HIROSHIMA MATH. J.
4 (1974), 685–742

Dirichlet Integrals of Functions on a Self-adjoint Harmonic Space

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(Received May 20, 1974)

Introduction

In the previous papers [9], the author introduced a notion of energy for functions on a self-adjoint harmonic space. Our model there was the harmonic space formed by solutions of the self-adjoint second order partial differential equation $\Delta u = Pu$ with $P \geqq 0$ on a Euclidean domain Ω. The energy of a function f with respect to this harmonic space is given by

$$E[f] = D[f] + \int_{\Omega} f^2 P dx, \tag{1}$$

where $D[f]$ denotes the ordinary Dirichlet integral of f over Ω.

For an abstract harmonic space $(\Omega, \mathfrak{H})$, its self-adjointness was defined as the property that it admits a symmetric Green function $G(x, y)$, provided that there is a positive potential on Ω. The condition $P \geqq 0$ in the above model was interpreted as the condition that the constant function 1 is superharmonic. On a self-adjoint harmonic space satisfying this condition, we defined the notion of energy of a function f in terms of potential representation of f with respect to the kernel $G(x, y)$, in such a way that it coincides with $E[f]$ in the special case of the above model.

The definition of energy in [9] also suggests how a value corresponding to the Dirichlet integral $D[f]$ should be defined on such a harmonic space; but it is not clear whether the value has such good properties as the ordinary Dirichlet integral enjoys — among others, whether it is always non-negative.

On the other hand, solutions of the equation $\Delta u = Pu$ form a harmonic space even if P is not necessarily non-negative on Ω (cf., e.g., [7, Théorème 34.1] and [8, Theorem 2.1]), so that one might ask if the method developed in [9] is applicable to the harmonic space on which 1 is not superharmonic. For such a harmonic space, there may not exist positive potentials even if the boundary is large, so that one had better consider the self-adjointness locally. However, in order to make a consistent definition of Dirichlet integrals, some global consideration is also necessary (see § 1.2).

For a self-adjoint harmonic space thus defined, we shal define (in § 4) the notion of *gradient measures* of certain locally bounded functions with the same idea as in the definition of energy measures in [9]; in fact the gradient measure δ_f is given as a generalization of the measure $|\mathrm{grad} f|^2 dx$ on a Euclidean domain, so that $\delta_f(A)$ (A: a Borel set) may be called the *Dirichlet integral* of f over A.

Verification of non-negativeness of energy in [9] was not an easy task. It requires more elaboration to verify that δ_f is a non-negative measure. For functions of potential type, we make a certain estimate (Theorem 1.2), which is a consequence of the energy-principle for Green functions (cf. § 1.3; also cf. [10]). To deal with gradient measures of harmonic functions, we consider (in § 3) a perturbation of the given harmonic space. Perturbations of harmonic spaces were first considered by B. Walsh [12] for a different purpose. What we need is a perturbed harmonic space for which 1 is harmonic; in the model mentioned above, the perturbed space should correspond to the harmonic space of solutions of $\Delta u=0$. With these extra considerations, the non-negativeness of δ_f can be shown by the method developed in [9].

For the equation $\Delta u=Pu$ with $P\geqq 0$, M. Nakai [11] studied the space of all Dirichlet-finite solutions (also cf. M. Glasner and M. Nakai [6]) and showed that it is a vector lattice as well as a Hilbert space with respect to the Dirichlet norm. In our axiomatic setting, we can prove Nakai's results in case 1 is superharmonic (§ 5); but we fail to verify these properties in the general case.

As we did in [9] for energy, we shall extend the definition of gradient measures to more general functions by functional completion (§ 6); the resulting class of functions is the space of Dirichlet functions. Also, along the same lines as in [9], we shall study the lattice structures of this space and the space of locally Dirichlet-finite functions (§ 7).

§ 1. Self-adjoint harmonic space

1.1. Brelot's harmonic space and P-domains

As a base space, we take a connected, locally compact Hausdorff space Ω with a countable base. On Ω, we consider a structure $\mathfrak{H}=\{\mathscr{H}(\omega)\}_{\omega:\mathrm{open}}$ of harmonic space satisfying Axioms 1, 2 and 3 of M. Brelot [3]. As usual, a function in $\mathscr{H}(\omega)$ will be called harmonic on ω. For notions of regular domains (regular open sets), superharmonic functions and potentials, one may refer to [3] (also, [1], [5]). The harmonic measure of a regular domain ω at $x\in\omega$ will be denoted by μ_x^ω. For a superharmonic function s on an open set ω in Ω, its harmonic support will be denoted by $S_h(s)$ in this paper; that is,

$$S_h(s) = \omega - \bigcup\{\omega';\, \mathrm{open},\, s|\omega' \in \mathscr{H}(\omega')\}.$$

Given a domain ω_0 in Ω, the restriction of $\mathfrak{H}$ to ω_0 will be denoted by $\mathfrak{H}_{\omega_0}$. $(\omega_0, \mathfrak{H}_{\omega_0})$ is again a harmonic space satisfying Brelot's Axioms $1\sim 3$. If f is a positive continuous function on ω_0, then

$$\mathfrak{H}_{\omega_0}/f = \{(\mathscr{H}/f)(\omega)\}_{\omega:\,\mathrm{open}\subset\omega_0}$$

defines a harmonic structure on ω_0, where

$$(\mathscr{H}/f)(\omega) = \{u/f;\, u \in \mathscr{H}(\omega)\}\,.$$

This structure also satisfies Brelot's Axioms $1\sim 3$ (cf. [3, Part IV, p. 68]). If, in particular, f is harmonic (resp. superharmonic) on ω_0, then the constant function 1 is harmonic (resp. superharmonic) on ω_0 with respect to $\mathfrak{H}_{\omega_0}/f$.

A domain ω in Ω is called a *P-domain* if it is non-compact and there is a positive potential on ω. The following properties are known in a general theory:

(P_1) Any subdomain of a P-domain is a P-domain (cf. [5, Corollary 2.3.3]).

(P_2) Ω has a covering by P-domains, namely, every $x \in \Omega$ is contained in a P-domain ([5, Theorem 2.3.3]).

(P_3) If ω is a P-domain, then there is a continuous positive potential on ω (cf. [3, Part IV, Proposition 11] or [5, Proposition 2.3.1]).

Furthermore, we have ([1, Satz 2.5.8] or [5, Corollary 2.3.1])

LEMMA 1.1. *Let ω be a P-domain and p be a positive potential on ω. Then there is an increasing sequence $\{p_n\}$ of positive potentials on ω such that each p_n is continuous, each $S_h(p_n)$ is compact in ω and $\lim_{n\to\infty}p_n = p$ on ω.*

1.2. Self-adjoint harmonic space

We shall assume

Axiom 4. On any P-domain ω, *the condition of proportionality* is satisfied, i.e., for each $y \in \omega$, if p_1, p_2 are two positive potentials on ω with $S_h(p_1) = S_h(p_2) = \{y\}$, then $p_1 = \alpha p_2$ for some constant $\alpha > 0$.

REMARK 1.1. The above axiom is equivalent to the following

Axiom 4'. There is a covering $\{\omega_\iota\}_{\iota\in I}$ of Ω by P-domains on each of which the condition of proportionality is satisfied.

The equivalence of these two axioms can be seen by using [7, Théorème 16.4 and its remark].

A harmonic space $(\Omega, \mathfrak{H})$ satisfying Axioms $1\sim 4$ is called *self-adjoint* if to each P-domain ω there corresponds a function $G_\omega(x, y)$: $\omega\times\omega\to(0, +\infty]$ having the following properties:

(a) $G_\omega(x, y) = G_\omega(y, x)$ for all $x, y \in \omega$;

(b) for each $y \in \omega$, $G_\omega(\cdot, y)$ is a potential on ω and $S_h(G_\omega(\cdot, y)) = \{y\}$;

(c) if ω' is a subdomain of ω and $y \in \omega'$, then there is $u_y \in \mathscr{H}(\omega')$ such that

$$G_\omega(x, y) = G_{\omega'}(x, y)+u_y(x)$$

for all $x \in \omega'$.

For a P-domain ω, a function G_ω: $\omega\times\omega\to(0, +\infty]$ satisfying (a) and (b) above is called a *Green function for* ω (or, more precisely, for $(\omega, \mathfrak{H}_\omega)$). Such a function, if exists, is positive and lower semicontinuous on $\omega\times\omega$ ([7, Proposition 18.1]). By Axiom 4, we can easily see that the system of Green functions $\{G_\omega(x, y)\}_{\omega:\text{P-domain}}$ satisfying (c) is uniquely determined up to a multiplicative constant independent of ω.

REMARK 1.2. If there is an exhaustion $\{\omega_n\}_{n=1}^\infty$ of Ω such that each ω_n is a P-domain with a Green function, then we can show that $(\Omega, \mathfrak{H})$ is self-adjoint. In particular, if Ω itself is a P-domain and has a Green function, then $(\Omega, \mathfrak{H})$ is self-adjoint (cf. [9, § 1.2; in particular, Proposition 1.2]).

REMARK 1.3. If, for every $x\in\Omega$, there is a P-domain containing x and possessing a Green function, then we may say that $(\Omega, \mathfrak{H})$ is locally self-adjoint. Obviously, a self-adjoint harmonic space is locally self-adjoint. We can show by examples that the converse is not true.

In the sequel, we shall always assume that $(\Omega, \mathfrak{H})$ *is a self-adjoint harmonic space and a system of Green functions* $\{G_\omega(x, y)\}_{\omega:\text{P-domain}}$ *satisfying* (c) *is fixed.*

1.3. Energy principle

Let ω be a P-domain. For a non-negative measure μ on ω, we denote by U_ω^μ its potential with respect to the kernel G_ω, i.e.,

$$U_\omega^\mu(x) = \int_\omega G_\omega(x, y)d\mu(y).$$

By a general theory of R.–M. Hervé [7, Théorèmes 18.2 and 18.3], we know that U_ω^μ is a potential on ω unless it is constantly infinite, and that any potential on ω is expressed as U_ω^μ by a uniquely determined measure μ. Let $I_\omega(\mu)$ be the G_ω-energy of μ, i.e., $I_\omega(\mu)=\int_\omega U_\omega^\mu(x)d\mu(x)$. We consider the following classes of measures:

$$\mathscr{M}_E^+(\omega) = \{\mu;\ \text{non-negative measure on } \omega \text{ such that } I_\omega(\mu)<+\infty\},$$

$$\mathscr{M}_E(\omega) = \{\sigma;\ \text{signed measure on } \omega \text{ such that } |\sigma|\in\mathscr{M}_E^+(\omega)\},$$

$$\mathscr{M}_B^+(\omega) = \left\{\mu;\ \begin{array}{l}\text{non-negative measure on } \omega \text{ such that}\\ \mu(\omega) < +\infty \text{ and } U_\omega^\mu \text{ is bounded on } \omega\end{array}\right\},$$

$$\mathscr{M}_B(\omega) = \{\sigma;\ \text{signed measure on } \omega \text{ such that } |\sigma| \in \mathscr{M}_B^+(\omega)\}.$$

Obviously, $\mathscr{M}_B^+(\omega) \subset \mathscr{M}_E^+(\omega)$ and $\mathscr{M}_B(\omega) \subset \mathscr{M}_E(\omega)$. For $\sigma \in \mathscr{M}_E(\omega)$, we denote its G_ω-energy by $I_\omega(\sigma)$, i.e., $I_\omega(\sigma) = I_\omega(\sigma^+) + I_\omega(\sigma^-) - 2\int U_\omega^{\sigma^+} d\sigma^-$.

