

$$0\leqq[u\vee(-u)]^2-u^2\leqq U^{\mu_u}.$$

On the other hand,

$$(U^{\nu})^2=\frac{1}{4}[u\vee(-u)-|u|]^2\leqq\frac{1}{4}\{[u\vee(-u)]^2-u^2\}.$$

Hence

$$(U^{\nu})^2\leqq\frac{1}{4}U^{\mu_u}. \tag{3.4}$$

Now, by (3.2) and (3.3) we see that $\min\{u\vee 0,\ (-u)\vee 0\}=U^{\nu}$. Thus (3.1) is equivalent to

$$\inf_{x\in\omega}[U^{\nu}(x)]^2\leqq\frac{\mu_u(\Omega)}{4\lambda_{\omega}(\Omega)}.$$

Now, using (3.4) and noting that $S(\lambda_{\omega})\subset\bar{\omega}$, we have

$$\mu_u(\Omega)\geqq\int_{\Omega}U^{\lambda_{\omega}}d\mu_u=\int_{\Omega}U^{\mu_u}d\lambda_{\omega}$$

$$\geqq 4\int_{\Omega}(U^{\nu})^2d\lambda_{\omega}\geqq 4\,\{\inf_{x\in\omega}[U^{\nu}(x)]^2\}\,\lambda_{\omega}(\Omega).$$

§ 3.3. Bounded family in $\boldsymbol{H}_E$.

The following results are known as consequences of Axioms 1, 2 and 3 (see [3] and [6]):

(A) *Harnack's inequality*: For a compact set K in Ω, there is $\alpha(K)\geqq 1$ such that

$$\sup_{x\in K}u(x)\leqq\alpha(K)\inf_{x\in K}u(x)$$

for all $u\in\mathscr{H}^+(\Omega)\equiv\{u\in\mathscr{H}(\Omega);\ u\geqq 0\}$.

(B) For a fixed $x_0\in\Omega$, $\mathscr{H}^+_{x_0}(\Omega)\equiv\{u\in\mathscr{H}^+(\Omega);\ u(x_0)\leqq 1\}$ is compact with respect to the locally uniform convergence topology. In particular, this family is locally uniformly bounded on Ω.

Now we consider the family

$$\boldsymbol{H}_E^1\equiv\{u\in\boldsymbol{H}_E;\ \|u\|\leqq 1\}.$$

THEOREM 3.2. *$\boldsymbol{H}_E^1$, $\{u\vee 0;\ u\in\boldsymbol{H}_E^1\}$ and $\{u\wedge 0;\ u\in\boldsymbol{H}_E^1\}$ are locally uniformly bounded on Ω.*

PROOF. Since $|u| \leqq \max\{u\vee 0, (-u)\vee 0\}$ and $u\wedge 0 = -[(-u)\vee 0]$, it is enough to show that $\{u\vee 0;\ u\in \boldsymbol{H}_E^1\}$ is locally uniformly bounded. Since this is a subfamily of $\mathscr{H}^+(\Omega)$, the above (B) shows that we only have to prove that $\{(u\vee 0)(x_0);\ u\in \boldsymbol{H}_E^1\}$ is bounded. Suppose it is not bounded. Then there are $u_n\in \boldsymbol{H}_E^1$ such that $(u_n\vee 0)(x_0)\geqq n$, $n=1, 2,\cdots$.

Case 1. $1\in\mathscr{H}(\Omega)$: In this case, since $|u(x_0)|\leqq\|u\|\leqq 1$,

$$[(-u_n)\vee 0](x_0)=(u_n\vee 0)(x_0)-u_n(x_0)\geqq n-1.$$

Let ω be any relatively compact open set containing x_0. Then, by (A), $\inf_{x\in\omega}(u_n\vee 0)(x)\geqq n/\alpha(\bar\omega)$ and $\inf_{x\in\omega}[(-u_n)\vee 0](x)\geqq(n-1)/\alpha(\bar\omega)$, so that

$$\inf_{x\in\omega}\min\{(u_n\vee 0)(x), [(-u_n)\vee 0](x)\}\geqq\frac{n-1}{\alpha(\bar\omega)}.$$

The left hand side is less than $\{\mu_{u_n}(\Omega)/4\lambda_\omega(\Omega)\}^{1/2}$ by Lemma 3.4. Since $\mu_{u_n}(\Omega)\leqq 2\|u_n\|^2\leqq 2$, we have

$$\frac{n-1}{\alpha(\bar\omega)}\leqq\left(\frac{1}{2\lambda_\omega(\Omega)}\right)^{1/2}$$

for all $n=1, 2,\cdots$, which is a contradiction since $\lambda_\omega(\Omega)>0$.

