HIROSHIMA MATH. J.
4 (1974), 375–396

Integral Representations of Beppo Levi Functions of Higher Order

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(Received January 10, 1974)

Introduction

If f is a C^1-function with compact support on the Euclidean space R^n ($n \geqq 3$), then it can be represented by its partial derivatives as follows:

$$(1) \qquad f(x) = -\frac{1}{a_n} \sum_{i=1}^{n} \int \frac{\partial}{\partial t_i} |x-t|^{2-n} \frac{\partial f}{\partial t_i}(t) dt .$$

There are many ways to represent a C^m-function (m: positive integer) with compact support on R^n ($n \geqq 2$) in terms of its partial derivatives of m-th order. Among them, the following two are regarded as generalizations of (1):

$$(2) \qquad \varphi(x) = \sum_{|\alpha|=m} a_\alpha \int \frac{(x-y)^\alpha D^\alpha \varphi(y)}{|x-y|^n} dy$$

(Yu. G. Reshetnyak [9]), and

$$(3) \qquad \varphi(x) = \begin{cases} \displaystyle\sum_{|\alpha|=m} c_\alpha \int D^\alpha(|x-y|^{2m-n}) D^\alpha \varphi(y) dy \\ \qquad \text{if } n-2m>0 \text{ or } n \text{ is odd} \\ \qquad \text{and } n-2m<0, \\ \displaystyle\sum_{|\alpha|=m} c'_\alpha \int D^\alpha(|x-y|^{2m-n} \log|x-y|) D^\alpha \varphi(y) dy \\ \qquad \text{if } n \text{ is even and } n-2m \leqq 0 \end{cases}$$

(H. Wallin [11]).

On the other hand, J. Deny and J. L. Lions [5] studied the space of Beppo Levi functions, e.g., the space $BL(L^p(R^n))$ of distributions on R^n whose partial derivatives belong to $L^p(R^n)$. They showed that any quasi continuous function f in $BL(L^2(R^n))$ ($n \geqq 3$) is represented as (1) quasi everywhere, with an additional constant. M. Ohtsuka [8] extended their results to p-precise functions, which belong to $BL(L^p(R^3))$, and gave many other properties of precise functions in his lectures at Hiroshima University.

In this paper, we consider the space $BL_m(L^p(R^n))$ of Beppo Levi functions of higher order m, that is the space consisting of distributions on R^n whose partial derivatives of m-th order all belong to $L^p(R^n)$. To obtain fine results, we need a concept of (m, p)-capacity. For our purpose, the (m, p)-capacity introduced by H. Wallin [12], which is denoted by $\Gamma_{m,p}$, is best suited. However, since we fail to verify whether it is subadditive or not, we also consider another capacity $\Gamma^+_{m,p}$, which is subadditive, and in fact, a true capacity in the sense of M. Brelot [2]. We shall see that it is equivalent to $\Gamma_{m,p}$. Through our capacity $\Gamma_{m,p}$, we shall define (m, p)-quasi continuity of functions.

It is known that functions in the Sobolev space $W^{m,p}(R^n)$ are represented as Bessel potentials (cf. [1], [3], [9]). Using Bessel potentials, Yu. G. Reshetnyak defined (l, p) capacity and then gave a characterization of (l, p) polar sets (see [9; Theorem 5.8]). We shall show that in case l is a positive integer m, his capacity is equivalent to $(\Gamma_{m,p})^p$ and that his characterization can be given by using our integral representations.

Then we shall show that integral representation of the form (2) is possible for certain (m, p)-quasi continuous functions in $BL_m(L^p(R^n))$. Integral representation of the form (3) was given by H. Wallin for functions in $BL_m(L^p(R^n))$ with compact supports (see [11; Lemmas 7 and 8]). We shall extend his result to the case where supports are not necessarily compact, and in fact we shall prove it in a way different from his.

In the final section, we shall discuss representation of (m, p)-quasi continuous functions in $BL_m(L^p(R^n))$ as Riesz potentials of functions in $L^p(R^n)$. It is an extension of M. Ohtsuka's result for p-precise functions given in [8]. To obtain our result we shall make use of the methods in the previous sections.

§1. Preliminaries

Let R^n be the n-dimensional Euclidean space with points $x=(x_1, x_2,\dots, x_n)$, $y=(y_1, y_2,\dots, y_n)$, etc. For a multi-index $\alpha=(\alpha_1, \alpha_2,\dots, \alpha_n)$, we set $|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n$, $x^\alpha=x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$ and

$$D^\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}}.$$

We shall use the following notations of L. Schwartz [10]: $\mathscr{D}(R^n)$, $\mathscr{S}(R^n)$.

In this paper, let $1<p<\infty$. For a non-negative integer m, we denote by $W^{m,p}(R^n)$ the Sobolev space, that is, the space of all distributions F such that $D^\alpha F\in L^p(R^n)$ for any α with $|\alpha|\leqq m$. The norm of F in $W^{m,p}(R^n)$ is defined by

$$\|F\|_{m,p}=\|(\sum_{|\alpha|\leqq m}|D^\alpha F|^2)^{1/2}\|_p$$

where $\|\cdot\|_p$ denotes the L^p-norm in R^n. It is well-known that $W^{m,p}(R^n)$ is a reflexive Banach space if $1<p<\infty$ (cf. [7]).

J. Deny and J. L. Lions introduced the following spaces ([5]): $BL_m(L^p(R^n))$ is the space of Beppo Levi functions of order m attached to $L^p(R^n)$, that is, the space of all distributions T such that $D^\alpha T\in L^p(R^n)$ for any $|\alpha|=m$ with a semi-norm $|T|_{m,p}=\|(\sum_{|\alpha|=m}|D^\alpha T|^2)^{1/2}\|_p$, and $BL_m\dot{}(L^p(R^n))$ is the quotient space of $BL_m(L^p(R^n))$ by the space of all polynomials of degree $\leqq m-1$. We note that if $F\in BL_m(L^p(R^n))$ has compact support, then $F\in W^{m,p}(R^n)$ by [10; Chapitre 6, Théorème XV (Kryloff)].

§2. (m, p)-capacity

Let m be a non-negative integer. We introduce the notion of (m, p)-capacity. First, for a compact set $e\subset R^n$, we define

$$\Gamma_{m,p}(e)=\inf\{\|\varphi\|_{m,p};\ \varphi\in\mathscr{D}(R^n),\ \varphi\geqq 1 \text{ on } e\},$$

$$\Gamma^+_{m,p}(e)=\inf\{\|\varphi\|_{m,p};\ \varphi\in\mathscr{D}_+(R^n),\ \varphi\geqq 1 \text{ on } e\},$$

where $\mathscr{D}_+(R^n)=\{\varphi\in\mathscr{D}(R^n);\ \varphi\geqq 0 \text{ on } R^n\}$. Next, for an open set $\omega\subset R^n$, we define

$$\Gamma_{m,p}(\omega)=\sup_{e\subset\omega,\, e:\text{compact}}\Gamma_{m,p}(e),$$

$$\Gamma^+_{m,p}(\omega)=\sup_{e\subset\omega,\, e:\text{compact}}\Gamma^+_{m,p}(e).$$

Then we note that $\Gamma_{m,p}(e)=\inf\{\Gamma_{m,p}(\omega);\ e\subset\omega,\ \omega \text{ is open}\}$ and $\Gamma^+_{m,p}(e)=\inf\{\Gamma^+_{m,p}(\omega);\ e\subset\omega,\ \omega \text{ is open}\}$, which allow us to define for an arbitrary set $A\subset R^n$ the following quantities:

$$\Gamma_{m,p}(A)=\inf_{A\subset\omega,\,\omega:\text{open}}\Gamma_{m,p}(\omega),$$

$$\Gamma^+_{m,p}(A)=\inf_{A\subset\omega,\,\omega:\text{open}}\Gamma^+_{m,p}(\omega).$$

$\Gamma_{m,p}(A)$ is called the (m, p)-capacity of A (cf. [12]).

REMARK 2.1. It is easy to see that $\Gamma_{0,p}(A)=\Gamma^+_{0,p}(A)=\{\text{outer Lebesgue measure of } A\}^{1/p}$. Furthermore, we have $\Gamma_{1,p}(A)=\Gamma^+_{1,p}(A)$, because, for $F\in BL_1(L^p(R^n))$, $|F|\in BL_1(L^p(R^n))$ and $|\operatorname{grad}|F||=|\operatorname{grad}F|$ a.e. on R^n (see [5; Théorème 3.2 in p. 316]).

