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An Integral Representation of an Eigenfunction of Invariant Differential Operators on a Symmetric Space

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1. Introduction

Let G be a connected semisimple Lie group with finite center and K a maximal compact subgroup of G. In [6], Harish-Chandra determined the Plancherel measure for the symmetric space G/K. The spherical Fourier transform may be regarded as a method of representing a more or less arbitrary spherical function as a linear combination of elementary spherical functions. On the other hand Ehrenpreis proved in [3], [4] and [5] that for various spaces W of functions or distributions on $\boldsymbol{R}^n$ (such as the space of solutions of linear constant coefficient partial differential equations) any $T \in W$ admits a representation

$$T(x) = \int \exp i < z, x > d\mu(z)/a(z)$$

where μ is a bounded measure on a "multiplicity variety", a is an element of an "analytic uniform structure" for W, and where the integral converges in a certain sense. Now, since elementary spherical functions are eigenfunctions of all invariant differential operators on the symmetric space G/K, the above result of Ehrenpreis suggests an analogous problem of representing an eigenfunction of a system of invariant differential operators as an integral of those elementary spherical functions which satisfy the same system of invariant differential equations. In this paper, we shall give a solution to this problem.

The authors are grateful to Professor L. Ehrenpreis for helpful discussions. He also raised the problem of extending the result of this paper to K-infinite eigenfunctions by using matrix coefficients of the principal series representations. We shall deal with this problem in the forthcoming paper.

2. Notation

We denote by $C^\infty(G)$ the space of all C^∞ functions on G with its usual topology. Let $\mathfrak{g}$ be the Lie algebra of all left invariant vector fields on G and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition where $\mathfrak{k}$ is the Lie algebra of K. If $x = k \exp X$ ($k \in$

K, $X \in \mathfrak{p}$), we denote by $\sigma(x)$ the norm of X with respect to the Killing form. We fix a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ such that $\mathfrak{a} \subset \mathfrak{p}$, once for all. Let $\mathfrak{a}^*$ be the dual space of $\mathfrak{a}$ and $\mathfrak{a}_C^*$ the complexification of $\mathfrak{a}^*$. Then the little Weyl group W acts on $\mathfrak{a}$, $\mathfrak{a}^*$ and $\mathfrak{a}_C^*$, canonically. For any $\lambda \in \mathfrak{a}_C^*$ we denote by $\|\lambda\|$ the norm of λ defined by the Killing form. Let $\mathfrak{U}$ be the universal enveloping algebra of the complexification of $\mathfrak{g}$. For $f \in C^\infty(G)$ and $D \in \mathfrak{U}$ we put $f(x; D) = (Df)(x)$ ($x \in G$). A continuous linear map f of a topological vector space E into a topological vector space F is called a homomorphism if it is an open map of E onto $f(E) = \mathrm{Im} f$.

3. The topology of $\hat{\mathscr{E}}'$

Let $\mathscr{E}$ be the set of functions $f \in C^\infty(G)$ such that $f(k_1 x k_2) = f(x)$ for any $k_1, k_2 \in K$ and $x \in G$. Then $\mathscr{E}$ is a Fréchet-Montel space for the topology induced by $C^\infty(G)$. We denote by $\mathscr{E}'$ the strong dual of $\mathscr{E}$. For an $S \in \mathscr{E}'$, the spherical Fourier transform of S is defined by

$$(\mathscr{F}S)(\lambda) = \hat{S}(\lambda) = S(\phi_\lambda) \qquad (\lambda \in \mathfrak{a}_C^*)$$

where ϕ_λ is the elementary spherical function. By Theorem 3 in [1], the space $\hat{\mathscr{E}}'$ of spherical Fourier transforms of $\mathscr{E}'$ is given by the set of all entire holomorphic functions F on $\mathfrak{a}_C^*$ satisfying the following conditions (a) and (b) viz.: (a) $F(s\lambda) = F(\lambda)$ *for any* $s \in W$ *and* $\lambda \in \mathfrak{a}_C^*$. (b) *There exist a constant* $R \geqq 0$ *and an integer* $m \geqq 0$ *such that*

$$\sup_{\lambda \in \mathfrak{a}_C^*} (1 + \|\lambda\|)^{-m} \exp\{-R\| \mathrm{Im}\, \lambda\|\} |F(\lambda)| < +\infty .$$