Case 2. $1\notin\mathscr{H}(\Omega)$, i.e., $\pi\neq 0$: Let $v_n=(u_n\vee 0)/(u_n\vee 0)(x_0)$. Then $v_n\in\mathscr{H}^+(\Omega)$ and $v_n(x_0)=1$. By (B), there is a subsequence $\{v_{n_j}\}$ which converges to $v\in\mathscr{H}(\Omega)$ locally uniformly on Ω. In particular, $v(x_0)=1$. Now, using Theorem 3.1, we have

$$\|v_n\|^2\leqq\frac{1}{n^2}\|u_n\vee 0\|^2=\frac{1}{n^2}E[u_n\vee 0]\leqq\frac{1}{n^2}E[u_n]=\frac{1}{n^2}\|u_n\|^2\leqq\frac{1}{n^2}.$$

Hence $\int v_n^2 d\pi\leqq 2\|v_n\|^2\leqq 2/n^2$. Thus, we may assume that $v_{n_j}\to 0$ π-almost everywhere on Ω. Hence $v=0$ π-almost everywhere on Ω. Since $\pi\neq 0$, $v\in\mathscr{H}(\Omega)$ and $v\geqq 0$, it follows that $v=0$, which contradicts the fact $v(x_0)=1$.

COROLLARY. *If $u_n\in\boldsymbol{H}_E$ and $\|u_n\|\to 0$ $(n\to\infty)$, then $u_n\to 0$, $u_n\vee 0\to 0$ and $u_n\wedge 0\to 0$ locally uniformly on Ω.*

LEMMA 3.5. *For any $u\in\boldsymbol{H}_E$, let h_u^* be the least harmonic majorant of u^2. Then $\{h_u^*;\ u\in\boldsymbol{H}_E^1\}$ is locally uniformly bounded on Ω.*

PROOF. Let ω be any non-empty relatively compact open set in Ω. By the above theorem, there is $M>0$ such that $|u(x)|\leqq M$ for all $x\in\bar\omega$ and $u\in\boldsymbol{H}_E^1$. Since $u^2=h_u^*-U^{\mu_u}$ and $\mu_u(\Omega)\leqq 2\|u\|^2\leqq 2$ for each $u\in\boldsymbol{H}_E^1$, we have, by using (A),

$$\begin{aligned}\sup_{x\in\bar\omega} h_u^*(x) &\leqq \alpha(\bar\omega)\inf_{x\in\bar\omega} h_u^*(x)\\ &\leqq \alpha(\bar\omega)\{M^2+\inf_{x\in\bar\omega} U^{\mu_u}(x)\}\\ &\leqq \alpha(\bar\omega)\left\{M^2+\frac{1}{\lambda_\omega(\Omega)}\int_\Omega U^{\mu_u}d\lambda_\omega\right\}\\ &=\alpha(\bar\omega)\left\{M^2+\frac{1}{\lambda_\omega(\Omega)}\int_\Omega U^{\lambda_\omega}d\mu_u\right\}\leqq\alpha(\bar\omega)\left\{M^2+\frac{2}{\lambda_\omega(\Omega)}\right\}.\end{aligned}$$

Hence $\{h_u^*;\ u\in \boldsymbol{H}_E^1\}$ is locally uniformly bounded.

§ 3.4. Completeness of the space $\boldsymbol{H}_E$.

PROPOSITION 3.3. *Let $\{u_n\}$ be a sequence in $\boldsymbol{H}_E$ such that $\{\|u_n\|\}$ is bounded. If $\{u_n\}$ converges to u locally uniformly on Ω, then $u\in\boldsymbol{H}_E$ and*