From the definitions, we can easily prove

LEMMA 2.1. (i) *$\Gamma_{m,p}$ and $\Gamma^+_{m,p}$ are monotone increasing and continuous from the right.*

(ii) $\Gamma^+_{m,p}$ *is countably subadditive, that is*

$$\Gamma^+_{m,p}(\bigcup_{j=1}^{\infty} A_j) \leqq \sum_{j=1}^{\infty} \Gamma^+_{m,p}(A_j)$$

for a countable family $\{A_j\}$ *of sets in* R^n.

For relationship between $\Gamma_{m,p}$ and $\Gamma^+_{m,p}$, we have

LEMMA 2.2. *There exists a constant* $C \geqq 1$ *such that*

$$\Gamma^+_{m,p}(e) \leqq C\,\Gamma_{m,p}(e) \qquad \text{for all compact set } e \text{ in } R^n.$$

From this lemma, the following theorem immediately follows:

THEOREM 2.1. *There exists a constant* $C \geqq 1$ *independent of* A *such that*

$$\Gamma_{m,p}(A) \leqq \Gamma^+_{m,p}(A) \leqq C\,\Gamma_{m,p}(A)$$

for any set A *in* R^n.

PROOF OF LEMMA 2.2. On account of Remark 2.1, it suffices to show the case $m \geqq 2$. We use the Bessel kernel G_m, which is determined by the following properties:

(i) G_m is a non-negative function belonging to $L^1(R^n)$,
(ii) the Fourier transform of G_m is $(1+4\pi^2|x|^2)^{-m/2}$.

It is known that a distribution F belongs to $W^{m,p}(R^n)$ if and only if there exists a function $f \in L^p(R^n)$ such that $F = G_m * f$ in $W^{m,p}(R^n)$, and that

$$C^{-1}\|f\|_p \leqq \|G_m * f\|_{m,p} \leqq C\|f\|_p \tag{2.1}$$

for some constant $C > 0$ independent of f (see [3; Theorem 7]).

Let $\varphi \in \mathscr{D}(R^n)$ and $\varphi > 1$ on e. We can write $\varphi = G_m * f$ for some $f \in \mathscr{S}(R^n)$. Then $G_m * f^+$ is obviously continuous and > 1 on e, where $f^+(x) = \max(0, f(x))$. If we show $\Gamma^+_{m,p}(e) \leqq \|G_m * f^+\|_{m,p}$, then

$$\Gamma^+_{m,p}(e) \leqq C\|f^+\|_p \leqq C\|f\|_p \leqq C^2\|\varphi\|_{m,p},$$

which implies $\Gamma^+_{m,p}(e) \leqq C^2\Gamma_{m,p}(e)$.

Take a function $\psi \in \mathscr{D}_+(R^1)$ which is equal to 1 on a neighborhood of 0. Set

$$\psi_j(t) = \begin{cases} 1 & \text{if } t \leqq j \\ \psi(t-j) & \text{if } t > j, \end{cases}$$

and set $\tilde{\psi}_j(x)=\psi_j(|x|)$. It is easy to check that $\tilde{\psi}_j F \to F$ in $W^{m,p}(R^n)$ as $j\to\infty$ for any $F\in W^{m,p}(R^n)$. Let $\{h_k\}$ be a sequence of functions belonging to $\mathscr{D}_+(R^n)$ such that $\mathrm{supp}(h_k)\subset\{x;\ |x|\leqq(1/k)\}$ and $\int h_k(x)dx=1$. For each j and k $\{\tilde{\psi}_j(G_m*f^+)\}*h_k\in\mathscr{D}_+(R^n)$ and for sufficiently large j and k, we have $\{\tilde{\psi}_j(G_m*f^+)\}*h_k \geqq 1$ on e. Therefore

$$\Gamma^+_{m,p}(e)\leqq\lim_{j\to\infty}\lim_{k\to\infty}\|\{\tilde{\psi}_j(G_m*f^+)\}*h_k\|_{m,p}=\|G_m*f^+\|_{m,p}.$$

Thus Lemma 2.2 is proved.

REMARK 2.2. In the above proof, we have also shown that $\mathscr{D}_+(R^n)$ is dense in $W^{m,p}_+(R^n)=\{f\in W^{m,p}(R^n);\ f\geqq 0 \text{ a.e.}\}$ with respect to the topology of $W^{m,p}(R^n)$.

A set $A\subset R^n$ is called (m, p)-polar if $\Gamma_{m,p}(A)=0$, or equivalently, $\Gamma^+_{m,p}(A)=0$. If a property is true on a set $A\subset R^n$ except for an (m, p)-polar set in A, then we say that this property is true (m, p)-quasi everywhere or (m, p)-*q.e.* on A.

A function f is called (m, p)-quasi continuous if given $\varepsilon>0$, there exists an open set $\omega\subset R^n$ such that $\Gamma_{m,p}(\omega)<\varepsilon$ and the restriction of f to $R^n-\omega$ is continuous.

By Lemma 2.1 and Theorem 2.1, we can prove the following lemma in the same manner as J. Deny J. L. Lions [5].

LEMMA 2.3. *For each $F\in W^{m,p}(R^n)$, we set $\Phi(F)=\{f;\ f$ is (m, p)-quasi continuous and equals F a.e. on $R^n\}$. Then we have the following assertions*:

(i) *$\Phi(F)$ is non-empty, two functions of $\Phi(F)$ are equal to each other (m, p)-q.e., and any function which equals some function of $\Phi(F)$ (m, p)-q.e. belongs to $\Phi(F)$ (cf. [5; Théorème 3.1 in p. 354]).*

(ii) *If a sequence $\{F_j\}$ converges to F in $W^{m,p}(R^n)$ as $j\to\infty$, then there exists a subsequence $\{F_{j_k}\}$ of $\{F_j\}$ such that for any $f_{j_k}\in\Phi(F_{j_k})$ and any $f\in\Phi(F)$, $\{f_{j_k}\}$ converges to f (m, p)-q.e. as $k\to\infty$ (cf. [5; Théorème 4.1 in p. 357]).*

A distribution $T\in BL_m(L^p(R^n))$ can be considered as a function f with $D^\alpha f\in L^p_{loc}(R^n)$ for any α with $|\alpha|\leqq m$. For this f, there exists an (m, p)-quasi continuous function equal to f a.e. (cf. [5; Théorème 3.1 in p. 354]).

Let ω be an open set in R^n. Denote by $\mathscr{W}^{m,p}(\omega)$ the class of all $F\in W^{m,p}(R^n)$ such that $F\geqq 1$ *a.e.* on ω. Then we show

LEMMA 2.4. $\Gamma_{m,p}(\omega)=\inf\{\|F\|_{m,p};\ F\in\mathscr{W}^{m,p}(\omega)\}$.

PROOF. Let e be a compact set in ω, and let $\{\tilde{\psi}_j\}$ and $\{h_k\}$ be the same as in the proof of Lemma 2.2. Then for any $F\in\mathscr{W}^{m,p}(\omega)$, we have $\Gamma_{m,p}(e)\leqq\lim_{j\to\infty}\lim_{k\to\infty}\|(\tilde{\psi}_j F)*h_k\|_{m,p}=\|F\|_{m,p}$. Therefore $\Gamma_{m,p}(\omega)\leqq\inf\{\|F\|_{m,p};\ F\in\mathscr{W}^{m,p}(\omega)\}$. To prove the converse inequality we may assume $\Gamma_{m,p}(\omega)<\infty$. Take a sequence

$\{e_j\}$ of compact sets such that $e_j \subset$ the interior of $e_{j+1} \subset \omega$ for any $j \geqq 1$, and $\cup_{j=1}^{\infty} e_j = \omega$. Next, for each j, choose $\{\varphi_j\} \subset \mathscr{D}(R^n)$ satisfying $\varphi_j \geqq 1$ on e_j and $\|\varphi_j\|_{m,p} < \Gamma_{m,p}(e_j) + (1/j)$. Then $\{\varphi_j\}$ is a bounded set in the reflexive Banach space $W^{m,p}(R^n)$. Hence there exists a subsequence $\{\varphi_{j_k}\}$ of $\{\varphi_j\}$ and $F \in W^{m,p}(R^n)$ such that $\varphi_{j_k} \to F$ weakly in $W^{m,p}(R^n)$ as $k \to \infty$. This implies $F \geqq 1$ *a.e.* on ω and $\|F\|_{m,p} \leqq \liminf_{k\to\infty} \|\varphi_{j_k}\|_{m,p} \leqq \Gamma_{m,p}(\omega)$. Thus our lemma is proved.