We equip $\hat{\mathscr{E}}'$ with the topology so that the spherical Fourier transform is a topological isomorphism of $\mathscr{E}'$ onto $\hat{\mathscr{E}}'$. The crucial point here is that the topology of $\hat{\mathscr{E}}'$ is given by an "analytic uniform structure" defined by Ehrenpreis as follows. We denote by $\mathscr{E}(\boldsymbol{R}^l)$ the space of all C^∞ functions on $\boldsymbol{R}^l$ where l is the rank of G/K and by $\mathscr{E}'(\boldsymbol{R}^l)$ the strong dual of $\mathscr{E}(\boldsymbol{R}^l)$. The Paley-Wiener theorem says that the space $\hat{\mathscr{E}}'(\boldsymbol{R}^l)$ of (usual) Fourier transforms of $\mathscr{E}'(\boldsymbol{R}^l)$ is the set of all entire holomorphic functions F on $\boldsymbol{C}^l$ satisfying the following condition: *There exist a constant* $R \geqq 0$ *and an integer* $m \geqq 0$ *such that*

$$\sup_{z \in \boldsymbol{C}^l} (1 + \|z\|)^{-m} \exp(-R\| \mathrm{Im}\, z\|) |F(z)| < +\infty.$$

Let A denote the set of all continuous positive functions $a(z) = a_1(\mathrm{Re}\, z) a_2(\mathrm{Im}\, z)$ ($z \in \boldsymbol{C}^l$), where a_1 dominates all polynomials and a_2 dominates all linear exponentials. For an $a \in A$ we denote by U_a the set of all elements $F \in \hat{\mathscr{E}}'(\boldsymbol{R}^l)$ such that

$|F(z)|\leqq a(z)$ for all $z\in \boldsymbol{C}^l$. We topologize $\hat{\mathscr{E}}'(\boldsymbol{R}^l)$ so that the family $\{U_a\}_{a\in A}$ is a fundamental system of neighborhoods of 0 and thus by Theorem 5.19 in [5] the (usual) Fourier transform is a topological isomorphism of $\mathscr{E}'(\boldsymbol{R}^l)$ onto $\hat{\mathscr{E}}'(\boldsymbol{R}^l)$. According to Ehrenpreis we call A an analytic uniform structure for $\mathscr{E}(\boldsymbol{R}^l)$. Since the dimension of $\mathfrak{a}$ is l, $\mathfrak{a}_{\boldsymbol{C}}^*$ is identified with $\boldsymbol{C}^l$. We remark that $\hat{\mathscr{E}}'$ is regarded as the set of all $F\in\hat{\mathscr{E}}'(\boldsymbol{R}^l)$ such that $F(s\lambda)=F(\lambda)$ for any $s\in W$ and $\lambda\in\mathfrak{a}_{\boldsymbol{C}}^*$.

LEMMA 1. *$\hat{\mathscr{E}}'$ is a closed subspace of $\hat{\mathscr{E}}'(\boldsymbol{R}^l)$.*

PROOF. For each $s\in W$ we define a linear operator Φ_s on $\hat{\mathscr{E}}'(\boldsymbol{R}^l)$ by

$$(\Phi_s F)(\lambda)=F(s^{-1}\lambda)\qquad(\lambda\in\mathfrak{a}_{\boldsymbol{C}}^*).$$

Then Φ_s is continuous. In fact, if we put $a^s(\lambda)=a(s\lambda)$ for $a\in A$, $s\in W$ and $\lambda\in\mathfrak{a}_{\boldsymbol{C}}^*$, then it is easy to see that $a^s\in A$ and that the inverse image of U_a by Φ_s is U_{a^s}. Next we define a continuous linear operator Φ on $\hat{\mathscr{E}}'(\boldsymbol{R}^l)$ by $\Phi=\frac{1}{[W]}\sum_{s\in W}\Phi_s$, where $[W]$ denotes the order of the Weyl group W. Then it is clear that Ker $(1-\Phi)=\hat{\mathscr{E}}'$. It follows that $\hat{\mathscr{E}}'$ is closed in $\hat{\mathscr{E}}'(\boldsymbol{R}^l)$.

LEMMA 2. *If B is a bounded subset of $\hat{\mathscr{E}}'$, then there exists a constant $a>0$ such that*

$$\sup_{\lambda\in\mathfrak{a}_{\boldsymbol{C}}^*}(1+\|\lambda\|)^{-a}\exp(-a\|\operatorname{Im}\lambda\|)|F(\lambda)|\leqq a$$

for any $F\in B$.