$$\|u\|\leqq\liminf_{n\to\infty}\|u_n\|.$$

PROOF. Obviously, $u\in\mathscr{H}(\Omega)$. By Lemma 3.5, we see that $\{h_{u_n}^*\}$ is locally uniformly bounded. Let $h_n\equiv h_{u_n}^*$. By (B) and the definition of liminf, we can choose a subsequence $\{u_{n_j}\}$ such that $\{h_{n_j}\}$ converges to $h^*\in\mathscr{H}(\Omega)$ locally uniformly on Ω and $\lim_{j\to\infty}\|u_{n_j}\|=\liminf_{n\to\infty}\|u_n\|$. Since $u_{n_j}^2\leqq h_{n_j}$, we have $u^2\leqq h^*$, so that u^2 has a harmonic majorant. Let $u^2=h_0^*-U^{\mu_u}$ with $h_0^*\in\mathscr{H}(\Omega)$. Obviously, $h_0^*\leqq h^*$. Now, for simplicity, let $\mu_j\equiv\mu_{u_{n_j}}$. Since $U^{\mu_j}=h_{n_j}-u_{n_j}^2$, $\{U^{\mu_j}\}$ converges to h^*-u^2 locally uniformly on Ω. Hence, for any relatively compact open set ω,

$$\begin{aligned}\mu_u(\omega)&\leqq\int_\Omega U^{\lambda_\omega}d\mu_u=\int_\Omega U^{\mu_u}d\lambda_\omega=\int_\Omega(h_0^*-u^2)d\lambda_\omega\\ &\leqq\int_\Omega(h^*-u^2)d\lambda_\omega=\lim_{j\to\infty}\int_\Omega U^{\mu_j}d\lambda_\omega=\lim_{j\to\infty}\int_\Omega U^{\lambda_\omega}d\mu_j\\ &\leqq\liminf_{j\to\infty}\mu_j(\Omega).\end{aligned}$$

Hence, $\mu_u(\Omega)\leqq\liminf_{j\to\infty}\mu_j(\Omega)$. Also, by Fatou's lemma, we have $\int_\Omega u^2d\pi \leqq\liminf_{j\to\infty}\int_\Omega u_{n_j}^2d\pi$. Therefore, $\|u\|\leqq\liminf_{j\to\infty}\|u_{n_j}\|=\liminf_{n\to\infty}\|u_n\|<+\infty$, and this also shows that $u\in\boldsymbol{H}_E$.

COROLLARY. *$\boldsymbol{H}_E^1$ is compact with respect to the locally uniform convergence topology.*

PROOF. The above proposition implies that $\boldsymbol{H}_E^1$ is closed with respect to the locally uniform convergence topology. On the other hand, Theorem 3.2 and (B) show that $\boldsymbol{H}_E^1$ is relatively compact with respect to this topology.

THEOREM 3.3. *$\boldsymbol{H}_E$ is complete with respect to the norm $\|\cdot\|$, so that it is a Hilbert space.*

PROOF. Let $\{u_n\}$ be a Cauchy sequence in $\boldsymbol{H}_E$ with respect to the norm $\|\cdot\|$. Then, it follows from Theorem 3.2 that $\{u_n\}$ converges locally uniformly on Ω. By Proposition 3.3, $u=\lim_{n\to\infty} u_n$ belongs to $\boldsymbol{H}_E$. Furthermore, applying Proposition 3.3 to u_n-u_m for any fixed m, we obtain

$$\|u-u_m\| \leqq \liminf_{n\to\infty} \|u_n-u_m\|$$

for each m. Since $\|u_n-u_m\| \to 0$ as $n, m\to\infty$, it follows that $\|u-u_m\| \to 0$ as $m \to \infty$. Hence $\boldsymbol{H}_E$ is complete.

§3.5. Density of $\boldsymbol{H}_{BE}$ in $\boldsymbol{H}_E$.

LEMMA 3.6. *Given $u \in \mathscr{H}(\Omega)$ and $\alpha>0$, let*

$$v_\alpha=[\max(u-\alpha, 0)]^2 \quad \text{and} \quad w_\alpha=\alpha^2-[\max(\alpha-u, 0)]^2.$$

Then $-v_\alpha$ and w_α are superharmonic functions on Ω. If u^2 has a harmonic majorant, then $-v_\alpha$ and w_α have harmonic minorants. Furthermore, if we express $v_\alpha=k_\alpha-U^{\nu_\alpha}$ and $w_\alpha=h_\alpha+U^{\tau_\alpha}$ with $k_\alpha, h_\alpha \in \mathscr{H}(\Omega)$, then $\nu_\alpha+\tau_\alpha=\mu_u$.