Let A be a set in R^n. Denote by $\mathscr{W}^{m,p}(A)$ the closure of the class of all $F \in W^{m,p}(R^n)$ such that $F \geqq 1$ *a.e.* on a neighborhood of A.

THEOREM 2.2. *If $\Gamma_{m,p}(A) < \infty$, then there exists a unique $F \in \mathscr{W}^{m,p}(A)$ such that*

(a) *if $f \in \Phi(F)$, then $f \geqq 1$ (m, p)-q.e. on A,*
(b) $\|F\|_{m,p} = \Gamma_{m,p}(A)$.

PROOF. By Lemma 2.3, the class of all $F \in W^{m,p}(R^n)$ such that $f \geqq 1$ (m, p)-*q.e.* on A for any $f \in \Phi(F)$ is a closed set in $W^{m,p}(R^n)$ and includes $\mathscr{W}^{m,p}(A)$. Therefore all $F \in \mathscr{W}^{m,p}(A)$ satisfy (a).

On the other hand $\mathscr{W}^{m,p}(A)$ is a closed convex subset of the reflexive Banach space $W^{m,p}(R^n)$, and by using Lemma 2.4, we see that $\Gamma_{m,p}(A) = \inf\{\|F\|_{m,p};\ F \in \mathscr{W}^{m,p}(A)\}$. This infimum is attained at a unique element $F \in \mathscr{W}^{m,p}(A)$ because of the uniform convexity of $W^{m,p}(R^n)$ (see [7; Chapitre 1, 3.3]). This F is the required one.

For any set A in R^n, we set $\mathscr{W}_+^{m,p}(A) = \{F \in \mathscr{W}^{m,p}(A);\ F \geqq 0$ a.e. on $R^n\}$. It is easy to see that $\mathscr{W}_+^{m,p}(A)$ is a closed convex set and consists of all F in $W^{m,p}(R^n)$ such that any function of $\Phi(F)$ is $\geqq 0$ (m, p)-*q.e.* on R^n and $\geqq 1$ (m, p)-*q.e.* on A.

LEMMA 2.5. $\Gamma_{m,p}^+(A) = \inf\{\|F\|_{m,p};\ F \in \mathscr{W}_+^{m,p}(A)\}$.

PROOF. Let $F \in \mathscr{W}_+^{m,p}(A)$. Then $\Gamma_{m,p}^+(A) \leqq \|F\|_{m,p}$ can be shown in the same way as J. Deny and J. L. Lions [5; Lemme 4.1 in p. 356]. Hence we have $\Gamma_{m,p}^+(A) \leqq \inf\{\|F\|_{m,p};\ F \in \mathscr{W}_+^{m,p}(A)\}$. The converse inequality can be shown in the same way as in Lemma 2.4 and Theorem 2.2.

By this lemma, we have the following theorem:

THEOREM 2.2′. *For an arbitrary set A with $\Gamma_{m,p}^+(A) < \infty$, there exists a unique $F \in W^{m,p}(R^n)$ such that*

(a) *for any $f \in \Phi(F)$, $f \geqq 0$ (m, p)-q.e. and $f \geqq 1$ (m, p)-q.e. on A,*
(b) $\|F\|_{m,p} = \Gamma_{m,p}^+(A)$.

DEFINITION. We shall denote by f_A any function in $\Phi(F)$ in Theorem 2.2′.

LEMMA 2.6. *Let $\{A_j\}$ be any increasing sequence of sets in R^n, and set $A=\bigcup_{j=1}^{\infty} A_j$. Then $\Gamma_{m,p}^{+}(A_j) \uparrow \Gamma_{m,p}^{+}(A)$ as $j \to \infty$.*

PROOF. Take $F_j \in W^{m,p}(R^n)$ with the properties in Theorem 2.2′ for each A_j. We may assume that $\lim_{j\to\infty}\Gamma_{m,p}^{+}(A_j)<\infty$. Then $\{F_j\}$ is bounded in $W^{m,p}(R^n)$. Therefore there exist a subsequence $\{F_{j_k}\}$ of $\{F_j\}$ and $F \in W^{m,p}(R^n)$ such that $F_{j_k} \to F$ weakly in $W^{m,p}(R^n)$ as $k \to \infty$. Moreover we have $F \in \bigcap_{j=1}^{\infty} \mathscr{W}_{+}^{m,p}(A_j)=\mathscr{W}_{+}^{m,p}(A)$. Hence

$$\Gamma_{m,p}^{+}(A) \leqq \|F\|_{m,p} \leqq \liminf_{k\to\infty} \|F_{j_k}\|_{m,p} = \lim_{j\to\infty} \Gamma_{m,p}^{+}(A_j) \leqq \Gamma_{m,p}^{+}(A),$$

which implies that $\lim_{j\to\infty}\Gamma_{m,p}^{+}(A_j)=\Gamma_{m,p}^{+}(A)$.

Lemmas 2.1 and 2.6 mean that $\Gamma_{m,p}^{+}$ is a true capacity in the sense of M. Brelot [2]. Thus we have

THEOREM 2.3. *Any analytic set in R^n is capacitable with respect to $\Gamma_{m,p}^{+}$.*

Yu. G. Reshetnyak defined the (l, p) capacity $\mathrm{Cap}_{(l,p)}E$ of sets E in R^n in case l is a positive number and $p>1$, as follows (see [9]):

$$\mathrm{Cap}_{(l,p)}E=\inf\{\|f\|_p; f \in L^p(R^n), \geqq 0 \quad \text{and} \quad G_l * f \geqq 1 \quad \text{on} \quad E\}.$$

As another application of Lemma 2.4 we give

THEOREM 2.4. *In case l is a positive integer m and $mp \leqq n$, there exists a positive constant C such that*

$$C^{-1}\{\Gamma_{m,p}(E)\}^p \leqq \mathrm{Cap}_{(m,p)}E \leqq C\{\Gamma_{m,p}(E)\}^p \tag{2.2}$$

for any set E in R^n.

PROOF. It suffices to show (2.2) for any open set ω in R^n because of the definition of $\Gamma_{m,p}$ and Lemma 2.2 in [9]. Recall that, if $f \in L^p(R^n)$, then $G_m * f$ belongs to $W^{m,p}(R^n)$ and satisfies (2.1). To show the left inequality of (2.2) for ω, choose a non-negative function f in $L^p(R^n)$ such that $G_m * f \geqq 1$ everywhere on ω. (If such an f does not exist, then $\mathrm{Cap}_{(m,p)}\omega=\infty$.) By (2.1) and Lemma 2.4 we have $C^{-p}\{\Gamma_{m,p}(\omega)\}^p \leqq \mathrm{Cap}_{(m,p)}\omega$.

To give the right inequality of (2.2), it suffices to show it only for any compact set e in R^n by the capacitability of e for $\mathrm{Cap}_{(m,p)}$ (Theorem 2.1 in [9]) and the definition of $\Gamma_{m,p}$. Let us choose $\varphi \in \mathscr{D}(R^n)$ so that $\varphi \geqq 1$ on e. As in the proof of Lemma 2.2 we write $\varphi=G_m * f$ for some $f \in \mathscr{S}(R^n)$ and have

$$\mathrm{Cap}_{(m,p)}e \leqq \|f\|_p^p \leqq C^p \|\varphi\|_{m,p}^p.$$

This yields $\mathrm{Cap}_{(m,p)}e \leqq C^p\{\Gamma_{m,p}(e)\}^p$.

§3. Integral representation I

Hereafter, let m be a positive integer.