THEOREM 1.1. *The Green function $G_\omega(x, y)$ for a P-domain ω satisfies the energy principle, i.e., it is of positive type*:

$$2\int_\omega U_\omega^\mu \, d\nu \leqq I_\omega(\mu) + I_\omega(\nu) \qquad \textit{for all} \quad \mu, \nu \in \mathscr{M}_E^+(\omega),$$

and the equality holds only when $\mu = \nu$.

PROOF. Consider a positive continuous potential p_0 on ω (cf. (P_3)) and let

$$G_{\omega, p_0}(x, y) \equiv \frac{G_\omega(x, y)}{p_0(x) p_0(y)}$$

for $x, y \in \omega$. It is a Green function for $(\omega, \mathfrak{H}_\omega/p_0)$. Since 1 is superharmonic with respect to $\mathfrak{H}_\omega/p_0$, $G_{\omega, p_0}(x, y)$ satisfies the energy principle by [10, Theorems 1 and 2]. Noting that $\mu \in \mathscr{M}_E^+(\omega)$ if and only if $p_0\mu$ (the measure defined by $d(p_0\mu) = p_0 d\mu$) has finite G_{ω, p_0}-energy, we obtain the theorem.

COROLLARY 1. *On any P-domain ω, the domination principle holds*; *in particular, Axiom D of Brelot* [3] *is fulfilled. Also the continuity principle holds on ω.*

For a proof, cf. [9, Theorem 4. 1].

COROLLARY 2. *If $\mu_n, \mu \in \mathscr{M}_E^+(\omega)$ $(n = 1, 2, \ldots)$ for a P-domain ω and if $U_\omega^{\mu_n} \uparrow U_\omega^\mu$, then $I_\omega(\mu_n - \mu) \to 0$ $(n \to \infty)$.*

1.4. Consequences of the domination principle

A set $e \subset \Omega$ is said to be *polar* if there is a covering $\{\omega_\iota\}_{\iota \in I}$ of Ω by P-domains such that for each $\iota \in I$ we find a positive superharmonic function s_ι on ω_ι with the property that $s_\iota(x) = +\infty$ for all $x \in e \cap \omega_\iota$. Using [7, Théorème 13.1], we can easily show that if e is polar then for any P-domain ω there is a positive potential p on ω such that $p(x) = +\infty$ for all $x \in e \cap \omega$. Let

$$\mathscr{N} = \{e \subset \Omega;\ e \colon \text{polar}\}.$$

We know: if $e \in \mathscr{N}$ and $e' \subset e$, then $e' \in \mathscr{N}$; if $e_n \in \mathscr{N}$, $n = 1, 2, \ldots$, then $\bigcup_{n=1}^\infty e_n \in \mathscr{N}$. As usual, "q.e." (quasi-everywhere) will mean "except on a set $e \in \mathscr{N}$".

Lemma 5.1 and its Corollary 1 in [9] are still valid in the present case.

Thus, by considering $\mathfrak{H}_\omega/s_0$ for a positive continuous superharmonic function s_0 on ω and applying [9, Corollary 2 to Lemma 5.1], we have (cf. Corollary 1 to Theorem 1.1 above)

LEMMA 1.2. *Let ω be a P-domain and p be a potential on ω which is locally bounded on $S_h(p)$. If s is a non-negative superharmonic function on ω such that $s \geqq p$ q.e. on $S_h(p)$, then $s \geqq p$ on ω.*

From this lemma, the next lemma follows in the same manner as [4, Hilfssatz 5.1]:

LEMMA 1.3. *If e is a polar set in Ω and ω is a P-domain, then $\mu(\omega \cap e)=0$ for any $\mu \in \mathscr{M}_E^+(\omega)$.*

If σ is a signed measure on a P-domain ω such that $U_\omega^{|\sigma|}$ is a potential, then $U_\omega^{\sigma^+} - U_\omega^{\sigma^-}$ is defined q.e. on ω. This function will again be denoted by U_ω^σ. By the above lemma, it is μ-measurable for any $\mu \in \mathscr{M}_E^+(\omega)$. It also follows that U_ω^σ is μ-measurable for any non-negative measure μ on ω for which U_ω^μ is locally bounded.

LEMMA 1.4. *Let ω be a P-domain on which there is a bounded positive superharmonic function. If p is a potential on ω such that $S_h(p)$ is compact in ω and p is bounded on $S_h(p)$, then it is bounded on ω.*

PROOF. Let s_0 be a bounded positive superharmonic function on ω. Since $\inf_{S_h(p)} s_0 > 0$, there is a constant $\alpha > 0$ such that $\alpha s_0 \geqq p$ on $S_h(p)$. Hence, by Lemma 1.2, $p \leqq \alpha s_0$ on ω.

LEMMA 1.5 (cf. [9, Lemma 4.5 and its corollary]). *Let ω be a P-domain and σ be a signed measure on ω such that $U_\omega^{|\sigma|}$ is a potential. Then, there are sequences $\{\mu_n\}$ and $\{\nu_n\}$ in $\mathscr{M}_E^+(\omega)$ such that their supports $S(\mu_n)$, $S(\nu_n)$ are compact in ω, $U_\omega^{\mu_n}$, $U_\omega^{\nu_n}$ are continuous on ω and $U_\omega^{\mu_n} \uparrow U_\omega^{\sigma^+}$, $U_\omega^{\nu_n} \uparrow U_\omega^{\sigma^-}$, $U_\omega^{\sigma_n} \to U_\omega^\sigma$ q.e. on ω, where $\sigma_n = \mu_n - \nu_n$. If, furthermore, $\sigma \in \mathscr{M}_E(\omega)$, then $I_\omega(\sigma_n - \sigma) \to 0$; if there is a bounded positive superharmonic function on ω, then $\sigma_n \in \mathscr{M}_B(\omega)$ for each n.*

PROOF. The first half is a consequence of Lemma 1.1 and Hervé's results. The second half follows from Corollary 2 to Theorem 1.1 and Lemma 1.4.

LEMMA 1.6. *Let ω be a P-domain on which there is a bounded positive superharmonic function. If μ is a non-negative measure on ω such that $\mu(\omega) < +\infty$, then U_ω^μ is a potential.*

The proof of this lemma may be carried out as in the classical theory by making use of [7, Lemma 3.1] and the above Lemma 1.4 (cf. [9, Lemmas 1.2 and 1.5]).

LEMMA 1.7. *Let ω be a P-domain, e be a subset of ω and s be a non-negative superharmonic function on ω. Then the reduced function*

$$R_s^{e,\omega} = \inf\{v;\ \text{superharmonic} \geqq 0 \text{ on } \omega, v \geqq s \text{ on } e\}$$

and its regularization $\hat{R}_s^{e,\omega}$ have the following properties:

(a) *$\hat{R}_s^{e,\omega} = R_s^{e,\omega}$ q.e. on ω; everywhere on ω if e is open;*

(b) *$\hat{R}_s^{e,\omega}$ is non-negative superharmonic on ω; it is a potential on ω if either e is relatively compact in ω or s is a potential on ω;*

(c) *$R_s^{e,\omega} = s$ on e (and hence $\hat{R}_s^{e,\omega} = s$ q.e. on e);*

(d) *$R_s^{e,\omega} = \hat{R}_s^{e,\omega}$ on $\omega - \bar{e}$ and is harmonic there, i.e., $S_h(\hat{R}_s^{e,\omega}) \subset \bar{e}$* ($\bar{e}$ denotes the closure of e in Ω).

For proofs, see [3, Part IV (§ 13, § 15–a, Proposition 10, p. 124 and Proposition 23)].

1.5. Inequalities

In this paragraph, we shall establish the following useful inequality:

THEOREM 1.2. *Let ω be a P-domain and μ be a non-negative measure on ω such that U_ω^μ is bounded on ω. Then*

$$\int_\omega (U_\omega^\sigma)^2 d\mu \leqq (\sup_\omega U_\omega^\mu) I_\omega(\sigma)$$

for all $\sigma \in \mathscr{M}_E(\omega)$.

To prove this theorem we prepare two lemmas, the first of which is quite elementary and is used to prove the second lemma.

LEMMA 1.8. *Let S be an abstract set, Φ be a non-negative real-valued function on S and A be a mapping of S into itself. If Φ is bounded on $A(S)$ and satisfies*

$$\Phi(Ax)^2 \leqq \Phi(x)\Phi(A^2x) \tag{1.1}$$

for all $x \in S$, then

$$\Phi(Ax) \leqq \Phi(x) \tag{1.2}$$

for all $x \in S$.

PROOF. Suppose (1.2) is not true for some $x_0 \in S$, i.e., $\Phi(x_0) < \Phi(Ax_0)$. By (1.1) and induction, we see that $\Phi(A^n x_0) > 0$ for all $n = 1, 2, \ldots$. Let $k = \Phi(Ax_0)/\Phi(x_0)$. Again by (1.1),

$$\frac{\Phi(A^n x_0)}{\Phi(A^{n-1}x_0)} \geqq \frac{\Phi(A^{n-1}x_0)}{\Phi(A^{n-2}x_0)} \geqq \cdots \geqq \frac{\Phi(Ax_0)}{\Phi(x_0)} = k\,.$$

Hence $\Phi(A^n x_0) \geqq k^n \Phi(x_0)$, $n=1, 2,\dots$. Since $k>1$, this contradicts the assumption that Φ is bounded on $A(S)$.

LEMMA 1.9. *Let ω be a P-domain and μ be a non-negative measure such that $U_\omega^\mu \leqq 1$. Then*

$$I_\omega(U_\omega^\sigma \mu) \leqq I_\omega(\sigma)$$

for any $\sigma \in \mathscr{M}_E(\omega)$ such that $U_\omega^{|\sigma|}$ is bounded and μ-integrable.