PROOF. It is easy to see that $-v_\alpha$ is superharmonic (cf. the proof of Lemma 2.1). Now, $w_\alpha(x)=\alpha^2$ if $u(x)\geqq\alpha$ and $w_\alpha(x)=2\alpha u(x)-u(x)^2$ if $u(x)<\alpha$. Since $2\alpha u-u^2\leqq\alpha^2$ and α^2, $2\alpha u-u^2$ are superharmonic on Ω, we see that w_α is superharmonic on Ω. If u^2 has a harmonic majorant, then $v_\alpha\leqq u^2$ implies that v_α has a harmonic majorant and $w_\alpha\geqq 2\alpha u-u^2$ implies that w_α has a harmonic minorant. Since $v_\alpha-w_\alpha=u^2-2\alpha u$, we have

$$k_\alpha-h_\alpha-U^{\nu_\alpha}-U^{\tau_\alpha}=h-2\alpha u-U^{\mu_u},$$

where h is the least harmonic majorant of u^2. This equality implies $\nu_\alpha+\tau_\alpha=\mu_u$ by Lemma 1.6.

PROPOSITION 3.4. *If $u \in \boldsymbol{H}_E$, $u\geqq 0$ and $\alpha>0$, then $u\wedge\alpha$* (= the greatest harmonic minorant of $\min(u, \alpha)$) *belongs to $\boldsymbol{H}_E$ and*

$$E[u\wedge\alpha]\leqq E[u].$$

PROOF. Let $f_\alpha=2\alpha(u\wedge\alpha)-(u\wedge\alpha)^2$. Then f_α is superharmonic on Ω and its associated measure is $\mu_{u\wedge\alpha}$. Since $f_\alpha=\alpha^2-(\alpha-u\wedge\alpha)^2\geqq 0$, $(u\wedge\alpha)^2$ has a harmonic majorant. Let $f_\alpha=h_\alpha^*+U^{\mu_{u\wedge\alpha}}$ with $h_\alpha^* \in \mathscr{H}(\Omega)$. Since $u\wedge\alpha\leqq\min(u, \alpha)\leqq\alpha$, we have $0\leqq\alpha-\min(u, \alpha)\leqq\alpha-u\wedge\alpha$, so that

$$f_\alpha=\alpha^2-(\alpha-u\wedge\alpha)^2\leqq\alpha^2-(\alpha-\min(u, \alpha))^2=w_\alpha,$$

or

$$h_\alpha^* + U^{\mu_{u\wedge\alpha}} \leqq h_\alpha + U^{\tau_\alpha}$$

in the notation of the above lemma. It follows that $h_\alpha^* \leqq h_\alpha$. On the other hand,

$$0 \leqq w_\alpha - f_\alpha = (\alpha - u\wedge\alpha)^2 - (\alpha - \min(u, \alpha))^2$$
$$= [2\alpha - u\wedge\alpha - \min(u, \alpha)][\min(u, \alpha) - u\wedge\alpha]$$
$$\leqq 2\alpha[\min(u, \alpha) - u\wedge\alpha].$$

Since $\min(u, \alpha) - u\wedge\alpha$ is a potential, the above inequality implies that $h_\alpha^* \geqq h_\alpha$. Thus, $h_\alpha^* = h_\alpha$, and hence $U^{\mu_{u\wedge\alpha}} \leqq U^{\tau_\alpha}$. Therefore $\mu_{u\wedge\alpha}(\Omega) \leqq \tau_\alpha(\Omega)$. By the above lemma, $\tau_\alpha(\Omega) \leqq \mu_u(\Omega)$. Hence $\mu_{u\wedge\alpha}(\Omega) \leqq \mu_u(\Omega)$. Since $0 \leqq u\wedge\alpha \leqq u$, $\int_\Omega (u\wedge\alpha)^2 d\pi \leqq \int_\Omega u^2 d\pi$. Hence $E[u\wedge\alpha] \leqq E[u] < +\infty$ and $u\wedge\alpha \in \boldsymbol{H}_E$.

