HIROSHIMA MATH. J.
3 (1973), 387–390

Remarks on Algebraic Hopf Subalgebras

Hiroshi YANAGIHARA

(Received May 14, 1973)

The aim of this note is to give a generalization of a theorem in the paper [2] which is concerned with algebraic Hopf subalgebras of the Hopf algebra attached to a group variety. In other words we show that a similar result to the theorem is obtained for not necessarily reduced group schemes over an algebraically closed field of a positive characteristic p, though the objects in [2] were group varieties exclusively. Moreover we give a corrected proof of Corollary to Lemma 12 in [2], because the previous proof is applicable only in the case where G is an affine algebraic group.

The terminologies are the same as in the papers [1] and [2].

1. In the following let k be an algebraically closed field of a positive characteristic p and G a group scheme of finite type over k. Let $\mathcal{O}=\mathcal{O}_{e,G}$ be the local ring of G at the neutral point e, that is, the stalk of the structure sheal of G at e. If $\mathcal{O}'$ is the local ring $\mathcal{O}_{e\times e,\, G\times G}$ of the product scheme $G\times G$ over k at the point $e\times e$, it is the quotient ring $(\mathcal{O}\otimes_k\mathcal{O})_S$ of $\mathcal{O}\otimes_k\mathcal{O}$ with respect to the multiplicatively closed set S which is the complement of the maximal ideal $\mathfrak{m}\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{m}$ of $\mathcal{O}\otimes_k\mathcal{O}$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$. Let R be the $\mathfrak{m}$-adic completion of $\mathcal{O}$. Then R has a natural structure of a formal group over k in the sense of §5 in [2], whose comultiplication Δ: $R\rightarrow R\hat{\otimes}_k R$ is given by the multiplication m of G. The antipode c of R is determined by the morphism $x\rightarrow x^{-1}$ of G to itself. Then R is called *the formalization of* G, and we remark that Proposition 7 of §5 in [2] is also true in this case. The proof is exactly the same.

First we give a corrected proof of the corollary to Lemma 12 in [2] in a slightly general form.

LEMMA 1. *Let G, $\mathcal{O}$ and $\mathcal{O}'$ be as above. Let $\mathfrak{a}$ be an ideal of $\mathcal{O}$ such that $\triangle(\mathfrak{a})\subset(\mathfrak{a}\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{a})\mathcal{O}'$ and $c(\mathfrak{a})=\mathfrak{a}$. Let G' be the closed subset of G defined by the ideal $\mathfrak{a}$. Then G' is the underlying space of an irreducible group k-subscheme of G.*

PROOF. We may assume that $\mathfrak{a}$ is equal to its radical, because the radical of $\mathfrak{a}$ also satisfies the same hypothesis as $\mathfrak{a}$. From our assumption, it follows that there exists an open subset V of $G'\times G'$ containing $e\times e$ such that the image of V by the morphism m of $G\times G$ onto G is contained in G'. Since each irreducible component of G' contains the point e, $(G'\times G')\cap V$ is a dense open subset of $G'\times G'$. Let x and y be any two points of G' and U any open subset of G containing e. Then there exists an open subset W of $G\times G$ containing the point $e\times e$ such that the image $m((x\times y)W)$ of the open set $(x\times y)W$ by the morphism m is contained in xyU. Since $V\cap(G'\times G')$ is a dense subset of $G'\times G'$, the intersection $(x\times y)W\cap V\cap(G'\times G')$ is not empty. Then we see that $xyU\cap G'$ is not empty. Since G' is a closed subset of G, xy belongs to G'. Similarly we see that x^{-1} is contained in G' if x is an element of G'. This means that G' is the underlying space of a group k-subscheme of G. Moreover G' is irreducible, because it is connected.

q.e.d.