PROOF. First we prove that for any neighborhood U of 0 in $\mathscr{E}$ there exists a constant $a>0$ such that

$$U\supset\left\{\frac{1}{a}(1+\|\lambda\|)^{-a}\exp(-a\|\operatorname{Im}\lambda\|)\phi_\lambda;\ \lambda\in\mathfrak{a}_{\boldsymbol{C}}^*\right\}.$$

By the definition of the topology of $\mathscr{E}$, there exist a compact subset Ω of G, a constant $b>0$ and $D_1,\dots,D_m\in\mathfrak{U}$ such that

$$U\supset\{f\in\mathscr{E};\ \max_{1\leqq j\leqq m}\ \max_{x\in\Omega}|f(x;D_j)|\leqq b\}.$$

By Lemma 46 in [6] there exist constants $c_j>0$ and $d_j>0$ such that

$$|\phi_\lambda(x;D_j)|\leqq c_j(1+\|\lambda\|)^{d_j}\exp(\sigma(x)\|\operatorname{Im}\lambda\|)$$

for any $x\in\Omega$ and $\lambda\in\mathfrak{a}_{\boldsymbol{C}}^*$.
Therefore if we set $a=\max\left\{\max_{1\leqq j\leqq m}\left(\frac{c_j}{b},\ d_j\right),\ \sup_{x\in\Omega}\sigma(x)\right\}$, we have

$$\max_{1 \leqq j \leqq m} \max_{x \in \Omega} \frac{1}{a}(1+\|\lambda\|)^{-a} \exp(-a\|\operatorname{Im}\lambda\|) |\phi_\lambda(x; D_j)|$$

$$\leqq \max_{1 \leqq j \leqq m} \max_{x \in \Omega} \frac{c_j}{a}(1+\|\lambda\|)^{d_j-a} \exp\{(\sigma(x)-a)\|\operatorname{Im}\lambda\|\} \leqq b$$

for any $\lambda \in \mathfrak{a}_C^*$. Next let B be a bounded subset of $\hat{\mathscr{E}}'$ and put $B_1 = \mathscr{F}^{-1}(B)$. Then since $\mathscr{E}$ is barrelled the polar B_1^0 of B_1 is a neighborhood of 0 in $\mathscr{E}$. It follows that there exists a constant $a > 0$ such that

$$B_1^0 \supset \left\{\frac{1}{a}(1+\|\lambda\|)^{-a} \exp(-a\|\operatorname{Im}\lambda\|)\phi_\lambda;\ \lambda \in \mathfrak{a}_C^*\right\}.$$

This implies that

$$\sup_{\lambda \in \mathfrak{a}_C^*} (1+\|\lambda\|)^{-a} \exp(-a\|\operatorname{Im}\lambda\|)|F(\lambda)| \leqq a$$

for any $F \in B$.

The following proposition plays an important role in this paper.

PROPOSITION 1. *The topology of $\hat{\mathscr{E}}'$ coincides with the relative topology induced by $\hat{\mathscr{E}}'(\boldsymbol{R}^l)$.*

PROOF. Let j be the canonical injection of $\hat{\mathscr{E}}'$ into $\hat{\mathscr{E}}'(\boldsymbol{R}^l)$. Since $\mathscr{E}$ is a Fréchet-Montel space, $\mathscr{E}'$ is bornologic according to the Grothendieck theorem (cf. [11]). By Lemma 5.18 in [5] and Lemma 2, j maps bounded sets into bounded sets. Hence, j is continuous. Now we identify $\hat{\mathscr{E}}'$ and $\hat{\mathscr{E}}'(\boldsymbol{R}^l)$ with $\mathscr{E}'$ and $\mathscr{E}'(\boldsymbol{R}^l)$, respectively. Since $\mathscr{E}$ and $\mathscr{E}(\boldsymbol{R}^l)$ are reflexive, the transpose ${}^t j$ of j is continuous. By Lemma 1 and the reflexivity of $\mathscr{E}(\boldsymbol{R}^l)$, $\operatorname{Im} j$ is closed in $\mathscr{E}'(\boldsymbol{R}^l)$ for the weak topology. Therefore by the Dieudonné-Schwartz theorem j is a homomorphism since $\mathscr{E}$ and $\mathscr{E}(\boldsymbol{R}^l)$ are Fréchet-Montel spaces. This completes the proof of the proposition.