PROOF. For simplicity, we omit the subscript ω in $U_\omega^\cdot$, $I_\omega(\cdot)$ and $\int_\omega$. Let

$$S = \{\sigma \in \mathscr{M}_E(\omega);\ |U^\sigma| \leqq 1, \int |U^\sigma| d\mu \leqq 1\}$$

and

$$\Phi(\sigma) = I(\sigma), \quad A\sigma = U^\sigma \mu \qquad \text{for} \quad \sigma \in S\,.$$

Then, for $\sigma \in S$, we have

$$|U^{A\sigma}| \leqq U^{|U^\sigma|\mu} \leqq U^\mu \leqq 1\,,$$

$$\int |U^{A\sigma}| d\mu \leqq \int U^{|U^\sigma|\mu} d\mu = \int U^\mu |U^\sigma| d\mu \leqq \int |U^\sigma| d\mu \leqq 1$$

and

$$I(|A\sigma|) = \int U^{|A\sigma|} d|A\sigma| = \int U^{|U^\sigma|\mu} |U^\sigma| d\mu \leqq \int U^\mu |U^\sigma| d\mu \leqq 1\,.$$

Hence A is a mapping of S into itself and $\Phi(A\sigma) \leqq I(|A\sigma|) \leqq 1$, i.e., Φ is bounded on $A(S)$. Furthermore,

$$\Phi(A\sigma) = I(A\sigma) = \int U^{A\sigma} U^\sigma d\mu = \int U^{A^2\sigma} d\sigma \leqq I(A^2\sigma)^{1/2} I(\sigma)^{1/2}\,,$$

where the last inequality follows from the energy principle. Thus, (1.1) in the above lemma is satisfied, and hence

$$I(U^\sigma \mu) \leqq I(\sigma)$$

for all $\sigma \in S$. If $\sigma \in \mathscr{M}_E(\omega)$ and $U^{|\sigma|}$ is bounded, μ-integrable, then, for some $\alpha>0$, $\alpha\sigma \in S$. Hence

$$I(U^{\sigma}\mu) = \frac{1}{\alpha^2} I(U^{\alpha\sigma}\mu) \leqq \frac{1}{\alpha^2} I(\alpha\sigma) = I(\sigma).$$

PROOF OF THEOREM 1.1. If $\mu=0$, then the theorem is trivial. Thus, assume $\mu\neq 0$. Then $\beta\equiv\sup_{\omega} U^{\mu}_{\omega}>0$. Since $U^{\mu/\beta}_{\omega}\leqq 1$, the above lemma implies that

$$I_{\omega}(U^{\sigma}_{\omega}\mu) \leqq \beta^2 I_{\omega}(\sigma)$$

for any $\sigma\in\mathcal{M}_E(\omega)$ such that $U^{|\sigma|}_{\omega}$ is bounded and μ-integrable. Hence, for such σ we have by the energy principle

$$\int_{\omega}(U^{\sigma}_{\omega})^2 d\mu \leqq I_{\omega}(\sigma)^{1/2} I_{\omega}(U^{\sigma}_{\omega}\mu)^{1/2} \leqq \beta I_{\omega}(\sigma). \tag{1.3}$$

Next, let $\sigma\in\mathcal{M}_E(\omega)$ be arbitrary. We choose a sequence $\{\sigma_n\}$ in $\mathcal{M}_E(\omega)$ as described in Lemma 1.5. Since there is a bounded positive superharmonic function U^{μ}_{ω}, $\sigma_n\in\mathcal{M}_B(\omega)$. Furthermore, since $S(\sigma_n)$ is compact, $\int_{\omega} U^{|\sigma_n|}_{\omega} d\mu = \int_{\omega} U^{\mu}_{\omega}\, d|\sigma_n| < +\infty$, i.e., $U^{|\sigma_n|}_{\omega}$ is μ-integrable for each n. Therefore, (1.3) holds for $\sigma=\sigma_n$ and $|\sigma_n|$, so that

$$\int_{\omega}(U^{|\sigma_n|}_{\omega})^2 d\mu \leqq \beta I_{\omega}(|\sigma_n|) \leqq \beta I_{\omega}(|\sigma|) < +\infty,$$

and hence

$$\int_{\omega}(U^{|\sigma|}_{\omega})^2 d\mu < +\infty.$$

Since $|U^{\sigma_n}_{\omega}|\leqq U^{|\sigma|}_{\omega}$, Lebesgue's convergence theorem implies $\int_{\omega}(U^{\sigma_n}_{\omega})^2 d\mu \to \int_{\omega}(U^{\sigma}_{\omega})^2 d\mu$ $(n\to\infty)$. On the other hand $I_{\omega}(\sigma_n)\to I_{\omega}(\sigma)$. Hence (1.3) holds for the given σ.

The next lemma, which is a consequence of the above theorem, will be used later (in § 7).

LEMMA 1.10. *Let ω be a P-domain and μ be a non-negative measure on ω such that U^{μ}_{ω} is bounded. Then, for any μ-square-integrable function f, $f\mu\in\mathcal{M}_E(\omega)$; in fact*

$$I_{\omega}(f\mu) \leqq (\sup_{\omega} U^{\mu}_{\omega})\int_{\omega} f^2 d\mu.$$

PROOF. Since $I_{\omega}(f\mu)\leqq I_{\omega}(|f|\mu)$, we may assume $f\geqq 0$. Let $\{\omega_n\}$ be an exhaustion of ω and let $f_n=\min(f, n)$ on ω_n, $f_n=0$ on $\omega-\omega_n$. Then $U^{f_n\mu}_{\omega}$ is bounded and $S(f_n\mu)\subset\bar{\omega}_n$. Therefore, $f_n\mu\in\mathcal{M}^+_E(\omega)$ and

$$I_\omega(f_n\mu) = \int_\omega U_\omega^{f_n\mu} f_n \, d\mu \leqq \left\{\int_\omega (U_\omega^{f_n\mu})^2 d\mu\right\}^{1/2} \left\{\int_\omega f^2 d\mu\right\}^{1/2}.$$

By the above theorem,

$$\int_\omega (U_\omega^{f_n\mu})^2 d\mu \leqq \beta I_\omega(f_n\mu),$$

where $\beta = \sup_\omega U_\omega^\mu$. Hence

$$I_\omega(f_n\mu) \leqq \beta \int_\omega f^2 d\mu.$$

Letting $n \to \infty$, we obtain the required inequality.

§ 2. Preliminary theory on locally bounded functions

2.1. The space $\mathscr{B}_{\mathrm{loc}}(\omega)$ and Axiom 5

A domain ω will be called a *PC-domain* if it is relatively compact and there is a P-domain ω^* such that $\bar{\omega} \subset \omega^*$. By (P_1) in § 1, a PC-domain is a P-domain. By (P_2), we also see that PC-domains form a base of open sets in Ω.

We consider the following space of locally bounded functions on an open set ω (cf. [9, § 6.1]):

$$\mathscr{B}_{\mathrm{loc}}(\omega) = \left\{ f;\ \begin{array}{l} \text{for any PC-domain } \omega' \text{ such that } \bar{\omega}' \subset \omega, \text{ there} \\ \text{are two non-negative bounded superhamonic} \\ \text{functions } s_1 \text{ and } s_2 \text{ such that } f|\omega' = s_1 - s_2 \end{array} \right\}.$$

For each $f \in \mathscr{B}_{\mathrm{loc}}(\omega)$, there is a unique signed measure σ_f on ω which has the following property: for any PC-domain ω' such that $\bar{\omega}' \subset \omega$, $U_{\omega'}^{|\sigma_f|}$ is bounded on ω' and

$$f|\omega' = u + U_{\omega'}^{\sigma_f}$$

with $u \in \mathscr{H}(\omega')$. We call σ_f *the associated measure of* f.

In this paper, we do not require that the constant function 1 is superharmonic; but we assume

Axiom 5. The constant function 1 belongs to $\mathscr{B}_{\mathrm{loc}}(\Omega)$ and $U_\omega^{|\pi|}$ is continuous for any PC-domain ω, where π is the associated measure of 1 (i.e., $\pi \equiv \sigma_1$).

REMARK 2.1. If 1 is superharmonic, then Axiom 5 is trivially satisfied. This case, in which $\pi \geqq 0$, was treated in [9].

REMARK 2.2. The above Axiom 5 is equivalent to the following

Axiom 5′. There is a covering $\{\omega_\iota\}_{\iota\in I}$ of Ω by domains on each of which there are two non-negative continuous superharmonic functions $s_\iota^{(1)}$ and $s_\iota^{(2)}$ such that $1=s_\iota^{(1)}-s_\iota^{(2)}$ on ω_ι.

2.2. PB-domains

A P-domain ω will be called a *PB-domain* if $U_\omega^{|\pi|}$ is bounded on ω. It is easy to see that a PC-domain is a PB-domain. Note that if 1 is superharmonic, then any P-domain is a PB-domain.

LEMMA 2.1. *If ω is a PB-domain, then $U_\omega^{\pi^+}$, $U_\omega^{\pi^-}$, and hence U_ω^π, are bounded continuous on ω and*

$$1 = u_\omega + U_\omega^\pi$$

with a bounded non-negative harmonic function u_ω on ω.

PROOF. It is easy to see by Axiom 5 that $U_\omega^{|\pi|}$ is continuous. Since $0\leqq U_\omega^{\pi^+}+U_\omega^{\pi^-}=U_\omega^{|\pi|}$ and $U_\omega^{|\pi|}$ is bounded, we see that $U_\omega^{\pi^+}$, $U_\omega^{\pi^-}$ are bounded continuous. Then $u_\omega=1-U_\omega^\pi$ is bounded harmonic on ω and $u_\omega\geqq -U_\omega^{\pi^+}$ implies that $u_\omega\geqq 0$ on ω.

By this lemma, for a PB-domain ω, $s_\omega\equiv 1+U_\omega^{\pi^-}=u_\omega+U_\omega^{\pi^+}$ is bounded superharmonic on ω. Obviously, $s_\omega\geqq 1$. Let

$$\beta_\omega = \sup_\omega s_\omega \quad (\geqq 1) \tag{2.1}$$

for any PB-domain ω. Then $U_\omega^{\pi^+}\leqq\beta_\omega$, $U_\omega^{\pi^-}\leqq\beta_\omega-1$, $U_\omega^{|\pi|}\leqq 2\beta_\omega-1$ and $|U_\omega^\pi|\leqq \beta_\omega$.

Using the functions s_ω for PC-domains ω, we see easily that $\mathscr{H}(\omega_0)\subset \mathscr{B}_{\mathrm{loc}}(\omega_0)$ for any open set ω_0.

LEMMA 2.2. *If ω is a PB-domain, then for any potential p on ω,*

$$\sup_\omega p \leqq \beta_\omega \sup_{S_h(p)} p. \tag{2.2}$$

PROOF. Let $M\equiv\sup_{S_h(p)}p$. If $M=+\infty$, then (2.2) is trivial. Suppose $M<+\infty$. Then $Ms_\omega\geqq p$ on $S_h(p)$. Hence, by Lemma 1.2, we see that $Ms_\omega\geqq p$ on ω, and hence (2.2).

LEMMA 2.3. *Let ω be a PB-domain and μ, ν be two non-negative measures on ω. If $U_\omega^\mu\leqq U_\omega^\nu$ on ω, then $\mu(\omega)\leqq\beta_\omega\nu(\omega)$.*

PROOF. $$\hat{G}_\omega(x, y) = \frac{G_\omega(x, y)}{s_\omega(x)s_\omega(y)}$$

is a Green function for $(\omega, \mathfrak{H}_\omega/s_\omega)$. For any non-negative measure μ on ω,

$$U_\omega^\mu(x) = s_\omega(x)\int_\omega \hat{G}_\omega(x, y)s_\omega(y)d\mu(y).$$

Hence, $U_\omega^\mu \leqq U_\omega^\nu$ implies $\int_\omega \hat{G}_\omega(x, y)s_\omega(y)d\mu(y) \leqq \int_\omega \hat{G}_\omega(x, y)s_\omega(y)d\nu(y)$. Applying [9, Lemma 1.10] with respect to the structure $\mathfrak{H}_\omega/s_\omega$, we see that $\int_\omega s_\omega d\mu \leqq \int_\omega s_\omega d\nu$. Therefore,

$$\mu(\omega) \leqq \int_\omega s_\omega d\mu \leqq \int_\omega s_\omega d\nu \leqq \beta_\omega \nu(\omega).$$

LEMMA 2.4. *Let ω be a PB-domain and ω' be a relatively compact open set such that $\bar{\omega}' \subset \omega$. Then, there is a signed measure $\lambda \equiv \lambda(\omega'; \omega)$ which has the following properties*:

(a) $U_\omega^\lambda = 0$ *on* ω' *and* $U_\omega^\lambda \geqq 0$ *on* ω;

(b) $S(\lambda) \subset \bar{\omega}'$;

(c) $U_\omega^{\lambda^-} \leqq \beta_\omega - 1$ *and* $U_\omega^{\lambda^+} \leqq \beta_\omega$ *on* ω.