LEMMA 2. *Let G, $\mathcal{O}$, $\mathcal{O}'$ and $\mathfrak{a}$ be the same as in Lemma 1. Then there exists a group k-subscheme G' of G such that the stalk at the point e of the defining ideal for G' is equal to $\mathfrak{a}$.*

PROOF. As seen in Lemma 1, the closed subset G' defined by the ideal $\mathfrak{a}$ is the underlying space of a group subvariety of G which is defined over k. Let U be an affine open subset of G containing e, and A the coordinates ring $\Gamma(U)$ of U. Then $U\times U$ is an affine open subset of $G\times G$ containing $e\times e$. Let S be the multiplicatively closed subset of $\mathcal{O}\otimes_k\mathcal{O}$ such that $\mathcal{O}'=(\mathcal{O}\otimes_k\mathcal{O})_S$. We see easily that there exists an element s of $(A\otimes_k A)\cap S^{(*)}$ such that $V=(U\times U)_s=\mathrm{Spec}(A\otimes_k A)_s=\mathrm{Spec}(B)$ is contained in the inverse image $m^{-1}(U)$ of U. Let $\mathfrak{a}_A$ and $\mathfrak{b}$ be the ideals $\mathfrak{a}\cap A$ and $(\mathfrak{a}\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{a})\mathcal{O}'\cap(A\otimes_k A)_s$ respectively. Then we have $\triangle(\mathfrak{a}_A)\subset\mathfrak{b}$. On the other hand if we put $V'\times e=V\cap(G\times e)$, V' is an open subset of G such that $U\supset V\ni e$. Let L_a (resp. R_a) be the morphism of G onto G such that $L_a(x)=ax$ (resp. $R_a(x)=xa$). Then we have $L_a=m\circ h_a$, where h_a is the closed immersion of G into $G\times G$ given by $h_a(x)=a\times x$. Therefore the comorphism $L_a^*|_A$ of L_a is equal to $h_a^*\circ m^*=h_a^*|_B\circ\triangle|_A$. Now $h_a^*|_B$ is defined by the natural homomorphism $B\to B/(\mathfrak{m}_a\otimes_k A)B$, where $\mathfrak{m}_a$ is the maximal ideal of A corresponding to a. Since $\triangle(\mathfrak{a}_A)\subset\mathfrak{b}$ and $h_a^*(\mathfrak{b}\mathcal{O}_{a\times e})=\mathfrak{a}_A\mathcal{O}_{e,G}$, we can easily see that $L_a^*((\mathfrak{a}_A\mathcal{O}_{e,G})\subset\mathfrak{a}_A\mathcal{O}_{e,G}=\mathfrak{a}$ if $\mathfrak{a}_A\mathcal{O}_{a,G}\neq\mathcal{O}_{a,G}$ and that $L_a^*(\mathfrak{a}_A\mathcal{O}_{a,G})=\mathcal{O}_{e,G}$ if $\mathfrak{a}_A\mathcal{O}_{a,G}=\mathcal{O}_{a,e}$. Similarly if U_0 is a sufficiently small neighbourhood of e such that $x\times x^{-1}$ is contained in V for any x in U_0, we see that $L_{a^{-1}}^*(\mathfrak{a}_A\mathcal{O}_{e,G})\subseteq\mathfrak{a}_A\mathcal{O}_{a,G}$ for a in U_0 and $\mathfrak{a}_A\mathcal{O}_{a,G}\neq\mathcal{O}_{a,G}$. Therefore we see that $L_a^*(\mathfrak{a}_A\mathcal{O}_{a,G})=\mathfrak{a}$ for any closed point a in $U_0\cap V'$ satisfying $\mathfrak{a}_A\mathcal{O}_{a,G}\neq\mathcal{O}_{a,G}$. In the same way we have an open subset U_1 of U such that $R_a^*(\mathfrak{a}_A\mathcal{O}_{a,G})=\mathfrak{a}$ for any closed point a in U_1 satisfying $\mathfrak{a}_A\mathcal{O}_{a,G}\neq\mathcal{O}_{a,G}$. Hence there exists an affine open subset $W=\mathrm{Spec}(C)$

(*) Let A be a commutative ring and let φ be the canonical homomorphism of A into the quotient ring A_T of A with respect to a multiplicatively closed subset T of A. If M is a subset of A_T, we understand by $A\cap M$ the subset $A\cap\varphi^{-1}(M)$ of A.