Let l be an integer and α a multi-index such that $m=|\alpha|-l+n\geqq 1$. We set $\kappa(x)=x^{\alpha}/|x|^{l}$. For a multi-index β with $|\beta|=m$, we can express

$$(3.1)\qquad (D^{\beta}\kappa)(x)=\sum_{k=0}^{m}\frac{a_k(x)}{|x|^{l+2k}},$$

where each $a_k(x)$ is a homogeneous polynomial of degree $(l+2k)-n$, or constantly zero. We shall show that $K=D^{\beta}\kappa$ fulfills the conditions for a kernel listed on p. 89 of [4]. In our case,

$$\Omega\left(\frac{x}{|x|}\right)=\sum_{k=0}^{m}\frac{a_k(x)}{|x|^{l+2k-n}}\qquad\text{and}\qquad K(x)=\frac{1}{|x|^{n}}\Omega\left(\frac{x}{|x|}\right).$$

Since $\sum_{k=0}^{m}a_k(x)/|x|^{l+2k-n}$ is a homogeneous function of degree 0, we can consider $\Omega(x)$ as a function on the unit sphere with center at the origin of R^n. If $|x|=|y|=1$, then

$$|\Omega(x)-\Omega(y)|\leqq\sum_{k=0}^{m}|a_k(x)-a_k(y)|\leqq c|x-y|$$

for some positive constant c. Next we show

LEMMA 3.1. $\int_{|x|=1}K(x)dS(x)=0.$

PROOF. First we observe

$$(3.2)\qquad \int_{|x|=1}x^{\gamma}dS(x)=\frac{2\prod_{i=1}^{n}\left(\frac{1+(-1)^{\gamma_i}}{2}\right)\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)}$$

for a multi-index γ, which can be obtained from an elementary calculus.

We prove the lemma by induction with respect to m. Let α, β and l be given so that $|\alpha|-l+n=|\beta|=1$. Then we have

$$D^{\beta}\left(\frac{x^{\alpha}}{|x|^{l}}\right)=\binom{\alpha}{\beta}\frac{x^{\alpha-\beta}}{|x|^{l}}-l\frac{x^{\alpha+\beta}}{|x|^{l+2}}$$

where

$$\binom{\alpha}{\beta}=\begin{cases}\prod_{i=1}^{n}\binom{\alpha_i}{\beta_i}=\prod_{i=1}^{n}\frac{\alpha_i!}{\beta_i!(\alpha_i-\beta_i)!} & \text{if } \alpha_i\geqq\beta_i \text{ for all } i,\\ 0 & \text{otherwise.}\end{cases}$$

Using (3.2), we have

$$\int_{|x|=1} D^{\beta}\Big(\frac{x^{\alpha}}{|x|^{l}}\Big)dS(x)=\binom{\alpha}{\beta}\frac{2\prod_{i=1}^{n}\Big(\frac{1+(-1)^{\alpha_i-\beta_i}}{2}\Big)\Gamma\Big(\frac{\alpha_i-\beta_i+1}{2}\Big)}{\Gamma\Big(\frac{n+|\alpha-\beta|}{2}\Big)}$$

$$-l\frac{2\prod_{i=1}^{n}\Big(\frac{1+(-1)^{\alpha_i+\beta_i}}{2}\Big)\Gamma\Big(\frac{\alpha_i+\beta_i+1}{2}\Big)}{\Gamma\Big(\frac{n+|\alpha+\beta|}{2}\Big)}$$

Since $|\beta|=1$, the right-hand side is seen to be zero.

Next we assume that the lemma is true for $|\alpha|-l+n=|\beta|=m$. Let α, β and l be given so that $|\alpha|-l+n=|\beta|=m+1$. Writing $\beta=\gamma+\delta$, where $|\gamma|=1$ and $|\delta|=m$, we have

$$D^{\beta}\Big(\frac{x^{\alpha}}{|x|^{l}}\Big)=\binom{\alpha}{\gamma}D^{\delta}\Big(\frac{x^{\alpha-\gamma}}{|x|^{l}}\Big)-lD^{\delta}\Big(\frac{x^{\alpha+\gamma}}{|x|^{l+2}}\Big).$$

Here if $\alpha-\gamma$ is not a multi-index, the first term of the right-hand side disappears, and if otherwise, $|\alpha-\gamma|-l+n=m$. Moreover $|\alpha+\gamma|-(l+2)+n=m$. Consequently, by the assumption of induction, we obtain

$$\int_{|x|=1} D^{\delta}\Big(\frac{\alpha^{\alpha-\gamma}}{|x|^{l}}\Big)dS(x)=0 \qquad \text{and} \qquad \int_{|x|=1} D^{\delta}\Big(\frac{x^{\alpha+\gamma}}{|x|^{l+2}}\Big)dS(x)=0,$$

i.e.,

$$\int_{|x|=1} D^{\beta}\Big(\frac{x^{\alpha}}{|x|^{l}}\Big)dS(x)=0.$$

Thus Lemma 3.1 is proved.

Let f be a function in $L^p(R^n)$. For a positive integer j, we set $K_{(1/j)}(x)=K(x)$ if $|x|\geqq 1/j$ and $=0$ if $|x|<1/j$. Then, we can apply the results of singular integrals in [4] and obtain:

(i) $K_{(1/j)}*f$ belongs to $L^p(R^n)$ for each j, and converges in $L^p(R^n)$ as $j\to\infty$ ([4; Theorems 1 and 7]),

(ii)

(3.3) $$\|K_{(1/j)}*f\|_p\leqq \text{const.}\,\|f\|_p \qquad \text{([4; Theorem 1])}.$$

Next, we consider $\kappa_j(x)=x^{\alpha}/(|x|^2+(1/j)^2)^{l/2}$. Let f be a function in $L^p(R^n)$ satisfying the following condition:

(3.4) $$\int(1+|x|)^{m-n}|f(x)|dx=\int(1+|x|)^{|\alpha|-l}|f(x)|dx<\infty,$$

or equivalently,

$$\int |x-y|^{m-n}|f(y)|dy \not\equiv \infty$$

(see Remark in p. 191 of [6] and also Lemma 9.1 of [8]). We set

$$(\kappa * f)(x) = \int \frac{(x-y)^\alpha}{|x-y|^l} f(y)dy$$

and $\kappa_j * f$ is similarly defined. By our assumptions, it is easy to see that $\kappa_j * f \in C^\infty$ and $D^\beta(\kappa_j * f) = (D^\beta \kappa_j) * f$ for any β. Furthermore we have

LEMMA 3.2. *For any multi-index β with $|\beta| = m$, $D^\beta(\kappa_j * f)$ converges in $L^p(R^n)$ as $j \to \infty$.*

PROOF. We can write

$$(D^\beta(\kappa_j * f))(x) - (K_{(1/j)} * f)(x) = j^n \int \theta_\beta(j(x-y)) f(y)dy$$

where $\theta_\beta = D^\beta \kappa_1 - K_{(1)}$. We shall show that $\theta_\beta \in L^1(R^n)$. First we notice that

$$D^\beta \kappa_1(x) = \sum_{k=0}^{m} \frac{a_k(x)}{(|x|^2+1)^{(l+2k)/2}}$$

for the same $a_k(x)$ as in (3.1). Therefore if $|x| \geqq 1$, then

$$\theta_\beta(x) = -\sum_{k=0}^{m} \frac{a_k(x)}{(|x|^2+1)^{(l+2k)/2}|x|^{l+2k}}((|x|^2+1)^{(l+2k)/2} - |x|^{l+2k})$$

and $(|x|^2+1)^{(l+2k)/2} - |x|^{l+2k} = O(|x|^{l+2k-2})$ as $|x| \to \infty$. Hence $\theta_\beta(x) = O(|x|^{-n-2})$ as $|x| \to \infty$, because each $a_k(x)$ is a homogeneous polynomial of degree $(l+2k)-\mathrm{n}$, or constantly zero. Thus $\theta_\beta \in L^1(R^n)$. We set $A_\beta = \int \theta_\beta(x)dx$. Then

$$\begin{aligned}
&(D^\beta(\kappa_j * f))(x) - (K_{(1/j)} * f)(x) - A_\beta f(x) \\
&= j^n \int \theta_\beta(j(x-y)) f(y)dy - \int \theta_\beta(y) f(x)dy \\
&= \int \theta_\beta(y)\{f(x-(y/j)) - f(x)\}dy.
\end{aligned}$$

Therefore we have by Hölder's inequality,

$$\begin{aligned}
&\|D^\beta(\kappa_j * f) - K_{(1/j)} * f - A_\beta f\|_p^p \\
&\leqq \left(\int |\theta_\beta(y)|dy\right)^{p/q} \int |\theta_\beta(y)| \left(\int |f(x-(y/j)) - f(x)|^p dx\right) dy,
\end{aligned}$$

where $(1/p)+(1/q)=1$. Noting that $\theta_\beta \in L^1(R^n)$, $\int |f(x-(y/j))|^p dx = \int |f(x)|^p dx$ and that $\int |f(x-(y/j))-f(x)|^p dx \to 0$ locally uniformly as $j\to\infty$ we obtain

$$\|D^\beta(\kappa_j * f) - K_{(1/j)} * f - A_\beta f\|_p \to 0 \tag{3.5}$$

as $j\to\infty$ by Lebesgue's convergence theorem. This yields Lemma 3.2.