PROOF. Let $v_1 = u_\omega + U_\omega^{\pi^+}$ and $v_2 = U_\omega^{\pi^-}$ $(= v_1 - 1)$. By Lemma 1.7, $p_i \equiv R_{v_i}^{\omega', \omega}$, $i = 1, 2$, are potentials on ω. Let λ_i, $i = 1, 2$, be the associated measures of p_i and let $\lambda = \lambda_1 - \lambda_2$. Since $v_1 \geqq v_2$, $p_1 \geqq p_2$. Hence $U_\omega^\lambda \geqq 0$. Then, by using Lemma 1.7 we see easily that this λ is the required measure.

2.3. Product of functions in $\mathscr{B}_{\mathrm{loc}}(\omega)$

LEMMA 2.5. *Let ω be a PB-domain and s be a bounded non-negative superharmonic function on ω. Then, for any constant α such that $\alpha \geqq \sup_\omega s$,*

$$v = 2\alpha s + \alpha^2 U_\omega^{\pi^-} - s^2$$

is a bounded non-negative superharmonic function on ω.

PROOF. Obviously, v is bounded. Writing

$$v = \alpha^2(1 + U_\omega^{\pi^-}) - (\alpha - s)^2,$$

we see that $v \geqq 0$. Furthermore, since $\alpha - s$ is non-negative upper semicontinuous, v is lower semicontinuous. Let ω' be any regular domain such that $\bar{\omega}' \subset \omega$ and let $x \in \omega'$. Then, since $\int d\mu_x^{\omega'} = u_{\omega'}(x)$ (see Lemma 2.1), we have

$$\left(\int s\, d\mu_x^{\omega'}\right)^2 \leqq \left(\int s^2 d\mu_x^{\omega'}\right)\left(\int d\mu_x^{\omega'}\right)$$

$$\leqq \left(\int s^2 d\mu_x^{\omega'}\right)\{1+U_{\omega'}^{\pi^-}(x)\}\,.$$

Hence,

$$\begin{aligned}\int v\,d\mu_x^{\omega'} &= \alpha^2\int U_\omega^{\pi^-}d\mu_x^{\omega'}+2\alpha\int s\,d\mu_x^{\omega'}-\int s^2 d\mu_x^{\omega'}\\ &\leqq \alpha^2\{U_\omega^{\pi^-}(x)-U_{\omega'}^{\pi^-}(x)\}+2\alpha\int s\,d\mu_x^{\omega'}-\left(\int s\,d\mu_x^{\omega'}\right)^2\{1+U_{\omega'}^{\pi^-}(x)\}^{-1}\\ &= \alpha^2\{1+U_\omega^{\pi^-}(x)\}-\left(\alpha-\int s\,d\mu_x^{\omega'}\right)^2\\ &\quad +[1-\{1+U_{\omega'}^{\pi^-}(x)\}^{-1}]\left(\int s\,d\mu_x^{\omega'}\right)^2-\alpha^2 U_{\omega'}^{\pi^-}(x)\,.\end{aligned}$$

Since $0\leqq\int s\,d\mu_x^{\omega'}\leqq s(x)\leqq\alpha$, $\left(\alpha-\int s\,d\mu_x^{\omega'}\right)^2\geqq(\alpha-s(x))^2$. Hence

$$\int v\,d\mu_x^{\omega'}\leqq v(x)+\alpha^2[1-U_{\omega'}^{\pi^-}(x)-\{1+U_{\omega'}^{\pi^-}(x)\}^{-1}]\leqq v(x)\,.$$

Therefore v is superharmonic on ω.

COROLLARY. *If ω is a PB-domain and s is a bounded non-negative superharmonic function on ω, then there are two bounded non-negative superharmonic functions v_1 and v_2 such that $s^2=v_1-v_2$ on ω. Thus, $\sigma\equiv\sigma_{s^2}$ is well-defined, $s^2=u+U_\omega^\sigma$ on ω with $u\in\mathscr{H}(\omega)$ and $U_\omega^{|\sigma|}$ is bounded. If, furthermore, $\sigma_s(\omega)<+\infty$ and $\pi^-(\omega)<+\infty$, then $\sigma^+(\omega)<+\infty$.*

PROOF. Let $\alpha\geq\sup_\omega s$ and $v_1=2\alpha s+\alpha^2 U_\omega^{\pi^-}$. Then v_1 is bounded non-negative superharmonic on ω. By the above lemma $v_2=v_1-s^2$ is bounded non-negative superharmonic on ω. Furthermore, it follows that $\sigma^+\leqq\sigma_{v_1}=2\alpha\sigma_s+\alpha^2\pi^-$. Hence we also have the last assertion in the corollary.

PROPOSITION 2.1. *If f, $g\in\mathscr{B}_{loc}(\omega)$, then $fg\in\mathscr{B}_{loc}(\omega)$.*

PROOF. Let ω' be any PC-domain such that $\bar{\omega}'\subset\omega$. Then, by definition $f|\omega'=s_1-s_2$ with bounded non-negative superharmonic functions s_1 and s_2 on ω. Since

$$f^2|\omega'=2(s_1^2+s_2^2)-(s_1+s_2)^2\,,$$

the above corollary implies that there are two bounded non-negative superharmonic functios v_1 and v_2 such that $f^2|\omega'=v_1-v_2$. Hence $f^2\in\mathscr{B}_{loc}(\omega)$. Then, it follows that $fg=\{(f+g)^2-f^2-g^2\}/2$ also belongs to $\mathscr{B}_{loc}(\omega)$.

2.4. Product of bounded potentials on a PB-domain

LEMMA 2.6. *Let ω be a PB-domain such that $\pi^-(\omega)<+\infty$. Then for any $\sigma\in\mathcal{M}_B(\omega)$, there is a $\sigma'\in\mathcal{M}_B(\omega)$ such that*

$$(U_\omega^\sigma)^2 = U_\omega^{\sigma'}.$$

PROOF. If $\mu\in\mathcal{M}_B^+(\omega)$, then by Lemma 2.5 $(U_\omega^\mu)^2=v_1-v_2$, where $v_1=2\alpha U_\omega^\mu+\alpha^2 U_\omega^{\pi^-}$ $(\alpha=\sup_\omega U_\omega^\mu)$ and v_2 is bounded non-negative superharmonic on ω. Thus we see that v_1 and v_2 are potentials on ω. Let ν_1 and ν_2 be their respective assoicated measures. Then $\nu_1=2\alpha\mu+\alpha^2\pi^-\in\mathcal{M}_B^+(\omega)$. Since $v_2\leqq v_1$, $\nu_2(\omega)<+\infty$ by Lemma 2.3, and hence $\nu_2\in\mathcal{M}_B^+(\omega)$. Thus $(U_\omega^\mu)^2=U_\omega^{\nu_1-\nu_2}$ and $\nu_1-\nu_2\in\mathcal{M}_B(\omega)$. For $\sigma\in\mathcal{M}_B(\omega)$, writing

$$(U_\omega^\sigma)^2 = 2\{(U_\omega^{\sigma^+})^2+(U_\omega^{\sigma^-})^2\}-(U_\omega^{|\sigma|})^2$$

and using the above result, we obtain the lemma.

REMARK 2.3. There are PB-domains ω for which $\pi^-(\omega)=+\infty$.

PROPOSITION 2.2. *Let ω be a PB-domain such that $\pi^-(\omega)<+\infty$. If $p=U_\omega^\sigma$ with $\sigma\in\mathcal{M}_B(\omega)$, then $\sigma_{p^2}\in\mathcal{M}_B(\omega)$ and*

$$\sigma_{p^2}(\omega) = \int_\omega p^2 d\pi.$$

PROOF. It is enough to prove the case $\sigma\in\mathcal{M}_B^+(\omega)$ (cf. the proof of the above lemma). First we note that p^2 is $|\pi|$-integrable, since

$$\int_\omega p^2 d|\pi| \leqq (\sup_\omega p)\int_\omega U_\omega^\sigma d|\pi| = (\sup_\omega p)\int_\omega U_\omega^{|\pi|} d\sigma < +\infty.$$

For $\alpha>0$, let $f_\alpha=\min(p/\alpha, 1)$ on ω. Then $0\leqq f_\alpha\leqq 1$ and $f_\alpha\uparrow 1$ as $\alpha\downarrow 0$. Let $1=u_\omega+U_\omega^\pi$ and

$$g_\alpha = \min(p/\alpha+U_\omega^{\pi^-}, u_\omega+U_\omega^{\pi^+}).$$

For each α, g_α is a bounded potential on ω (in fact, $g_\alpha\leqq\beta_\omega$) and $f_\alpha=g_\alpha-U_\omega^{\pi^-}$. Let $\mu_\alpha=\sigma_{g_\alpha}$, i.e., $g_\alpha=U_\omega^{\mu_\alpha}$. Since $g_\alpha\leqq p/\alpha+U_\omega^{\pi^-}$, we see that $\mu_\alpha\in\mathcal{M}_B^+(\omega)$ by Lemma 2.3. The above lemma implies that $p^2=U_\omega^{\sigma'}$ with $\sigma'\equiv\sigma_{p^2}\in\mathcal{M}_B(\omega)$. Hence, by Lebesgue's convergence theorem,

$$\sigma'(\omega) = \lim_{\alpha\to 0}\int_\omega f_\alpha d\sigma' = \lim_{\alpha\to 0}\int_\omega (U_\omega^{\mu_\alpha}-U_\omega^{\pi^-})d\sigma'$$

$$= \lim_{\alpha\to 0}\int_\omega p^2 d\mu_\alpha-\int_\omega p^2 d\pi^-.$$

Let $\omega_\alpha=\{x\in\omega\,;\,p(x)>\alpha\}$. Then ω_α is an open set and $f_\alpha=1$ on ω_α. It follows that $\mu_\alpha|\omega_\alpha=\pi^+|\omega_\alpha$. Hence

$$\int_\omega p^2 d\mu_\alpha=\int_{\omega_\alpha} p^2 d\pi^+ +\int_{\omega-\omega_\alpha} p^2 d\mu_\alpha .$$

Since $\omega_\alpha\uparrow\omega$ as $\alpha\downarrow 0$,

$$\lim_{\alpha\to 0}\int_{\omega_\alpha} p^2 d\pi^+ =\int_\omega p^2 d\pi^+ .$$

On the other hand,

$$0\leqq\int_{\omega-\omega_\alpha} p^2 d\mu_\alpha \leqq \alpha\int_{\omega-\omega_\alpha} p\, d\mu_\alpha$$

$$\leqq \alpha\int_\omega U_\omega^{\mu_\alpha} d\sigma \leqq \alpha\beta_\omega\sigma(\omega)\to 0 \ \ (\alpha\to 0).$$

Thus we obtain the required equality.