of G contained in U and an ideal $\mathfrak{a}_C$ of C such that $L_a^*(\mathfrak{a}_C\mathcal{O}_{a,G})=R_a^*(\mathfrak{a}_C\mathcal{O}_{a,G})=\mathfrak{a}$ for any closed point a in $W\cap G'$ and $\mathfrak{a}_C\mathcal{O}_{a,G}=\mathcal{O}_{a,G}$ for any point a in W but not in G'. Next we show that there exists a coherent sheaf $\mathfrak{c}$ of ideals of $\mathcal{O}_G$ satisfying the following conditions: the closed subset of G defined by this sheaf $\mathfrak{c}$ is G' and $\mathfrak{a}=\mathfrak{c}_e=L_x^*(\mathfrak{c}_x)$ for any closed point x in G', where $\mathfrak{c}_x$ is the stalk of $\mathfrak{c}$ at x. To see this let x be any closed point of G'. Then xW is an affine open subset of G containing x, and we have $W=L_{x^{-1}}(xW)$ and hence $xW=\mathrm{Spec}(L_{x^{-1}}^*(C))$. Therefore we put $\Gamma(xW,\mathfrak{c})=L_{x^{-1}}^*(\mathfrak{a}_C)$. Since we have $\mathfrak{a}_C\mathcal{O}_{y,G}=L_{y^{-1}}^*(\mathfrak{a})$ for any closed point y in $W\cap G'$, we can see easily that the ideal of $\mathcal{O}_{xy,G}$ generated by $\Gamma(xW,\mathfrak{c})$ is equal to $L_{(xy)^{-1}}(\mathfrak{a})$.. This implies the existence of a coherent sheaf $\mathfrak{c}$ of ideals of $\mathcal{O}_G$ satisfying the above conditions. Moreover we have $\mathfrak{a}=\mathfrak{c}_e=R_x^*(\mathfrak{c}_x)$ for any closed point x in G'. In fact if x is any closed point of G', there exist closed points y and z in W such that $x=yz$, because $W\cap G'$ and $xW^{-1}\cap G'$ have a common closed point y of G. Therefore we have $L_x=L_{yz}=L_y\circ L_z$ and $R_x=R_{yz}=R_z\circ R_y$. Since $L_{x^{-1}}^*(\mathfrak{a})=R_{x^{-1}}^*(\mathfrak{a})$ for any closed point x in $W\cap G'$ and $R_yL_z=L_zR_y$ for any closed points y and z in G, it follows easily that $L_{x^{-1}}^*(\mathfrak{a})=L_{y^{-1}}^*(L_{z^{-1}}^*(\mathfrak{a}))$ $=R_{z^{-1}}^*(R_{y^{-1}}^*(\mathfrak{a}))=R_{x^{-1}}^*(\mathfrak{a})$. Now we show the morphism m gives naturally the multiplication of the subscheme $(G',\mathcal{O}_G/\mathfrak{c})$. Let a and b be any closed points of G'. Then the morphism m is equal to the composition $R_b\circ L_a\circ m\circ(L_{a^{-1}}\times R_{b^{-1}})$ and hence $m^*(\mathfrak{c}_{ab})=(L_{a^{-1}}\times R_{b^{-1}})^*\circ m^*\circ L_a^*\circ R_b^*(\mathfrak{c}_{ab})=(L_{a^{-1}}\times R_{b^{-1}})^*m^*(\mathfrak{c}_e)$ $\subset(L_{a^{-1}}^*\otimes R_{b^{-1}}^*)((\mathfrak{c}_e\otimes\mathcal{O}_{e,G}+\mathcal{O}_{e,G}\otimes\mathfrak{c}_e)\mathcal{O}')=(\mathfrak{c}_a\otimes\mathcal{O}_b+\mathcal{O}_a\otimes\mathfrak{c}_b)\mathcal{O}_{a\times b}$. This means that m induces naturally a morphism of $(G',\mathcal{O}_G/\mathfrak{c})\times(G',\mathcal{O}_G/\mathfrak{c})$ to $(G',\mathcal{O}_G/\mathfrak{c})$. Similarly we can see that the morphism c of G to G induces a morphism of $(G',\mathcal{O}_G/\mathfrak{c})$ to itself. It is easy to see that these morphisms give the structure of a group subscheme of G to $(G',\mathcal{O}_G/\mathfrak{c})$, and hence the proof is completed.