Let us denote by $\hat{\mathscr{E}}$ the strong dual of $\hat{\mathscr{E}}'$. For an $f \in \mathscr{E}$ we put $\hat{f} = {}^t\mathscr{F}^{-1}(f)$. Then we have $\hat{f}(\hat{S}) = ({}^t\mathscr{F}^{-1}f)(\hat{S}) = f(\mathscr{F}^{-1}\hat{S}) = f(S)$ for any $f \in \mathscr{E}$ and $S \in \mathscr{E}'$. Using the Hahn-Banach theorem, from Theorem 1.5 in [5] and Proposition 1 there exist a bounded measure μ on $\mathfrak{a}_C^*$ and an $a \in A$ such that

$$\begin{aligned} \hat{f}(\hat{S}) &= \int \hat{S}(\lambda) d\mu(\lambda)/a(\lambda) \\ &= \int S(\phi_\lambda) d\mu(\lambda)/a(\lambda). \end{aligned}$$

Choosing the Dirac measure placed at the point $x \in G$, we have

$$f(x)=\int \phi_\lambda(x)d\mu(\lambda)/a(\lambda).$$

Thus we obtain the following theorem.

THEOREM 1. *For any $f\in\mathscr{E}$, we can find a bounded measure μ on $\mathfrak{a}_C^*$ and an $a\in A$ such that*

$$f(S)=\int S(\phi_\lambda)d\mu(\lambda)/a(\lambda)$$

for any $S\in\mathscr{E}'$, where the integral exists in the sense of Lebesgue-Stieltjes. In particular, we have

$$f(x)=\int \phi_\lambda(x)d\mu(\lambda)/a(\lambda) \qquad (x\in G).$$

4. Integral representation of solutions of invariant differential equations

Fix a finite number of invariant differential operators $D_1,\dots, D_r$ on G/K. We wish to obtain an integral representation of a solution $f\in\mathscr{E}$ of differential equations:

$$D_1 f=0,\dots, D_r f=0.$$

We put $D=(D_1,\dots, D_r)$. For any vector space V we denote by V^r the r-fold direct sum of V with itself. For any $f\in\mathscr{E}$ we put $Df=(D_1 f,\dots, D_r f)$. Then D defines a continuous linear map of $\mathscr{E}$ into $\mathscr{E}^r$. The transpose ${}^tD$ of D is a continuous linear map of $(\mathscr{E}^r)'=(\mathscr{E}')^r$ into $\mathscr{E}'$ and it is clear that $\operatorname{Im}{}^tD=\sum_{i=1}^r {}^tD_i\mathscr{E}'$. On the other hand by [8] (p. 70) there exist polynomials $P_1,\dots, P_r$ such that $\mathscr{F}({}^tD_iS)=P_i\mathscr{F}(S)$ for any $S\in\mathscr{E}'$ and that $P_i(s\lambda)=P_i(\lambda)$ for any $s\in W$ and $\lambda\in\mathfrak{a}_C^*$. Thus we have $\mathscr{F}(\operatorname{Im}{}^tD)=\sum_{i=1}^r P_i\hat{\mathscr{E}}'$. We set $P=(P_1,\dots, P_r)$, $P(\hat{\mathscr{E}}')^r=\sum_{i=1}^r P_i\hat{\mathscr{E}}'$ and $P\hat{\mathscr{E}}'(\boldsymbol{R}^l)^r=\sum_{i=1}^r P_i\hat{\mathscr{E}}'(\boldsymbol{R}^l)$.

LEMMA 3. $\hat{\mathscr{E}}'\cap P\hat{\mathscr{E}}'(\boldsymbol{R}^l)^r=P(\hat{\mathscr{E}}')^r$.

PROOF. Let $F_1,\dots, F_r\in\hat{\mathscr{E}}'(\boldsymbol{R}^l)$ such that $\sum_{i=1}^r P_iF_i\in\hat{\mathscr{E}}'$. Then we have

$$\begin{aligned}\sum_{i=1}^r P_iF_i&=\Phi(\sum_{i=1}^r P_iF_i)\\ &=\sum_{i=1}^r P_i\Phi(F_i)\in\sum_{i=1}^r P_i\hat{\mathscr{E}}'.\end{aligned}$$

This shows that $\hat{\mathscr{E}}'\cap P\hat{\mathscr{E}}'(\boldsymbol{R}^l)^r\subset P(\hat{\mathscr{E}}')^r$. The converse is obvious.