Corollary. *Let ω be a PB-domain such that $\pi^-(\omega)<+\infty$. If $p_i=U_\omega^{\sigma_i}$ with $\sigma_i\in\mathscr{M}_B(\omega)$, $i=1, 2$, then $\sigma_{p_1p_2}\in\mathscr{M}_B(\omega)$ and*

$$\sigma_{p_1p_2}(\omega)=\int_\omega p_1p_2\, d\pi .$$

2.5. The space $\mathscr{H}_{BE}(\omega)$

Lemma 2.7. *If ω is a PB-domain such that $\pi^-(\omega)<+\infty$, then for any bounded $u\in\mathscr{H}(\omega)$, $\sigma^+_{u^2}(\omega)<+\infty$.*

Proof. Let $\alpha=\sup_\omega|u|$ and consider the function

$$v=\alpha^2\beta_\omega U_\omega^{\pi^-}-u^2$$

on ω. It is obviously a continuous function. Let ω' be any regular domain such that $\bar{\omega}'\subset\omega$ and let $x\in\omega'$. As in the proof of Lemma 2.5, we have

$$u^2(x)=\left(\int u\ d\mu_x^{\omega'}\right)^2 \lneqq \left(\int u^2 d\mu_x^{\omega'}\right)\{1+U_{\omega'}^{\pi^-}(x)\} .$$

Since

$$\int u^2 d\mu_x^{\omega'}\leqq\alpha^2\int d\mu_x^{\omega'}\leqq\alpha^2\{1+U_{\omega'}^{\pi^-}(x)\}\leqq\alpha^2\beta_\omega ,$$

we have

$$u^2(x) \leqq \int u^2 d\mu_x^{\omega'} + \alpha^2 \beta_\omega U_{\omega'}^{\pi^-}(x).$$

Hence

$$\begin{aligned}\int v\, d\mu_x^{\omega'} &= -\int u^2 d\mu_x^{\omega'} + \alpha^2\beta_\omega \int U_\omega^{\pi^-} d\mu_x^{\omega'} \\ &\leqq -u^2(x) + \alpha^2\beta_\omega U_{\omega'}^{\pi^-}(x) + \alpha^2\beta_\omega\{U_\omega^{\pi^-}(x) - U_{\omega'}^{\pi^-}(x)\} \\ &= v(x).\end{aligned}$$

Therefore v is superharmonic, that is, $\sigma_v \geqq 0$. Hence $\sigma_{u^2} \leqq \alpha^2\beta_\omega\pi^-$, which implies $\sigma_{u^2}^+(\omega) \leqq \alpha^2\beta_\omega\pi^-(\omega) < +\infty$.

For an open set ω, let

$$\mathscr{H}_{BE}(\omega) = \{u \in \mathscr{H}(\omega);\ \text{bounded}, \sigma_{u^2}^-(\omega) < +\infty\}.$$

PROPOSITION 2.3. *If ω is a PB-domain such that $\pi^-(\omega) < +\infty$, then $\mathscr{H}_{BE}(\omega)$ is a linear subspace of $\mathscr{H}(\omega)$ and is a vector lattice with respect to the natural order.*

PROOF. It is obvious that $u \in \mathscr{H}_{BE}(\omega)$ implies $\alpha u \in \mathscr{H}_{BE}(\omega)$ for any real α. Let $u, v \in \mathscr{H}_{BE}(\omega)$. Obviously, $u+v$ and $u-v$ are bounded. Since $(u+v)^2 + (u-v)^2 = 2(u^2+v^2)$,

$$\sigma_{(u+v)^2}^- \leqq 2(\sigma_{u^2}^- + \sigma_{v^2}^-) + \sigma_{(u-v)^2}^+.$$

By the above lemma, $\sigma_{(u-v)^2}^+(\omega) < +\infty$. Hence $\sigma_{(u+v)^2}^-(\omega) < +\infty$, so that $u+v \in \mathscr{H}_{BE}(\omega)$.

Next, let $u \in \mathscr{H}_{BE}(\omega)$ and $\alpha = \sup_\omega |u|$. $-|u|$ is superharmonic on ω and $0 \leqq |u| \leqq \alpha s_\omega$ ($s_\omega = 1 + U_\omega^{\pi^-}$). Hence the least harmonic majorant w of $|u|$ exists and $|u| \leqq w \leqq \alpha s_\omega$. It follows that w is also bounded. For simplicity, let $\sigma = \sigma_{u^2}$ and $\tau = \sigma_{w^2}$. Since $w - |u|$ is a potential and $0 \leqq w^2 - u^2 \leqq 2\alpha\beta_\omega(w - |u|)$, we see that $U_\omega^\sigma \leqq U_\omega^\tau$. Therefore, $U_\omega^{\tau^-} \leqq U_\omega^{\tau^+} + U_\omega^{\sigma^-}$. By assumption $\sigma^-(\omega) < +\infty$ and by the above lemma $\tau^+(\omega) < +\infty$. Hence Lemma 2.3 implies that $\tau^-(\omega) < +\infty$. Therefore $w \in \mathscr{H}_{BE}(\omega)$. Since $\mathscr{H}_{BE}(\omega)$ is a linear subspace as proved above, it follows that $\mathscr{H}_{BE}(\omega)$ is a vector lattice.

The next lemma will be used in the later sections.

LEMMA 2.8. *If $f \in \mathscr{B}_{loc}(\omega_0)$ (ω_0: an open set) and ω is a PC-domain such that $\bar{\omega} \subset \omega_0$, then $f|\omega - U_\omega^{\sigma_f} \in \mathscr{H}_{BE}(\omega)$.*

PROOF. First, note that $\pi^-(\omega) < +\infty$ if ω is a PC-domain. For simplicity, let $\sigma \equiv \sigma_f$. Let $u = f|\omega - U_\omega^\sigma$. It is a bounded harmonic function on ω. We can choose another PC-domain ω' such that $\bar{\omega}\subset\omega'$, $\bar{\omega}'\subset\omega_0$. $u'=f|\omega'-U^{\sigma}_{\omega'}$ is also bounded harmonic on ω'. We can write

$$u = u'|\omega+(U^{\sigma}_{\omega'}|\omega-U^{\sigma}_{\omega}).$$

Since $\sigma_{(u')^2}$ is a signed measure on ω', $\sigma^-_{(u')^2}(\omega)<+\infty$. Thus $u'|\omega\in\mathscr{H}_{BE}(\omega)$. Next, we consider $v=U^{\sigma}_{\omega'}|\omega-U^{\sigma}_{\omega}$. It is bounded harmonic on ω. Since $\sigma|\omega'\in\mathscr{M}_B(\omega')$, there is a $\sigma'\in\mathscr{M}_B(\omega')$ such that $(U^{\sigma}_{\omega'})^2=U^{\sigma'}_{\omega'}$ by Lemma 2.6. Now,

$$v^2 = U^{\sigma'}_{\omega'}\,|\,\omega-2(U^{\sigma}_{\omega'}\,|\,\omega)U^{\sigma}_{\omega}+(U^{\sigma}_{\omega})^2.$$

Let $\tau=\sigma_{v^2}$. By the corollary to Lemma 2.5, we see that $v^2=h+U^{\tau}_{\omega}$ with $h\in\mathscr{H}(\omega)$ (cf. the proof of Proposition 2.1). Since $|2(U^{\sigma}_{\omega'}|\omega)U^{\sigma}_{\omega}+(U^{\sigma}_{\omega})^2|$ is majorized by a potential on ω, it follows that

$$U^{\tau}_{\omega} = U^{\sigma'}_{\omega}-2(U^{\sigma}_{\omega'}\,|\,\omega)U^{\sigma}_{\omega}+(U^{\sigma}_{\omega})^2\,.$$

Hence

$$U^{\tau^-}_{\omega} \leqq U^{\tau^+}_{\omega}+U^{\sigma'^-}_{\omega}+2\alpha U^{|\sigma|}_{\omega}\,,$$

where $\alpha=\sup_{\omega}|U^{\sigma}_{\omega'}|$. By Lemma 2.7, $\tau^+(\omega)<+\infty$. Obviously, $\sigma'^-(\omega)<+\infty$ and $|\sigma|(\omega)<+\infty$. Hence, $\tau^-(\omega)<+\infty$ by Lemma 2.3, so that $v\in\mathscr{H}_{BE}(\omega)$. Therefore $u\in\mathscr{H}_{BE}(\omega)$.

2.6. Product of a bounded harmonic function and a bounded potential

LEMMA 2.9. *Let ω be a PB-domain. If $\sigma\in\mathscr{M}_B(\omega)$ and $u\in\mathscr{H}(\omega)$ is bounded, then there is a signed measure σ' on ω such that $U^{|\sigma'|}_{\omega}$ is bounded and $uU^{\sigma}_{\omega}=U^{\sigma'}_{\omega}$. If, in addition, $\pi^-(\omega)<+\infty$ and $u\in\mathscr{H}_{BE}(\omega)$, then $\sigma'\in\mathscr{M}_B(\omega)$.*

PROOF. As in the proof of Proposition 2.3, the least harmonic majorant of $|u|$ on ω exists and is bounded, and hence $u=u_1-u_2$ with non-negative bounded harmonic functions u_1 and u_2. Thus we may assume that $u\geqq 0$ and $\sigma\in\mathscr{M}^+_B(\omega)$. Since

$$uU^{\sigma}_{\omega} = \frac{1}{2}\{(u+U^{\sigma}_{\omega})^2-u^2-(U^{\sigma}_{\omega})^2\}\,,$$

it follows from the corollary to Lemma 2.5 that $uU^{\sigma}_{\omega}=h+U^{\sigma'}_{\omega}$ with a signed measure σ' on ω such that $U^{|\sigma'|}_{\omega}$ is bounded and $h\in\mathscr{H}(\omega)$. Since uU^{σ}_{ω} is dominated by a potential, $h=0$, so that $uU^{\sigma}_{\omega}=U^{\sigma'}_{\omega}$.

Next, suppose $\pi^-(\omega)<+\infty$ and $u\in\mathscr{H}_{BE}(\omega)$. For simplicity, put $s=u+U^{\sigma}_{\omega}$ and $p=U^{\sigma}_{\omega}$. Then $\sigma'=\frac{1}{2}(\sigma_{s^2}-\sigma_{u^2}-\sigma_{p^2})$. Since $\sigma_s=\sigma$, the corollary to

Lemma 2.5 implies that $\sigma^+_{s^2}(\omega) < +\infty$. By Lemma 2.6, $\sigma_{p^2} \in \mathcal{M}_B(\omega)$ and by assumption $\sigma^-_{u^2}(\omega) < +\infty$. Therefore,

$$\sigma'^+(\omega) \leqq \frac{1}{2}\{\sigma^+_{s^2}(\omega) + \sigma^-_{u^2}(\omega) + \sigma^-_{p^2}(\omega)\} < +\infty .$$

Since $U^{\sigma'}_\omega \geqq 0$, $U^{\sigma'^-}_\omega \leqq U^{\sigma'^+}_\omega$. Hence, by Lemma 2.3, we also have $\sigma'^-(\omega) < +\infty$. Therefore $\sigma' \in \mathcal{M}_B(\omega)$.

The rest of this section is devoted to the proof of the following proposition (cf. [9, § 2.3]):

PROPOSITION 2.4. *Let ω be a PB-domain such that $\pi^-(\omega) < +\infty$. If $p = U^\sigma_\omega$ with $\sigma \in \mathcal{M}_B(\omega)$ and if $u \in \mathcal{H}_{BE}(\omega)$, then*

$$\sigma_{up}(\omega) = \int_\omega u\, d\sigma + \int_\omega up\, d\pi .$$

Given an open set ω in Ω, if $\bar{\omega}$ is not compact, then let ω^a be the closure of ω in the one point compactification of Ω; otherwise, let $\omega^a \equiv \bar{\omega}$.