LEMMA 3.3. *Let f be a function in $L^p(R^n)$ satisfying* (3.4). *Suppose $|\beta| = m$. Then*

(i) $D^\beta(\kappa_j * f) \to D^\beta(\kappa * f)$ *in $L^p(R^n)$ as $j\to\infty$,*

(ii) $\|D^\beta(\kappa * f)\|_p \leqq \text{const.} \|f\|_p$,

(iii) *$\kappa * f$ is (m, p)-quasi continuous.*

PROOF. From Lemma 3.2, it follows that $\kappa_j * f$ is a Cauchy sequence in $BL_m(L^p(R^n))$. Then there exist a sequence $\{P_j\}$ of polynomials of degree $\leqq m-1$ and $u \in BL_m(L^p(R^n))$ such that $\kappa_j * f \to u$ in $BL_m(L^p(R^n))$ as $j\to\infty$ and $D^{\beta'}(\kappa_j * f + P_j) \to D^{\beta'} u$ in $L^p_{\mathrm{loc}}(R^n)$ as $j\to\infty$ for any β' with $|\beta'| \leqq m$ (see [5; Théorème 2.1 in Chap. III]).

First we consider the special case:

$$\kappa(x) = |x|^{m-n} \quad \text{and} \quad \kappa_j(x) = (|x|^2 + (1/j)^2)^{(m-n)/2}.$$

Since $\kappa_j * |f| \to \kappa * |f|$ pointwise as $j\to\infty$, there exists a polynomial P_0 of degree $\leqq m-1$ such that $\kappa * |f| = u - P_0$ *a.e.* on R^n. Moreover, for any $\varphi \in \mathscr{D}(R^n)$, $\varphi(\kappa_j * |f|) \to \varphi(\kappa * |f|)$ in $W^{m,p}(R^n)$ as $j\to\infty$. It follows from Lemma 2.3, (ii), that $\varphi(\kappa * |f|)$ is (m, p)-quasi continuous, which means that $\kappa * |f|$ is (m, p)-quasi continuous and that $\left\{x ; \int |x-y|^{m-n} |f(y)| dy = \infty\right\}$ is (m, p)-polar.

Now we consider the general case. We observe that $\kappa_j * f$ converges to $\kappa * f$ except on an (m, p)-polar set, in fact, except on the set $\left\{x ; \int |x-y|^{m-n} |f(y)| dy = \infty\right\}$. Therefore, in a way similar to the above, we obtain (i) and (iii). Moreover,

$$\begin{aligned} \|D^\beta(\kappa * f)\|_p \leqq{} & \|D^\beta(\kappa_j * f) - D^\beta(\kappa * f)\|_p \\ & + \|D^\beta(\kappa_j * f) - K_{(1/j)} * f - A_\beta f\|_p \\ & + \|K_{(1/j)} * f\|_p + |A_\beta|\, \|f\|_p \end{aligned}$$

for any j. Letting $j\to\infty$ and using (3.3) and (3.5), we have (ii) of the lemma.

LEMMA 3.4. (*Yu. G. Reshetnyak* [9; *Lemma* 6.2]) *For $\varphi \in \mathscr{D}(R^n)$, we can write*

$$\varphi(x)=\sum_{|\alpha|=m} a_\alpha \int \frac{(x-y)^\alpha D^\alpha \varphi(y)}{|x-y|^n} dy$$

where $a_\alpha=(-1)^m m/(\alpha!\,\omega_n)$, ω_n *being the surface area of the unit sphere in* R^n.

THEOREM 3.1. *Let f be an (m, p)-quasi continuous function belonging to $BL_m(L^p(R^n))$ such that*

$$\int (1+|x|)^{m-n}|D^\alpha f(x)|dx<\infty \qquad \text{for any } \alpha \text{ with } |\alpha|=m. \tag{3.6}$$

If there exists a sequence $\{\varphi_j\}$ of functions in $\mathcal{D}(R^n)$ such that $\varphi_j \to f$ in $BL_m(L^p(R^n))$ as $j\to\infty$, then

$$f(x)=\sum_{|\alpha|=m} a_\alpha \int \frac{(x-y)^\alpha D^\alpha f(y)}{|x-y|^n} dy + P(x) \qquad (m, p)\text{-}q.e., \tag{3.7}$$

where P is a polynomial of degree $\leqq m-1$.

PROOF. By Lemma 3.4, we have

$$\varphi_j(x)=\sum_{|\alpha|=m} a_\alpha \int \frac{(x-y)^\alpha D^\alpha \varphi_j(y)}{|x-y|^n} dy.$$

Here we set

$$G_f(x)=\sum_{|\alpha|=m} a_\alpha \int \frac{(x-y)^\alpha D^\alpha f(y)}{|x-y|^n} dy.$$

From Lemma 3.3, it follows that G_f is (m, p)-quasi continuous and that $\varphi_j \to G_f$ in $BL_m(L^p(R^n))$ as $j\to\infty$. Therefore $G_f^{\boldsymbol\cdot}=f^{\boldsymbol\cdot}$ in $BL_m^{\boldsymbol\cdot}(L^p(R^n))$. Hence there exists a polynomial P of degree $\leqq m-1$ such that $f=G_f+P$ *a.e.* on R^n, which implies Theorem 3.1 by virtue of Lemma 2.3.

REMARK 3.1. Let f be an (m, p)-quasi continuous function in $W^{m,p}(R^n)$ satisfying (3.6). Then we have (3.7), because there exists a sequence $\{\varphi_j\}$ of functions in $\mathcal{D}(R^n)$ such that $\varphi_j \to f$ in $W^{m,p}(R^n)$ as $j\to\infty$ (cf. Remark 2.2).

REMARK 3.2. Let f be an (m, p)-quasi continuous function belonging to $BL_m(L^p(R^n))$. If f has compact support, then we have (3.7). Moreover if $(1\leqq)$ $m<n$, then $P=0$.

REMARK 3.3. In Theorem 3.1, if $(m<)$ $mp<n$, we can omit (3.6). In fact, in this case,

$$\int \frac{|D^\alpha f(x)|}{(1+|x|)^{n-m}} dx \leqq \left(\int |D^\alpha f(x)|^p dx\right)^{1/p} \left(\int \frac{dx}{(1+|x|)^{q(n-m)}}\right)^{1/q} <\infty$$

where $q=p/(p-1)$.

REMARK 3.4. In case $m=1$, these Remarks and Theorem 3.1 were given by M. Ohtsuka [8; Theorem 9.11].

As a consequence of Theorem 3.1, we have

THEOREM 3.2. *(cf. [9; Theorem 5.8] and [12; Theorem 1]) A set A in R^n is (m, p)-polar if and only if there exists a non-negative function f in $L^p(R^n)$ satisfying (3.4) such that $\int|x-y|^{m-n}f(y)dy=\infty$ for every $x\in A$.*

PROOF. The "if" part was observed in the proof of Lemma 3.3. We prove the "only if" part. Suppose $m<n$. First we consider the case where A is bounded. Take a sequence $\{\omega_j\}$ of open sets in R^n such that ω_1 is bounded, $\omega_j\supset\omega_{j+1}\supset A$ and $\Gamma^+_{m,p}(\omega_j)<1/2^j$ for each $j\geqq 1$. Let φ be a non-negative function in $\mathcal{D}(R^n)$ such that $\varphi=1$ on ω_1. By Theorem 3.1 and Remark 3.2, we have

$$(\varphi f_{\omega_j})(x)=\sum_{|\alpha|=m}a_\alpha\int\frac{(x-y)^\alpha D^\alpha(\varphi f_{\omega_j})(y)}{|x-y|^n}dy \qquad (m,\ p)\text{-}q.e.$$

(for the notation f_{ω_j} see the Definition given after Theorem 2.2′). This implies that

$$\sum_{|\alpha|=m}|a_\alpha|\int|x-y|^{m-n}|D^\alpha(\varphi f_{\omega_j})(y)|dy\geqq 1 \qquad \text{for } x\in\omega_j.$$

We set $f=\sum_{j=1}^\infty\{\sum_{|\alpha|=m}|a_\alpha||D^\alpha(\varphi f_{\omega_j})|\}$. Then f is a non-negative function in $L^p(R^n)$ with compact support. Moreover, for $x\in A$, we obtain $\int|x-y|^{m-n}f(y)dy=\infty$.