2. PROPOSITION. *Let G be a group k-scheme and $\mathcal{O}=\mathcal{O}_{e,G}$ the local ring of G at the neutral point e. Let R be the formalization of G and $\bar{\mathfrak{a}}$ an ideal of R such that $(\bar{\mathfrak{a}}\cap\mathcal{O})R=\bar{\mathfrak{a}}$. If $R/\bar{\mathfrak{a}}$ is a formal subgroup of R, then $\mathfrak{a}=\bar{\mathfrak{a}}\cap\mathcal{O}$ corresponds to a group k-subscheme G' of G such that the formalization of G' is $R/\bar{\mathfrak{a}}$.*

PROOF. If $\mathfrak{H}$ is the Hopf subalgebra of $\mathfrak{H}(R)$ corresponding to the formal subgroup $R/\bar{\mathfrak{a}}$, $\mathfrak{H}$ satisfies the condition of Proposition 7 in [2]. Therefore $\Delta(\mathfrak{a})$ is contained in the ideal of $\mathcal{O}'$ generated by $\mathfrak{a}\otimes\mathcal{O}+\mathcal{O}\otimes\mathfrak{a}$ and $c(\mathfrak{a})$ is equal to $\mathfrak{a}$. Hence it follows from Lemma 2 that $\mathfrak{a}$ corresponds to a group k-subscheme $(G',\mathcal{O}_G/\mathfrak{c})$ such that the stalk $\mathfrak{c}_e$ of the sheaf at the neutral point e is $\mathfrak{a}$. Then it is clear that the formalization of $(G',\mathcal{O}_G/\mathfrak{c})$ is $R/\bar{\mathfrak{a}}$. q.e.d.

If G' is a group k-subscheme of G whose local ring at e is given by $\mathcal{O}/\mathfrak{a}$, the formalization of G' is $R/\mathfrak{a}R$ and the Hopf algebra $\mathfrak{H}(R/\mathfrak{a}R)$ attached to the formal group $R/\mathfrak{a}R$ in the sense of § 5 in [2] is canonically identified with the subset of the elements D in the Hopf algebra $\mathfrak{H}(R)$ attached to R such that $D(\mathfrak{a})=0$. We call such a Hopf subalgebra $\mathfrak{H}(R/\mathfrak{a}R)$ of $\mathfrak{H}(R)$ *algebraic in wider sense.* Now we obtain the following generalization of Theorem 6 in [2].

THEOREM. *Let G, $\mathcal{O}$ and R be the same as in the above proposition. Let $\mathfrak{H}(R)$ be the Hopf algebra attached to the formal group R. Let $\mathfrak{H}$ be a Hopf subalgebra of $\mathfrak{H}(R)$ and $\mathfrak{a}$ the ideal of $\mathcal{O}$ consisting of the elements x such that $D(x)=0$ for any element D in $\mathfrak{H}$. Then H is algebraic in wider sense if and only if $\mathfrak{H}$ is the set of the elements D in $\mathfrak{H}(R)$ such that $D(\mathfrak{a})=0$.*

PROOF. Let $\mathfrak{a}_{\mathfrak{H}}$ be the ideal of R consisting of the element x in R such that $D(x)=0$ for any element D in H. Then we see easily in the same way as in Proposition 7 in [2] that $(\mathfrak{a}_{\mathfrak{H}} \cap \mathcal{O})R$ is $\mathfrak{a}_{\mathfrak{H}}$ if and only if $\mathfrak{H}$ is the set of the elements D of $\mathfrak{H}(R)$ such that $D(\mathfrak{a}_{\mathfrak{H}} \cap \mathcal{O})=0$. Since $\mathfrak{a}_{\mathfrak{H}} \cap \mathcal{O}$ is $\mathfrak{a}$, our assertion follows from Proposition.

References

[1] Oort, R., "Commutative group schemes", Lecture Notes in Mathematics, no. 15. (1966), Springer-Verlag.

[2] Yanagihara, H., Some Properties of Hopf Algebras Attached to Group Varieties, Hiroshima Math. J. **1** (1971), 227–251.

Department of Mathematics,
Faculty of Science,
Hiroshima University