We fix a PB-domain ω_0 such that $\pi^-(\omega_0) < +\infty$. For $y \in \omega_0$ and $\alpha > 0$ ($\alpha < G_{\omega_0}(y, y)$), consider the open set

$$\omega_{\alpha,y} = \{x \in \omega_0 ; G_{\omega_0}(x, y) > \alpha\} .$$

By using [2, Corollary 3 and Lemma 1], we see easily that $\omega^a_{\alpha,y}$ is a resolutive compactification of $\omega_{\alpha,y}$. Let $H^{\omega_{\alpha,y}}_\psi$ be the Dirichlet solution of $\omega_{\alpha,y}$ for the boundary function $\psi \in \mathbf{C}(\partial^a\omega_{\alpha,y})$, where $\partial^a\omega_{\alpha,y} = \omega^a_{\alpha,y} - \omega_{\alpha,y}$ and $\mathbf{C}(X)$ means the set of continuous functions on X. We shall denote by $\mu_{\alpha,y}$ the harmonic measure at y for the open set $\omega_{\alpha,y}$. By [2, Lemma 1], we see that $\mu_{\alpha,y}(\partial^a\omega_{\alpha,y} - \omega_0) = 0$ (cf. [9, Lemma 2.6]). We note that each component ω' of $\omega_{\alpha,y}$ is a PB-domain and $1 = H^{\omega_{\alpha,y}}_1 + U^\pi_{\omega'}$ on ω'. On account of the fact that $U^{\pi^+}_{\omega_0} \leqq \beta_{\omega_0}$, we obtain the following lemma in the same way as [9, Lemma 2.5]:

LEMMA 2.10. $\pi^+(\omega_{\alpha,y}) \leqq \dfrac{\beta_{\omega_0}}{\alpha}$ *and* $\lim_{\alpha\to 0} \alpha\pi^+(\omega_{\alpha,y}) = 0$.

By virtue of this lemma and our assumption that $\pi^-(\omega_0) < +\infty$, we see that

$$\psi \longrightarrow \int_{\omega_{\alpha,y}} H^{\omega_{\alpha,y}}_\psi d\pi$$

is a bounded linear functional on $\mathbf{C}(\partial^a\omega_{\alpha,y})$. Hence, there is a signed measure $\nu_{\alpha,y}$ on $\partial^a\omega_{\alpha,y}$ such that

$$\int_{\omega_{\alpha,y}} H^{\omega_{\alpha,y}}_\psi d\pi = \int \psi\, d\nu_{\alpha,y}$$

for all $\psi \in \mathbf{C}(\partial^a \omega_{\alpha,y})$. Since $\mu_{\alpha,y}(\partial^a \omega_{\alpha,y} - \omega_0) = 0$ and hence $\nu_{\alpha,y}(\partial^a \omega_{\alpha,y} - \omega_0) = 0$, we may regard $\mu_{\alpha,y}$ and $\nu_{\alpha,y}$ as measures on ω_0.

LEMMA 2.11. *With the notation given above, let*

$$w_{\alpha,y} = \frac{1}{\alpha} U_{\omega_0}^{\mu_{\alpha,y}} - U_{\omega_0}^{\nu_{\alpha,y}} + U_{\omega_0}^{\pi|\omega_{\alpha,y}}.$$

Then $w_{\alpha,y} = 1$ on $\omega_{\alpha,y}$ and $|w_{\alpha,y}(x)| \leqq 4\beta_{\omega_0} - 1$ for all $x \in \omega_0$.

PROOF. Fix α and y and let $\mu = \mu_{\alpha,y}$, $\nu = \nu_{\alpha,y}$, $\omega = \omega_{\alpha,y}$ and $w = w_{\alpha,y}$. Also, let $\beta = \beta_{\omega_0}$. We first remark that $U_{\omega_0}^{\mu}(x) \leqq G_{\omega_0}(x, y)$ for all $x \in \omega_0$ and $U_{\omega_0}^{\mu}(x) = \alpha H_1^{\omega}(x)$ for $x \in \omega$ (cf. [9, Lemma 1.4]). Hence

$$U_{\omega_0}^{\mu}(x) \leqq G_{\omega_0}(x, y) \leqq \alpha$$

for $x \notin \omega$ and

$$U_{\omega_0}^{\mu}(x) = \alpha H_1^{\omega}(x) \leqq \alpha\{1 + U_{\omega_0}^{\pi^-}(x)\} \leqq \alpha\beta$$

for $x \in \omega$. Therefore, $U_{\omega_0}^{\mu} \leqq \alpha\beta$ on ω_0.

Next, as in the proof of [9, Lemma 2.8], we have

$$U_{\omega_0}^{\nu}(x) = \int_{\omega} H_{\psi_x}^{\omega} d\pi,$$

where $\psi_x(\xi) = G_{\omega_0}(x, \xi)$ if $\xi \in \partial^a \omega \cap \omega_0$ and $\psi_x(\xi) = 0$ if $\xi \in \partial^a \omega - \omega_0$. Since $H_{\psi_x}^{\omega}(z) \leqq G_{\omega_0}(x, z)$ for $z \in \omega$, we have

$$|U_{\omega_0}^{\nu}| \leqq U_{\omega_0}^{|\pi|} \leqq 2\beta - 1.$$

Also $|U_{\omega_0}^{\pi|\omega}| \leqq \beta$. Thus

$$|w| \leqq \beta + (2\beta - 1) + \beta = 4\beta - 1.$$

If $x \in \omega$, then let ω' be the component of ω containing x. Then, again as in the proof of [9, Lemma 2.8], we see that

$$U_{\omega_0}^{\nu}(x) = U_{\omega_0}^{\pi|\omega}(x) - U_{\omega'}^{\pi}(x).$$

Therefore,

$$w(x) = H_1^{\omega}(x) + U_{\omega'}^{\pi}(x) = 1.$$

By virtue of this lemma, we obtain the following lemma in the same way as [9, Lemma 2.9]:

LEMMA 2.12. *With the same notation as above, if σ is a signed measure on*

ω_0 such that $|\sigma|(\omega_0)<+\infty$, then

$$\sigma(\omega_0)=\lim_{\alpha\to 0}\left\{\frac{1}{\alpha}\int_{\omega_0}U^{\sigma}_{\omega_0}d\mu_{\alpha,y}-\int_{\omega_0}U^{\sigma}_{\omega_0}d\nu_{\alpha,y}\right\}+\int_{\omega_0}U^{\sigma}_{\omega_0}d\pi$$

for any $y\in\omega_0$.

PROOF OF PROPOSITION 2.4 (cf. the proof of [9, Lemmas 2.10 and 2.11]). Let $\sigma'=\sigma_{up}$. By Lemma 2.9, $\sigma'\in\mathscr{M}_B(\omega)$ and $up=U^{\sigma'}_{\omega}$. It follows that up is $|\pi|$-integrable. Let $\{\omega_n\}$ be an exhaustion of ω and consider the signed measures $\lambda_n\equiv\lambda(\omega_n;\omega)$ given in Lemma 2.4. Then $\{U^{\lambda_n}_{\omega}\}$ is uniformly bounded and $U^{\lambda_n}_{\omega}\to 1$ on ω. Therefore, by Lebesgue's convergence theorem,

$$\sigma'(\omega)=\lim_{n\to\infty}\int_{\omega}U^{\lambda_n}_{\omega}d\sigma'=\lim_{n\to\infty}\int_{\omega}up\,d\lambda_n\,.$$

Since $\lambda_n|\omega_n=\pi|\omega_n$ and $\int_{\omega_n}up\,d\pi\to\int_{\omega}up\,d\pi$,

$$\sigma'(\omega)=\lim_{n\to\infty}\int_{\omega-\omega_n}up\,d\lambda_n+\int_{\omega}up\,d\pi\,.$$

Thus, it is enough to show that

$$\lim_{n\to\infty}\int_{\omega-\omega_n}up\,d\lambda_n=\int_{\omega}u\,d\sigma\,. \tag{2.3}$$

Consider any $y\in\omega$ and fix it for a while. Choose m such that $y\in\omega_m$. Let $\gamma=\sup_{x\in\omega-\omega_m}G_{\omega}(x,y)$ and $p_y(x)=\min(G_{\omega}(x,y),\gamma)$. As in the proof of Lemma 1.4, we see that $\gamma<+\infty$. It follows that $p_y+\gamma U^{\pi^-}_{\omega}$ is a potential whose associated measure belongs to $\mathscr{M}^+_B(\omega)$. Hence, by Lemma 2.9, $up_y=U^{\tau_y}_{\omega}$ for some $\tau_y\in\mathscr{M}_B(\omega)$. By the same argument as above, we have

$$\begin{aligned}\tau_y(\omega)&=\lim_{n\to\infty}\int_{\omega-\omega_n}up_y\,d\lambda_n+\int_{\omega}up_y\,d\pi\\&=\lim_{n\to\infty}\int_{\omega-\omega_n}uG_{\omega}(\cdot,y)d\lambda_n+\int_{\omega}up_y\,d\pi\,.\end{aligned}\tag{2.4}$$

On the other hand, letting $\omega_0=\omega$ and using the notation introduced above, we obtain from Lemma 2.12 the equality

$$\tau_y(\omega)=\lim_{\alpha\to 0}\left\{\frac{1}{\alpha}\int_{\omega}up_y\,d\mu_{\alpha,y}-\int_{\omega}up_y\,d\nu_{\alpha,y}\right\}+\int_{\omega}up_y\,d\pi\,.$$

Now, if $0<\alpha\leqq\gamma$, then $p_y=\alpha$ on $\partial\omega_{\alpha,y}$ $(=\bar{\omega}_{\alpha,y}-\omega_{\alpha,y})$. Since $S(\mu_{\alpha,y})\subset\partial\omega_{\alpha,y}$ and $S(\nu_{\alpha,y})\subset\partial\omega_{\alpha,y}$ when we regard $\mu_{\alpha,y}$ and $\nu_{\alpha,y}$ as measures on ω, we have

$$\frac{1}{\alpha}\int_{\omega} u p_y \, d\mu_{\alpha,y} = \int_{\omega} u \, d\mu_{\alpha,y} = u(y)$$

and

$$\int_{\omega} u p_y \, d\nu_{\alpha,y} = \alpha \int_{\omega} u \, d\nu_{\alpha,y} = \alpha \int_{\omega_{\alpha,y}} u \, d\pi \to 0 \ (\alpha \to 0),$$

where the last convergence follows from Lemma 2.10. Hence

$$\tau_y(\omega) = u(y) + \int_{\omega} u p_y \, d\pi,$$

so that, by (2.4), we have

$$\lim_{n\to\infty}\int_{\omega-\omega_n} uG_{\omega}(\cdot, y) d\lambda_n = u(y).$$

Since this is valid for any $y \in \omega$, integrating both sides by σ and using Lebesgue's convergence theorem as well as Fubini's theorem, we obtain (2.3).

§ 3. Perturbation theory

The theory in this section may be regarded as a special case of the perturbation theory developed by B. Walsh [12]. Since our formulation is slightly different from his, we shall give some of the details.

3.1. The operator G_{ω}

For an open set ω, let

$\mathbf{B}(\omega)$ = the linear space of all bounded Borel measurable functions on ω,

$\mathbf{C}_b(\omega) = \{f \in \mathbf{B}(\omega); f \text{ is continuous on } \omega\}$

and for a relatively compact open set ω, let

$\mathbf{C}(\bar{\omega})$ = the linear space of all continuous functions on $\bar{\omega}$,

$\mathbf{C}_0(\bar{\omega}) = \{f \in \mathbf{C}(\bar{\omega}); f = 0 \text{ on } \partial\omega\}$.