By Theorem 6.2 in [5] $P\hat{\mathscr{E}}'(\boldsymbol{R}^l)^r$ is closed in $\hat{\mathscr{E}}'(\boldsymbol{R}^l)$ and thus $P(\hat{\mathscr{E}}')^r$ is closed in $\hat{\mathscr{E}}'$ from Proposition 1 and Lemma 3. Therefore the strong dual of Ker D is identified with $\mathscr{E}'/\mathrm{Im}\,{}^tD$. Furthermore by Lemma 3 there exists a continuous injective linear map j^* of $\hat{\mathscr{E}}'/P(\hat{\mathscr{E}}')^r$ into $\hat{\mathscr{E}}'(\boldsymbol{R}^l)/P\hat{\mathscr{E}}'(\boldsymbol{R}^l)^r$ such that the following diagram is commutative:

$$\begin{array}{ccc} \hat{\mathscr{E}}' & \xrightarrow{j} & \hat{\mathscr{E}}'(\boldsymbol{R}^l) \\ \downarrow & & \downarrow \\ \hat{\mathscr{E}}'/P(\hat{\mathscr{E}}')^r & \xrightarrow{j^*} & \hat{\mathscr{E}}'(\boldsymbol{R}^l)/P\hat{\mathscr{E}}'(\boldsymbol{R}^l)^r . \end{array}$$

PROPOSITION 2. *j^* is an injective homomorphism.*

PROOF. First we remark that the inverse image of $P\hat{\mathscr{E}}'(\boldsymbol{R}^l)^r$ by the map I–Φ coincides with $\hat{\mathscr{E}}'+P\hat{\mathscr{E}}'(\boldsymbol{R}^l)^r$. It follows that $\mathrm{Im}\,j^*=\{\hat{\mathscr{E}}'+P\hat{\mathscr{E}}'(\boldsymbol{R}^l)^r\}/P\hat{\mathscr{E}}'(\boldsymbol{R}^l)^r$ is a closed subspace of $\hat{\mathscr{E}}'(\boldsymbol{R}^l)/P\hat{\mathscr{E}}'(\boldsymbol{R}^l)^r$. By arguments similar to the proof of Proposition 1 we see that j^* is a homomorphism.

Under these preparations, our problem can be solved as follows. The method is that, reducing the case of a symmetric space to that of the Euclidean space, we apply the result of Ehrenpreis in [5].

Let $D_1,\dots, D_r$ be a finite number of invariant differential operators on G/K. We consider the system of differential equations:

$$D_1f=0,\dots, D_rf=0 \qquad (f\in\mathscr{E}).$$

As we remarked above, f is canonically identified with a continuous linear form on $\mathscr{E}'/\mathrm{Im}\,{}^tD$ where $D=(D_1,\dots, D_r)$. Therefore, $\hat{f}$ is regarded as a continuous linear form on $\hat{\mathscr{E}}'/P(\hat{\mathscr{E}}')^r$. In view of Proposition 2, by the Hahn-Banach theorem, $\hat{f}$ can be extended to a continuous linear form on $\hat{\mathscr{E}}'(\boldsymbol{R}^l)/P\hat{\mathscr{E}}'(\boldsymbol{R}^l)^r$. Hence, from Theorems 4.2 and 7.1 in [5] we obtain the following

THEOREM 2. *For any $f\in\mathscr{E}$ satisfying $D_1f=\cdots=D_rf=0$, there exist a finite number of complex algebraic varieties $V_j(1\leqq j\leqq m)$, linear partial differential operators ∂_j $(1\leqq j\leqq m)$ with polynomial coefficients, bounded measures μ_j on V_j* and *$a\in A$ such that*

$$f(S)=\sum_{j=1}^{m}\int_{V_j}(\partial_j)_\lambda S(\phi_\lambda)d\mu_j(\lambda)/a(\lambda)$$

for any $S\in\mathscr{E}'$.
Furthermore we have

$$f(x)=\sum_{j=1}^{m}\int_{V_j}(\partial_j)_\lambda\phi_\lambda(x)d\mu_j(\lambda)/a(\lambda)$$

for any $x \in G$.

REMARK. As Ehrenpreis stated in [5], the measures μ_j $(1 \leqq j \leqq m)$ in Theorem 2 are not unique. To obtain the uniqueness, even for the (usual) Laplacian on $\boldsymbol{R}^n$, one must consider certain "functionals" on S^{n-1} which are, in general, not measures any more (cf. [7], [10]). In case the eigenvalue is equal to zero, the situation is more complicated (see [12]).

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