Next we consider the general case. For each j, we set $A_j=A\cap\{x;|x|\leqq j\}$. Then from the above argument, for each j, there exists a non-negative function $f_j\in L^p(R^n)$ satisfying (3.4) such that $\int|x-y|^{m-n}f_j(y)dy=\infty$ for every $x\in A_j$. By Lemmas 3.3 and 2.1, the set $B=\cup_{j=1}^\infty\{x;\int|x-y|^{m-n}f_j(y)dy=\infty\}$ is seen to be (m, p)-polar. Hence there exists a point $x_0\notin B$. Set $c_j=\int|x_0-y|^{m-n}f_j(y)dy$, $\tilde{c}_j=2^j\max\{c_j, \|f_j\|_p, 1\}$ and $f=\sum_{j=1}^\infty(1/\tilde{c}_j)f_j$. Then $\int|x_0-y|^{m-n}f(y)dy<\infty$ and $\int|x-y|^{m-n}f(y)dy=\infty$ for any $x\in A$. Thus f is the required function.

If $m\geqq n$, then $A=\emptyset$ on account of the next proposition, so that we may take $f=0$.

PROPOSITION 3.1. *Any non-empty set A in R^n is not (m, p)-polar if and only if $mp>n$.*

This can be proved in the same way as H. Wallin [12; Proposition 2].

By using our integral representation, we can prove the following theorem; cf. Theorem 13.5 in [1].

THEOREM 3.3. *Let f be an (m, p)-quasi continuous function belonging to $BL_m(L^p(R^n))$. Then any partial derivative of f of order α with $|\alpha| \leqq m$ exists $(m-|\alpha|, p)$-q.e. and is $(m-|\alpha|, p)$-quasi continuous.*

§ 4. Integral representation II

In this section, we study a representation of the form (3) (see Introduction). We denote by Δ_m the Laplace operator iterated m times. First we show

LEMMA 4.1. *Let $H \in BL_m(L^p(R^n))$. If $\Delta_m H = 0$, then H is a polynomial of degree $\leqq m-1$.*

PROOF. Let α be any multi-index with $|\alpha| = m$, and set $T = D^\alpha H$. By our assumptions, $T \in L^p(R^n)$ and $\Delta_m T = 0$ in the distribution sense. Then the Fourier transform of T exists and

$$(-4\pi^2|x|^2)^m \mathscr{F}(T) = \mathscr{F}(\Delta_m T) = 0,$$

where $\mathscr{F}(T)$ denotes the Fourier transform of T. Hence $\mathscr{F}(T)$ is supported by $\{0\}$, so that we can write $\mathscr{F}(T) = \Sigma_\beta c_\beta D^\beta \delta$, where δ is the Dirac measure and constants c_β are equal to 0 except for a finite number of β. Therefore T is a polynomial. Noting that $T \in L^p(R^n)$, we have $T=0$. Thus H is seen to be a polynomial of degree $\leqq m-1$.

We note the following well-known representation of $\varphi \in \mathscr{D}(R^n)$: If $n-2m > 0$ or n is odd and $n-2m < 0$, then

$$\varphi(x) = c\int |x-y|^{2m-n} \Delta_m \varphi(y) dy$$

and if $n-2m \leqq 0$ and n is even, then

$$\varphi(x) = c'\int |x-y|^{2m-n} \log|x-y| \Delta_m \varphi(y) dy$$

where c and c' are certain constants. Furthermore notice that Δ_m is of the form $\sum_{|\alpha|=m} \tilde{c}_\alpha D^{2\alpha}$ for suitable constants $\tilde{c}_\alpha$. Setting $c_\alpha = (-1)^m c\tilde{c}_\alpha$ and $c'_\alpha = (-1)^m c'\tilde{c}_\alpha$, we have

LEMMA 4.2. (*H. Wallin* [11; *p.* 71]) *Let $\varphi \in \mathscr{D}(R^n)$. If either $n-2m > 0$ or n is odd and $n-2m < 0$, then*

$$\varphi(x)=\sum_{|\alpha|=m} c_\alpha \int D^\alpha(|x-y|^{2m-n})D^\alpha\varphi(y)dy,$$

and if $n-2m\leqq 0$ *and* n *is even, then*

$$\varphi(x)=\sum_{|\alpha|=m} c'_\alpha \int D^\alpha(|x-y|^{2m-n}\log|x-y|)D^\alpha\varphi(y)dy,$$

where c_α *and* c'_α *are constants.*

THEOREM 4.1. *Let* f *be an* (m, p)*-quasi continuous function in* $BL_m(L^p(R^n))$ *such that* $\int(1+|x|)^{m-n}|D^\alpha f(x)|dx<\infty$ *for any* α *with* $|\alpha|=m$. *If either* $n-2m>0$ *or* n *is odd and* $n-2m<0$, *then*

$$f(x)=\sum_{|\alpha|=m} c_\alpha \int D^\alpha(|x-y|^{2m-n})D^\alpha f(y)dy+P(x) \qquad (m, p)\text{-q.e.},$$

and if n *is even and* $m<n\leqq 2m$, *then*

$$f(x)=\sum_{|\alpha|=m} c'_\alpha \int D^\alpha(|x-y|^{2m-n}\log|x-y|)D^\alpha f(y)dy+P(x) \quad (m, p)\text{-q.e.},$$

where c_α *and* c'_α *are the same constants as in Lemma* 4.2 *and* P *is a polynomial of degree* $\leqq m-1$.

PROOF. First, suppose $n-2m>0$ or n is odd and $n-2m<0$. We set $G_f(x)=\sum_{|\alpha|=m}c_\alpha\int D^\alpha(|x-y|^{2m-n})D^\alpha f(y)dy$. By Lemma 3.3, G_f is seen to be an (m, p)-quasi continuous function belonging to $BL_m(L^p(R^n))$. Let $\varphi\in\mathscr{D}(R^n)$. In view of our assumption that $\int(1+|x|)^{m-n}|D^\alpha f(x)|dx<\infty$ for any α with $|\alpha|=m$, we can apply Fubini's theorem, and have

$$\begin{aligned}\int G_f(x)\Delta_m\varphi(x)dx &= \sum_{|\alpha|=m} c_\alpha\int D^\alpha f(y)dy\int D^\alpha_y(|x-y|^{2m-n})\Delta_m\varphi(x)dx\\ &= \sum_{|\alpha|=m} c_\alpha\int D^\alpha f(y)dyD^\alpha_y\int|x-y|^{2m-n}\Delta_m\varphi(x)dx\\ &= \sum_{|\alpha|=m} c_\alpha\int D^\alpha f(y)\frac{1}{c}D^\alpha\varphi(y)dy\\ &= \int f(y)\{\sum_{|\alpha|=m}(-1)^m\frac{c_\alpha}{c}D^{2\alpha}\varphi(y)\}dy\\ &= \int f(y)\Delta_m\varphi(y)dy,\end{aligned}$$

where c is the same constant as given after Lemma 4.1. Therefore, $\Delta_m(f-G_f)=0$

in the distribution sense. By Lemma 4.1, there exists a polynomial P of degree $\leqq m-1$ such that $f-G_f=P$ *a.e.* on R^n. Thus $f=G_f+P$ (m, p)-*q.e.* on account of Lemma 2.3.

The second half of the theorem is similarly obtained, since if $m<n$ and n is even then $D^\alpha(|x|^{2m-n}\log|x|)$ is a linear combination of functions like κ in § 3.

To consider the remaining case, we first prove the following lemma similar to Lemma 3.3.

LEMMA 4.3. *Let* $|\alpha|=m-n\geqq 0$, *and set* $\kappa(x)=x^\alpha\log|x|$. *Let* f *be a non-negative function in* $L^p(R^n)$ *such that*

$$\int(1+|x|)^{m-n}\log(1+|x|)f(x)dx<\infty. \tag{4.1}$$

Then $\kappa*f$ *is a continuous function belonging to* $BL_m(L^p(R^n))$.