The space $\mathbf{B}(\omega)$ is a Banach space with respect to the sup-norm: $\|f\|_{\omega} = \sup_{\omega}|f|$; $\mathbf{C}_b(\omega)$ is a closed subspace of $\mathbf{B}(\omega)$. In case ω is relatively compact, $\mathbf{C}(\bar{\omega})$ and $\mathbf{C}_0(\bar{\omega})$ can be regarded as closed subspaces of $\mathbf{B}(\omega)$ (or of $\mathbf{C}_b(\omega)$).

Given a PB-domain ω, we define an operator G_{ω} by

$$(G_{\omega}f)(x) = \int_{\omega} G_{\omega}(x, y) f(y) d\pi(y).$$

When π is replaced by π^+ (resp. π^-), the corresponding operator is denoted by G_{ω}^+ (resp. G_{ω}^-). These are bounded linear operators of $\mathbf{B}(\omega)$ into $\mathbf{C}_b(\omega)$ and their operator norms are evaluated as

$$\|G_\omega\| \leqq \|U_\omega^{|\pi|}\|_\omega, \ \|G_\omega^+\| \leqq \|U_\omega^{\pi^+}\|_\omega \text{ and } \|G_\omega^-\| \leqq \|U_\omega^{\pi^-}\|_\omega .$$

If ω is a regular PB-domain, then these operators map $\mathbf{B}(\omega)$ into $\mathbf{C}_0(\bar{\omega})$.

LEMMA 3.1. *Let ω be a PB-domain. If $f \in \mathbf{C}_b(\omega)$ and $f - G_\omega f \in \mathscr{H}(\omega)$, then for any regular domain ω' such that $\bar{\omega}' \subset \omega$,*

$$f = H_f^{\omega'} + G_{\omega'} f \qquad on \quad \omega' .$$

PROOF. $G_\omega f - G_{\omega'} f$ is continuous on $\bar{\omega}'$ and harmonic on ω'. Hence $v = f - G_{\omega'} f$ is continuous on $\bar{\omega}'$ and harmonic on ω'. Since $v = f$ on $\partial\omega'$, $v = H_f^{\omega'}$.

3.2. Perturbed sheaf $\mathfrak{H}^\sim$

For each open set ω in Ω, we define

$$\mathscr{H}^\sim(\omega) = \left\{ v \in \mathbf{C}(\omega) ; \begin{array}{l} \text{for each } x \in \omega, \text{ there is a regular} \\ \text{PB-domain } \omega' \text{ such that } x \in \omega', \ \bar{\omega}' \subset \omega \\ \text{and } v = H_v^{\omega'} + G_{\omega'} v \quad \text{on} \quad \omega' \end{array} \right\} .$$

PROPOSITION 3.1. *For each open set ω, $\mathscr{H}^\sim(\omega)$ is a linear subspace of $\mathbf{C}(\omega)$ and $\mathfrak{H}^\sim = \{\mathscr{H}^\sim(\omega)\}_{\omega:\text{open}}$ satisfies Axiom 1 of Brelot* [3].

This proposition is easily verified by the definition of $\mathscr{H}^\sim(\omega)$, Lemma 3.1 and Axiom 2 for $\mathfrak{H}$.

PROPOSITION 3.2. *$1 \in \mathscr{H}^\sim(\omega)$ for any open set ω.*

PROOF. If ω' is a PB-domain, then $1 = H_1^{\omega'} + G_{\omega'} 1$.

PROPOSITION 3.3. *Let ω be a PB-domain. If $v \in \mathscr{H}^\sim(\omega)$ and v is bounded, then $v - G_\omega v \in \mathscr{H}(\omega)$.*

PROOF. Let $u = v - G_\omega v$. For each $x \in \omega$, there is a regular domain ω' such that $x \in \omega'$, $\bar{\omega}' \subset \omega$ and $v = H_v^{\omega'} + G_{\omega'} v$ on ω'. Hence

$$u = H_v^{\omega'} + G_{\omega'} v - G_\omega v \qquad \text{on} \quad \omega' ,$$

so that $u | \omega' \in \mathscr{H}(\omega')$. Since x is arbitrary, $u \in \mathscr{H}(\omega)$.

LEMMA 3.2 (cf. [12, p. 342]). *Given $x \in \Omega$ and $\delta > 0$, there is a PB-domain ω containing x such that $\|U_\omega^{|\pi|}\|_\omega < \delta$.*

PROOF. Fix $x_0 \in \Omega$ and let ω_0 be a PB-domain containing x_0. If $|\pi||\omega_0 = 0$, then we may take $\omega = \omega_0$. Suppose $|\pi||\omega_0 \neq 0$. Then $p_0 \equiv U_{\omega_0}^{|\pi|}$ is positive continuous on ω_0. Let

$$0 < \varepsilon < \min\left\{1, \frac{\delta}{3p_0(x_0)}\right\}.$$

By continuity, there is a regular neighborhood ω' of x_0 such that $\bar{\omega}' \subset \omega_0$ and $|p_0(x) - p_0(x_0)| < \varepsilon p_0(x_0)$ for all $x \in \bar{\omega}'$. Since $u \equiv H_1^{\omega'}$ is positive continuous on $\bar{\omega}'$, there is a domain ω such that $x_0 \in \omega \subset \omega'$ and

$$\inf_{\omega} u \leqq \frac{1}{1+\varepsilon} \sup_{\omega} u .$$

Since $H_u^{\omega} = u$ on ω, we see that $\|1 - H_1^{\omega}\|_{\omega} < \varepsilon$. Then

$$H_{p_0}^{\omega} \geqq (1-\varepsilon)p_0(x_0)H_1^{\omega} \geqq (1-\varepsilon)^2 p_0(x_0) \qquad \text{on} \quad \omega .$$

Hence

$$U_{\omega}^{|\pi|} = p_0 - H_{p_0}^{\omega} \leqq (1+\varepsilon)p_0(x_0) - (1-\varepsilon)^2 p_0(x_0) \leqq 3\varepsilon p_0(x_0) < \delta \qquad \text{on} \quad \omega .$$

A PB-domain ω will be called a *small domain* if

$$\|U_{\omega}^{\pi^+}\|_{\omega} + \|U_{\omega}^{\pi^-}\|_{\omega} < 1.$$

By the above lemma, small domains form a base of open sets in Ω. If ω is a small domain, then $(I - G_{\omega}^{-})^{-1}$ exists as an operator of $\mathbf{C}_b(\omega)$ into itself and

$$\|G_{\omega}^{+}\| \cdot \|(I - G_{\omega}^{-})^{-1}\| \leqq \|U_{\omega}^{\pi^+}\|_{\omega}(1 - \|U_{\omega}^{\pi^-}\|_{\omega})^{-1} < 1 .$$

Therefore, [12, Lemma 3.2.1] asserts the following

PROPOSITION 3.4. *If ω is a small domain, then $(I - G_{\omega})^{-1}$ exists as an operator on $\mathbf{C}_b(\omega)$ and for any non-negative bounded continuous superharmonic function s on ω, $(I - G_{\omega})^{-1}s \geqq 0$.*

From this proposition and Lemma 3.1, the next proposition immediately follows:

PROPOSITION 3.5. *Let ω be a small domain. If $u \in \mathscr{H}(\omega)$ and u is bounded, then $(I - G_{\omega})^{-1}u \in \mathscr{H}^{\sim}(\omega)$.*

Let ω be a small regular domain. Then, for each $\phi \in \mathbf{C}(\partial\omega)$,

$$\tilde{H}_{\phi}^{\omega} \equiv (I - G_{\omega})^{-1} H_{\phi}^{\omega}$$

makes sense and it is continuous on $\bar{\omega}$ if extended by ϕ on $\partial\omega$. By Propositions

3.3, 3.4 and 3.5, we see that $\tilde{H}_\phi^\omega \in \mathscr{H}^\sim(\omega)$, $\phi \geqq 0$ implies $\tilde{H}_\phi^\omega \geqq 0$ and that if $v \in \mathbf{C}(\bar{\omega})$, $v = \phi$ on $\partial\omega$ and $v|\omega \in \mathscr{H}^\sim(\omega)$ then $v = \tilde{H}_\phi^\omega$. Thus we have

PROPOSITION 3.6 ([12, Proposition 3.2.2]). *Small regular domains are regular with respect to $\mathfrak{H}^\sim$, so that $\mathfrak{H}^\sim$ satisfies Axioms* 2 *of Brelot* [3].

REMARK 3.1. We know ([12, Proposition 3.2.2]) that $\mathfrak{H}^\sim$ has the Bauer convergence property in the sense of [5, § 1.1]. But it is not clear whether $\mathfrak{H}^\sim$ satisfies Axiom 3 of Brelot [3] even in our special case. In this connection, we note the following: in case $\pi \geqq 0$, i.e., 1 is superharmonic, any non-negative $\mathfrak{H}^\sim$-harmonic function is superharmonic; and hence $\mathfrak{H}^\sim$ is elliptic in the sense of [5, p. 66] by virtue of Axiom 3 for $\mathfrak{H}$.

3.3. $\mathfrak{H}^\sim$-superharmonic functions

We shall restrict $\mathfrak{H}^\sim$-superharmonic functions (superharmonic functions with respect to $\mathfrak{H}^\sim$) to continuous ones; namely, a $\mathfrak{H}^\sim$-superharmonic function on an open set ω is a continuous function s on ω such that for each small regular domain ω' with $\bar{\omega}' \subset \omega$, $s \geqq \tilde{H}_s^{\omega'}$ on ω'.

PROPOSITION 3.7 (cf. [12, Proposition 3.3.1]). *Let ω be an open set and f be a continuous function on ω. Then f is $\mathfrak{H}^\sim$-superharmonic on ω if and only if $f \in \mathscr{B}_{\mathrm{loc}}(\omega)$ and $\sigma_f \geqq f\pi$ on ω.*

PROOF. First suppose $f \in \mathscr{B}_{\mathrm{loc}}(\omega)$ and $\sigma_f \geqq f\pi$ on ω. Let ω' be any small regular domain such that $\bar{\omega}' \subset \omega$. Then

$$f = H_f^{\omega'} + U_{\omega'}^{\sigma_f} \geqq H_f^{\omega'} + G_{\omega'} f$$

on ω'. Put $v = (I - G_{\omega'})f - H_f^{\omega'}$. Then v is a non-negative bounded continuous function on ω' and $\sigma_v = \sigma_f - f\pi \geqq 0$. Therefore v is superharmonic. Hence, by Proposition 3.4, $(I - G_{\omega'})^{-1} v \geqq 0$, so that $f - \tilde{H}_f^{\omega'} \geqq 0$. Thus f is $\mathfrak{H}^\sim$-superharmonic on ω.