PROPOSITION 3.5. *If $u \in \boldsymbol{H}_E$ and $u \geqq 0$, then $\lim_{\alpha\to+\infty} \|u - u\wedge\alpha\| = 0$ and $u\wedge\alpha \to u$ locally uniformly on Ω as $\alpha \to +\infty$.*

PROOF. First note that $u - u\wedge\alpha = (u-\alpha)\vee 0$. By Lemma 3.1,

$$[(u-\alpha)\vee 0]^2 = k_\alpha^* - U^{\mu_{u-u\wedge\alpha}}$$

with $k_\alpha^* \in \mathscr{H}(\Omega)$. Since $0 \leqq \max(u-\alpha, 0) \leqq (u-\alpha)\vee 0$, $k_\alpha^* - U^{\mu_{u-u\wedge\alpha}} \geqq v_\alpha$ for the function v_α given in Lemma 3.6. Hence, in the notation of Lemma 3.6, we have $k_\alpha^* \geqq k_\alpha$. On the other hand, $k_\alpha^{1/2} \geqq \max(u-\alpha, 0)$ and $k_\alpha^{1/2}$ is superharmonic (cf. the proof of Lemma 2.2). Hence $k_\alpha^{1/2} \geqq (u-\alpha)\vee 0$, i.e., $k_\alpha \geqq [(u-\alpha)\vee 0]^2$. Thus $k_\alpha \geqq k_\alpha^*$, and hence $k_\alpha = k_\alpha^*$. Therefore, for the measure ν_α in Lemma 3.6, we have $U^{\mu_{u-u\wedge\alpha}} \leqq U^{\nu_\alpha}$, so that $\mu_{u-u\wedge\alpha}(\Omega) \leqq \nu_\alpha(\Omega)$. Since $v_\alpha = 0$ on $\{u < \alpha\}$, $S(\nu_\alpha) \subset \{u \geqq \alpha\}$. Therefore

$$\mu_{u-u\wedge\alpha}(\Omega) \leqq \nu_\alpha(\{u \geqq \alpha\}).$$

By Lemma 3.6, $\nu_\alpha(\{u \geqq \alpha\}) \leqq \mu_u(\{u \geqq \alpha\})$. Hence

$$\mu_{u-u\wedge\alpha}(\Omega) \leqq \mu_u(\{u \geqq \alpha\}) \to 0 \qquad (\alpha \to +\infty).$$

Then, by Proposition 2.2, we have

$$E[u - u\wedge\alpha] \to 0 \qquad (\alpha \to +\infty).$$

In case $1 \notin \mathscr{H}(\Omega)$, it means that $\|u - u\wedge\alpha\| \to 0$, and it follows from the Corollary to Theorem 3.2 that $u\wedge\alpha \to u$ locally uniformly on Ω. Thus what remains to show is $u = \lim_{\alpha\to+\infty} u\wedge\alpha$ in case $1 \in \mathscr{H}(\Omega)$. Since $u\wedge\alpha$ is monotone increasing with α and $u\wedge\alpha \leqq u$, $v = \lim_{\alpha\to+\infty} u\wedge\alpha$ exists, $v \in \mathscr{H}^+(\Omega)$ and $v \leqq u$. Let $c = u(x_0) - v(x_0)$. Then $c \geqq 0$. Since $E[u - c - u\wedge\alpha] = E[u - u\wedge\alpha] \to 0$ $(\alpha \to +\infty)$ and $(u\wedge\alpha)(x_0) \to u(x_0) - c$, $u\wedge\alpha$ tends to $u - c$ locally uniformly on Ω by the Corollary to Theorem 3.2. It follows that $v = u - c$. Now, $u\wedge\alpha$

$=(u\wedge\alpha)\wedge\alpha\leqq v\wedge\alpha\leqq u\wedge\alpha$, so that $u\wedge\alpha=v\wedge\alpha$. Hence $u\wedge(\alpha+c)=(v+c)\wedge(\alpha+c)=(v\wedge\alpha)+c=(u\wedge\alpha)+c$. Letting $\alpha\to+\infty$, we have $v=v+c=u$.

COROLLARY 1. *$\boldsymbol{H}_{BE}$ is dense in $\boldsymbol{H}_E$; $\boldsymbol{H}_E$ is a completion of $\boldsymbol{H}_{BE}$.*

COROLLARY 2 (Virtanen-Ozawa) *If $\boldsymbol{H}_E$ contains a non-constant function, then it contains a non-constant bounded function.*

PROOF. Let $u\in\boldsymbol{H}_E$ be non-constant. Then either $u\vee 0$ or $u\wedge 0$ is non-constant. Thus we may assume that $u>0$. Then $E[u]>0$. By the above proposition, there is $\alpha>0$ such that $E[u-u\wedge\alpha]<E[u]$. Then $E[u\wedge\alpha]>0$, so that $u\wedge\alpha$ is non-constant (cf. Proposition 3.1), while it is bounded.

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