PROOF. Set $\kappa_j(x)=x^\alpha\log(|x|^2+(1/j)^2)^{1/2}$. Then $\kappa_j*f\in C^\infty$. Moreover, recalling the discussions in § 3, we infer that $\{\kappa_j*f\}$ is a Cauchy sequence in $BL_m(L^p(R^n))$. If $\alpha=0$, then we have

$$(\kappa_j*f)(x)=\log 2\int_{|x-y|<1}f(y)dy-\int_{|x-y|<1}\log\frac{2}{\sqrt{|x-y|^2+(1/j)^2}}f(y)dy$$
$$+\int_{|x-y|\geqq 1}\log\sqrt{|x-y|^2+(1/j)^2}\,f(y)dy\,.$$

By Lebesgue's convergence theorem, the second term of the right-hand side increases to $\int_{|x-y|<1}(\log 2/|x-y|)f(y)dy$ as $j\to\infty$ and the last term decreases to $\int_{|x-y|\geqq 1}\log|x-y|f(y)dy$ as $j\to\infty$ because of (4.1). Therefore $(\kappa_j*f)(x)\to(\kappa*f)(x)$ as $j\to\infty$. If $|\alpha|\geqq 1$, then since $|(x-y)^\alpha\log(|x-y|^2+(1/j)^2)^{1/2}|\leqq$const. $(1+|y|)^{m-n}$ $\log(2+|y|)$, $(\kappa_j*f)(x)\to(\kappa*f)(x)$ as $j\to\infty$ by Lebesgue's convergence theorem. Hence, in a way similar to the proof of Lemma 3.3 $\kappa*f$ is shown to be (m, p)-quasi continuous. Because of Proposition 3.1, any (m, p)-quasi continuous function is continuous for $mp>n$. Thus we obtain the lemma.

On account of this lemma we can prove the following theorem in the same way as Theorem 4.1:

THEOREM 4.2. *Let* n *be even and* $n\leqq m$. *Let* f *be an* (m, p)-*quasi continuous function in* $BL_m(L^p(R^n))$ *such that*

$$\int(1+|x|)^{m-n}\log(1+|x|)|D^\alpha f(x)|dx<\infty \qquad \textit{for any } \alpha \textit{ with } |\alpha|=m.$$

Then we have the following representation of f:

$$f(x)=\sum_{|\alpha|=m} c'_\alpha \int D^\alpha(|x-y|^{2m-n}\log|x-y|)D^\alpha f(y)dy+P(x),$$

where P is a polynomial of degree $\leqq m-1$, and c'_α are the same constants as in Lemma 4.2.

REMARK 4.1. The function f in the above theorem is continuous by Proposition 3.1.

§5. A representation by Riesz potentials of functions in $L^p(R^n)$

Given a multi-index α and a number l, we set $\kappa(x)=x^\alpha/|x|^l$ and $\kappa_j(x)=x^\alpha/(|x|^2+(1/j)^2)^{l/2}$ for each positive integer j. Let β be any multi-index with $|\beta|=m$ and set $K=D^\beta\kappa$. For a function f in $L^p(R^n)$, the convolutions $\kappa*f$, κ_j*f and $K_{(1/j)}*f$ make sense (see §3).

Suppose that $|\alpha|-l+n=m$. Then we see from (i) and (ii) stated after Lemma 3.1 in §3 that $K_{(1/j)}*f$ converges to a function $R^\beta_{\alpha,l}f$ in $L^p(R^n)$ as $j\to\infty$ such that

$$\|R^\beta_{\alpha,l}f\|_p\leqq \text{const.}\,\|f\|_p. \tag{5.1}$$

First we show

LEMMA 5.1. *Let α, l be given so that $m\leqq|\alpha|-l+n<m+2$. If a function f in $L^p(R^n)$ satisfies $\int(1+|x|)^{|\alpha|-l}|f(x)|dx<\infty$, then $\kappa*f$ is an (m, p)-quasi continuous function in $BL_m(L^p(R^n))$ and $D^\beta(\kappa*f)=R^\beta_{\alpha,l}f+a^\beta_{\alpha,l}f$ for any β with $|\beta|=m$, where*

$$a^\beta_{\alpha,l}=\begin{cases}0 & \text{if } m<|\alpha|-l+n<m+2,\\ A_\beta & \text{defined in the proof of Lemma 3.2}\\ & \text{if } |\alpha|-l+n=m.\end{cases}$$

PROOF. First consider the case $|\alpha|-l+n=m$. From Lemma 3.3, we see that $\kappa*f$ is an (m, p)-quasi continuous function in $BL_m(L^p(R^n))$ and that $\kappa_j*f\to\kappa*f$ in $BL_m(L^p(R^n))$ as $j\to\infty$. In the proof of Lemma 3.2, we showed that $D^\beta(\kappa_j*f)-K_{(1/j)}*f-a^\beta_{\alpha,l}f$ tends to 0 in $L^p(R^n)$ as $j\to\infty$. Hence we have $D^\beta(\kappa*f)=R^\beta_{\alpha,l}f+a^\beta_{\alpha,l}f$.

Next let us consider the case where $m<|\alpha|-l+n<m+2$. We note

$$|D^\beta\kappa(x)|\leqq C|x|^{|\alpha|-l-m} \qquad \text{for all} \quad x$$

and

$$|D^{\beta}\kappa_j(x)-D^{\beta}\kappa(x)|\leqq C|x|^{|\alpha|-l-m-2} \qquad \text{for all } x \text{ with } |x|\geqq N$$

where C and N are constants. Hence by using Lebesgue's dominated convergence theorem we have

$$\int |D^{\beta}\kappa_j - D^{\beta}\kappa| dx \to 0 \qquad \text{as} \quad j\to\infty,$$

so that $\{\kappa_j * f\}$ is a Cauchy sequence in $BL_m(L^p(R^n))$. In a way similar to the proof of Lemma 3.3, we see that $\kappa * f$ is an (m, p)-quasi continuous function in $BL_m(L^p(R^n))$ and that $\kappa_j * f$ converges to $\kappa * f$ in $BL_m(L^p(R^n))$ as $j\to\infty$. On the other hand,

$$\| D^{\beta}(\kappa_j * f) - K_{(1/j)} * f \|_p^p$$
$$\leqq 2^{p-1}\left\{\left(\int | D^{\beta}\kappa_j - D^{\beta}\kappa | dx\right)^p + \left(\int_{|x|\leqq 1/j} | D^{\beta}\kappa | dx\right)^p\right\} \| f \|_p^p .$$

The right-hand side tends to 0 as $j\to\infty$. Therefore we obtain $D^{\beta}(\kappa * f) = R^{\beta}_{\alpha,l} f$ and the lemma is proved.

For a number l and a function f, we set

$$U_l^f(x) = \int |x-y|^{l-n} f(y) dy .$$

By the above lemma we have

THEOREM 5.1. *Suppose that $m\leqq l<m+2$. If a function f in $L^p(R^n)$ satisfies $\int (1+|x|)^{l-n}|f(x)|dx<\infty$, then U_l^f is an (m, p)-quasi continuous function in $BL_m(L^p(R^n))$ and*

$$D^{\beta}(U_l^f) = R^{\beta}_{0,n-l} f + a^{\beta}_{0,n-l} f \tag{5.2}$$

for any β with $|\beta| = m$.

REMARK 5.1. In case $m=1$, Theorem 5.1 was given by M. Ohtsuka [8; Theorem 9.6].

Let $2m<n$. As was seen in § 4, any $\varphi \in \mathscr{D}(R^n)$ is written in the form: $\varphi(x) = c\int |x-y|^{2m-n} \Delta_m \varphi(y) dy$, where $\Delta_m = \sum_{|\alpha|=m} \tilde{c}_\alpha D^{2\alpha}$. By Riesz's composition formula, we have

$$\varphi(x) = \frac{c}{c(m, m)} \int | x-z |^{m-n} dz \int | z-y |^{m-n} \Delta_m \varphi(y) dy,$$

where

$$c(m, m)=\pi^{n/2}\frac{\Gamma\left(\frac{m}{2}\right)^2\Gamma\left(\frac{n-2m}{2}\right)}{\Gamma\left(\frac{n-m}{2}\right)^2\Gamma(m)}.$$

Setting $\psi(z)=\frac{c}{c(m, m)}\int|z-y|^{m-n}\Delta_m\varphi(y)dy$, we have by (5.2)

$$\psi(z)=\frac{c}{c(m, m)}\sum_{|\alpha|=m}\tilde{c}_\alpha\int|z-y|^{m-n}D^{2\alpha}\varphi(y)dy$$

$$=\frac{c}{c(m, m)}\sum_{|\alpha|=m}\tilde{c}_\alpha\{(R^\alpha_{0,n-m}D^\alpha\varphi)(z)+a^\alpha_{0,n-m}D^\alpha\varphi(z)\}.$$

For simplicity we write R_α and a_α for $R^\alpha_{0,n-m}$ and $a^\alpha_{0,n-m}$ respectively. Then we obtain

LEMMA 5.2. *Let $2m<n$, and let $\varphi\in\mathcal{D}(R^n)$. Then $\varphi=U^\psi_m$, where $\psi=\frac{c}{c(m, m)}\sum_{|\alpha|=m}\tilde{c}_\alpha(R_\alpha+a_\alpha)D^\alpha\varphi$.*

LEMMA 5.3. *Let $m\leqq|\alpha|-l+n<m+1$. If $\varphi\in C^m$ satisfies $|D^\gamma\varphi|=O(|x|^{-|\gamma|-1})$ as $|x|\to\infty$ for each γ with $|\gamma|\leqq m$, then*

$$\int_{|x-y|>r}(D^\beta\kappa)(x-y)\varphi(y)dy\to(\kappa*D^\beta\varphi)(x)-a^\beta_{\alpha,l}\varphi(x)\quad\text{as}\quad r\to 0$$

for all x, where $|\beta|=m$.

PROOF. We write $\beta=\sum_{i=1}^m\beta_i$ where $|\beta_i|=1$ for $i=1, 2,\dots, m$ and set $\gamma_0=0$, $\gamma_i=\sum_{j=1}^i\beta_j$ for $i=1, 2,\dots, m$. Then we have

$$\int_{|x-y|>r}(D^\beta\kappa)(x-y)\varphi(y)dy$$

$$=\lim_{R\to\infty}\int_{r<|x-y|<R}(D^\beta\kappa)(x-y)\varphi(y)dy$$

$$=\lim_{R\to\infty}\left[\int_{r<|x-y|<R}\kappa(x-y)D^\beta\varphi(y)dy\right.$$

$$\left.-\sum_{i=1}^m\int_{\{y;|x-y|=r\}\cup\{y;|x-y|=R\}}(D^{\gamma_{i-1}}\kappa)(x-y)(D^{\beta-\gamma_i}\varphi)(y)n_y\cdot\beta_i dS(y)\right]$$

$$=\int_{|x-y|>r}\kappa(x-y)D^\beta\varphi(y)dy$$

$$-\sum_{i=1}^m\int_{|x-y|=r}(D^{\gamma_{i-1}}\kappa)(x-y)(D^{\beta-\gamma_i}\varphi)(y)n_y\cdot\beta_i dS(y),$$

where n_y means the outward normal on the boundary of the domain $\{y;\ r<|x-y|<R\}$. Hence we obtain

$$\int_{|x-y|>r}(D^\beta\kappa)(x-y)\varphi(y)dy\to\int\kappa(x-y)D^\beta\varphi(y)dy-c\varphi(x)\quad\text{as}\quad r\to 0,$$

where $c=\int_{|x-y|=r}(D^{\gamma_{m-1}}\kappa)(x-y)n_y\cdot\beta_m dS(y)=\int_{|y|=1}(D^{\gamma_{m-1}}\kappa)(y)y^{\beta_m}dS(y)$ if $|\alpha|-l+n=m$ and $=0$ otherwise. This c is just equal to $a^\beta_{\alpha,l}$. In fact, if $|\alpha|-l+n=m$, then

$$\begin{aligned}a^\beta_{\alpha,l}&=\lim_{R\to\infty}\int_{|x|<R}\theta_\beta(x)dx\\&=\lim_{R\to\infty}\int_{|x|=R}[(D^{\beta-\beta_m}\kappa_1)(x)-(D^{\beta-\beta_m}\kappa)(x)]n_x\cdot\beta_m dS(x)\\&\quad+\int_{|x|=1}(D^{\beta-\beta_m}\kappa)(x)x^{\beta_m}dS(x)\\&=c.\end{aligned}$$

LEMMA 5.4. *Let $2m<n$. Then for a function f in $L^p(R^n)$ we have*

$$\frac{c}{c(m,m)}\sum_{|\alpha|=m}\tilde{c}_\alpha(R_\alpha+a_\alpha)^2 f=f. \tag{5.3}$$

PROOF. It suffices to show (5.3) for $f=\varphi\in\mathscr{D}(R^n)$ on account of (5.1). We note that $R_\alpha\varphi=|x|^{m-n}*D^\alpha\varphi-a_\alpha\varphi\in C^\infty$ and that $|D^\gamma(R_\alpha\varphi)|=O(|x|^{-m-1})$ as $|x|\to\infty$ for any γ with $|\gamma|\leqq m$. From Lemma 5.3 it follows that $R_\alpha(R_\alpha\varphi)=|x|^{m-n}*D^\alpha(R_\alpha\varphi)-a_\alpha R_\alpha\varphi$. Using (5.2) and Riesz's composition formula, we have

$$(R_\alpha+a_\alpha)^2\varphi=c(m,m)|x|^{2m-n}*D^{2\alpha}\varphi,$$

which yields (5.3) with $f=\varphi\in\mathscr{D}(R^n)$. Thus the lemma is shown.

THEOREM 5.2. *Let $2m<n$, and let f be an (m,p)-quasi continuous function such that there exists a sequence $\{\varphi_j\}$ in $\mathscr{D}(R^n)$ converging to f in $BL_m(L^p(R^n))$. If*

$$\int(1+|x|)^{m-n}\Big|\sum_{|\alpha|=m}\tilde{c}_\alpha(R_\alpha+a_\alpha)D^\alpha f\Big|dx<\infty, \tag{5.4}$$

then there exists a function $g\in L^p(R^n)$ such that

$$\int(1+|x|)^{m-n}|g(x)|dx<\infty \tag{5.5}$$

and

$$f=U_m^g+P \qquad (m,\ p)\text{-}q.e.$$

for some polynomial P of degree $\leqq m-1$; actually

$$(5.6) \qquad g=\frac{c}{c(m,\ m)}\sum_{|\alpha|=m}\tilde{c}_\alpha(R_\alpha+a_\alpha)D^\alpha f\,.$$

Conversely if there exists a function $g\in L^p(R^n)$ satisfying (5.5) *and $f-U_m^g$ is equal $(m,\ p)$-$q.e.$ to a polynomial of degree $\leqq m-1$, then* (5.4) *and* (5.6) *are fulfilled.*

PROOF. Assume (5.4). By Lemma 5.2, we can write $\varphi_j=U_m^{\psi_j}$, where $\psi_j=\frac{c}{c(m,\ m)}\sum_{|\alpha|=m}\tilde{c}_\alpha(R_\alpha+a_\alpha)D^\alpha\varphi_j$. Denote by g the right-hand side of (5.6). Then $g\in L^p(R^n)$ and by (5.1), $\|\psi_j-g\|_p$ tends to 0 as $j\to\infty$. Therefore from Theorem 5.1, it follows that U_m^g is an $(m,\ p)$-quasi continuous function in $BL_m(L^p(R^n))$ and that $\varphi_j=U_m^{\psi_j}\to U_m^g$ in $BL_m(L^p(R^n))$ as $j\to\infty$. Thus $f^\cdot=(U_m^g)^\cdot$ in $BL_m^\cdot(L^p(R^n))$, so that there exists a polynomial P of degree $\leqq m-1$ such that $f=U_m^g+P\,(m,\ p)$-$q.e.$

Conversely suppose that $g\in L^p(R^n)$ satisfies (5.5) and that $f-U_m^g$ is equal $(m,\ p)$-$q.e.$ to a polynomial of degree at most $m-1$. By (5.2) and (5.3) we have

$$\begin{aligned}\sum_{|\alpha|=m}\tilde{c}_\alpha(R_\alpha+a_\alpha)D^\alpha f &= \sum_{|\alpha|=m}\tilde{c}_\alpha(R_\alpha+a_\alpha)D^\alpha U_m^g\\ &= \sum_{|\alpha|=m}\tilde{c}_\alpha(R_\alpha+a_\alpha)^2 g\\ &= \frac{c(m,\ m)}{c}\,g\,.\end{aligned}$$

Hence (5.6) is fulfilled and then so is (5.4) by assumption (5.5).

REMARK 5.2. If $mp<n$, then condition (5.4) is satisfied.

REMARK 5.3. In case the support of f is compact, then condition (5.4) is satisfied. Moreover, in this case, $f=U_m^g$ $(m,\ p)$-$q.e.$, where g is the right-hand side of (5.6).

REMARK 5.4. In case $m=1$, these Remarks and Theorem 5.2 were given by M. Ohtsuka [8; Theorem 9.7].

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