Conversely, suppose f is $\mathfrak{H}^\sim$-superharmonic on ω. Let $\varepsilon > 0$. Since f is continuous, for each $x \in \omega$ there is a PC-domain ω_x such that $x \in \omega_x \subset \bar{\omega}_x \subset \omega$ and $(0 \leqq) f - \tilde{H}_f^{\omega'} < \varepsilon$ on ω' for any small regular domain ω' with $\bar{\omega}' \subset \omega_x$. Consider the function

$$s = f - G_{\omega_x} f + \varepsilon G_{\omega_x}^{+} 1$$

on ω_x. For any small regular domain ω' with $\bar{\omega}' \subset \omega_x$, since

$$H_f^{\omega'} = \tilde{H}_f^{\omega'} - G_{\omega'} \tilde{H}_f^{\omega'} \leqq f - G_{\omega'} \tilde{H}_f^{\omega'},$$

we have

$$\begin{aligned} H_s^{\omega'} &= H_f^{\omega'} - G_{\omega_x} f + G_{\omega'} f + \varepsilon(G_{\omega_x}^+ 1 - G_{\omega'}^+ 1) \\ &\leqq s + G_{\omega'}(f - \tilde{H}_f^{\omega'}) - \varepsilon G_{\omega'}^+ 1 . \end{aligned}$$

Now,

$$G_{\omega'}(f - \tilde{H}_f^{\omega'}) \leqq G_{\omega'}^+(f - \tilde{H}_f^{\omega'}) \leqq \varepsilon G_{\omega'}^+ 1 .$$

Hence $H_s^{\omega'} \leqq s$. This means that s is superharmonic on ω_x, so that $f \in \mathscr{B}_{\mathrm{loc}}(\omega_x)$ and

$$\sigma_f - f\pi + \varepsilon\pi^+ \geqq 0$$

on ω_x. Since ω_x's cover ω, $f \in \mathscr{B}_{\mathrm{loc}}(\omega)$ and the above inequality holds on ω. Thus, ε being arbitrary, we conclude that $\sigma_f - f\pi \geqq 0$ on ω.

COROLLARY. *If $u \in \mathscr{H}^{\sim}(\omega)$, then $\sigma_{u^2} \leqq u^2\pi$ on ω.*

PROOF. Since $1 \in \mathscr{H}^{\sim}(\omega)$, we see easily that $-u^2$ is $\mathfrak{H}^{\sim}$-superharmonic on ω.

§ 4. Gradient measures of locally bounded functions

4.1. Gradient measures

Let ω be an open set in Ω. For f, $g \in \mathscr{B}_{\mathrm{loc}}(\omega)$, we define their *mutual gradient measure* on ω by

$$\delta_{[f,g]} = \frac{1}{2}\{f\sigma_g + g\sigma_f - \sigma_{fg} - fg\pi\}$$

and the *gradient measure* of $f \in \mathscr{B}_{\mathrm{loc}}(\omega)$ by

$$\delta_f \equiv \delta_{[f,f]} = \frac{1}{2}\{2f\sigma_f - \sigma_{f^2} - f^2\pi\} .$$

By virtue of Proposition 2.1, these are well-defined signed measures on ω. Note that if c denotes a constant, then

$$\delta_{[c,f]} = \frac{1}{2}\{c\sigma_f + f\sigma_c - \sigma_{cf} - cf\pi\} = \frac{1}{2}\{c\sigma_f + cf\pi - c\sigma_f - cf\pi\} = 0$$

for any $f \in \mathscr{B}_{\mathrm{loc}}(\omega)$, and hence $\delta_c = 0$ and $\delta_{c+f} = \delta_f$ for any $f \in \mathscr{B}_{\mathrm{loc}}(\omega)$.

REMARK 4.1. In case Ω is a Euclidean domain and $\mathfrak{H}$ is defined by solutions of $\Delta u = Pu$, the measure δ_f is nothing but $|\mathrm{grad} f|^2 dx$ provided that f is continuously differentiable. (Cf. the introduction of [9]-*I*.)

THEOREM 4.1. *Let ω_0 be an open set. For any $f \in \mathscr{B}_{loc}(\omega_0)$, δ_f is a non-negative measure on ω_0. In case ω_0 is a domain, $\delta_f = 0$ if and only if $f \equiv const.$ on ω_0.*

PROOF. Let ω be any small PC-domain such that $\bar{\omega} \subset \omega$. Then $f = u + U_\omega^{\sigma_f}$ on ω with $u \in \mathscr{H}(\omega)$. Since u is bounded and ω is a small domain, $v = (I - G_\omega)^{-1} u$ exists and belongs to $\mathscr{H}^\sim(\omega)$ by Proposition 3.5. Let $p = U_\omega^{\sigma_f} - G_\omega v$. Then $f = v + p$, so that

$$\delta_f = \delta_v + 2\delta_{[v,p]} + \delta_p . \tag{4.1}$$

Since $v = u + G_\omega v$, $\sigma_v = v\pi$. Hence

$$\delta_v = \frac{1}{2}\{2v^2\pi - \sigma_{v^2} - v^2\pi\} = \frac{1}{2}\{v^2\pi - \sigma_{v^2}\} .$$

By the corollary to Proposition 3.7, we see that $\delta_v \geqq 0$. Next we have

$$\begin{aligned} 2\delta_{[v,p]} &= v\sigma_p + p\sigma_v - \sigma_{vp} - vp\pi \\ &= (u + G_\omega v)\sigma_p + vp\pi - \sigma_{vp} - vp\pi \\ &= u\sigma_p + (G_\omega v)\sigma_p - \sigma_{up} - \sigma_{(G_\omega v)p}. \end{aligned} \tag{4.2}$$

Since ω is a PC-domain, $|\sigma_f|(\omega) < +\infty$ and $|\pi|(\omega) < +\infty$. From the boundedness of v it follows that $\sigma_{(G_\omega v)p} \in \mathscr{M}_B(\omega)$ and $\sigma_p \in \mathscr{M}_B(\omega)$. Moreover, by Lemma 2.8, $u \in \mathscr{H}_{BE}(\omega)$. Therefore, we can apply Propositions 2.3 and 2.6 and obtain

$$\begin{aligned} \sigma_{(G_\omega v)p}(\omega) &= \int_\omega (G_\omega v)p \, d\pi \\ &= \int_\omega vp \, d\pi - \int_\omega up \, d\pi \\ &= \int_\omega (G_\omega v) d\sigma_p - \int_\omega up \, d\pi \end{aligned}$$

and

$$\sigma_{up}(\omega) = \int_\omega u \, d\sigma_p + \int_\omega up \, d\pi .$$

Therefore (4.2) implies

$$\delta_{[v,p]}(\omega) = 0 . \tag{4.3}$$

Also, by Proposition 2.3, $\sigma_{p^2}(\omega) = \int_\omega p^2 d\pi$, so that

(4.4) $$\delta_p(\omega) = \int_\omega p\, d\sigma_p - \int_\omega p^2 d\pi\,.$$

Since $U_\omega^{\pi+} < 1$, using Theorem 1.2 we have

(4.5) $$\int_\omega p^2 d\pi \leqq \int_\omega p^2 d\pi^+ \leqq \|U_\omega^{\pi+}\|_\omega I_\omega(\sigma_p) \leqq \int_\omega p\, d\sigma_p\,.$$

Therefore, $\delta_p(\omega) \geqq 0$ by (4.4), and hence by (4.1),

(4.6) $$\delta_f(\omega) = \delta_v(\omega) + 2\delta_{[v,p]}(\omega) + \delta_p(\omega) \geqq 0\,.$$

Since this is true for any small PC-domain ω such that $\bar{\omega} \subset \omega_0$ and such domains form a base of open sets in ω_0, we conclude that $\delta_f \geqq 0$.

If $f \equiv c$ (const.), then $\delta_c = 0$ as remarked before. Conversely, suppose ω_0 is a domain, $f \in \mathscr{B}_{\mathrm{loc}}(\omega_0)$ and $\delta_f = 0$. Let ω be any small PC-domain such that $\bar{\omega} \subset \omega_0$ and use the same notation as above. Since $\delta_v \geqq 0$ and $\delta_p \geqq 0$ on ω as we have shown above, (4.3) and (4.6) imply that $\delta_v = 0$ and $\delta_p = 0$ on ω. It follows from (4.4) that inequalities in (4.5) become equalities, in particualr,

$$\|U_\omega^{\pi+}\|_\omega I_\omega(\sigma_p) = I_\omega(\sigma_p)\,.$$

Since $\|U_\omega^{\pi+}\|_\omega < 1$, we have $I_\omega(\sigma_p) = 0$; hence $p = 0$ on ω by the energy principle.

Next we shall show that $\delta_v = 0$ on ω implies $v \equiv$ const. on ω. Since $\delta_{v+\alpha g} \geqq 0$ on ω for any $g \in \mathscr{B}_{\mathrm{loc}}(\omega)$ and for any real number α, we see that $\delta_{[v,g]} = 0$ for any $g \in \mathscr{B}_{\mathrm{loc}}(\omega)$. In particular, if $h \in \mathscr{H}(\omega)$, then

$$0 = \delta_{[v,h]} = \frac{1}{2}\{h\sigma_v - \sigma_{vh} - vh\pi\} = -\frac{1}{2}\sigma_{vh}\,.$$

This means that $vh \in \mathscr{H}(\omega)$ for any $h \in \mathscr{H}(\omega)$, and hence $v^2 h \in \mathscr{H}(\omega)$ for any $h \in \mathscr{H}(\omega)$. Since ω is a PC-domain, there is $h_0 \in \mathscr{H}(\omega)$ which is positive on ω (see [3, p. 94]). Let $x_0 \in \omega$ be fixed and consider the function $w = (v - v(x_0))^2 h_0$. By the above observation, $w \in \mathscr{H}(\omega)$. Since $w \geqq 0$, $w(x_0) = 0$ and $h_0 > 0$, we conclude that $v \equiv v(x_0)$ on ω. Thus we have seen that $f \equiv$ const. on ω. Since ω_0 is connected, it follows that $f \equiv$ const. on ω_0.

COROLLARY. *Let ω_0 be any open set in Ω.*

(a) *If $f, g \in \mathscr{B}_{\mathrm{loc}}(\omega_0)$, then*

$$|\delta_{[f,g]}| \leqq \frac{1}{2}(\delta_f + \delta_g) \quad \textit{and} \quad \delta_{f+g} \leqq 2(\delta_f + \delta_g)\,.$$

(b) *If $f, g \in \mathscr{B}_{\mathrm{loc}}(\omega_0)$ and A is a relatively compact Borel set such that $\bar{A} \subset \omega_0$, then*

$$|\delta_{[f,g]}(A)| \leqq \delta_f(A)^{1/2}\,\delta_g(A)^{1/2}$$

and

$$\delta_{f+g}(A)^{1/2} \leqq \delta_f(A)^{1/2}+\delta_g(A)^{1/2}.$$

The value $\delta_f(A)$ may be called the *Dirichlet integral* of f over A (cf. Remark 4.1).

REMARK 4.2. If $u \in \mathscr{H}(\omega)$, then $\delta_u = -\frac{1}{2}(\sigma_{u^2}+u^2\pi)$. Hence if $u \in \mathscr{H}_{BE}(\omega)$ and $\pi^-(\omega)<+\infty$, then $\delta_u(\omega)<+\infty$.

4.2. Gradient measures of max. and min. of functions

LEMMA 4.1. *$\mathscr{B}_{\mathrm{loc}}(\omega_0)$ is a vector lattice with respect to the max. and min. operations for any open set ω_0.*

PROOF. Let $f \in \mathscr{B}_{\mathrm{loc}}(\omega_0)$ and let ω be any PC-domain such that $\bar{\omega} \subset \omega_0$. Then $f|\omega = s_1 - s_2$ with bounded non-negative superharmonic functions s_1 and s_2 on ω. Then

$$\max(f, 0) = s_1 - \min(s_1, s_2)$$

and $\min(s_1, s_2)$ is bounded non-negative superharmonic on ω. Hence $\max(f, 0) \in \mathscr{B}_{\mathrm{loc}}(\omega_0)$. Since $\mathscr{B}_{\mathrm{loc}}(\omega_0)$ is a linear space, it follows that it is a vector lattice with respect to the max. and min. operations.

LEMMA 4.2. *If $f \in \mathscr{B}_{\mathrm{loc}}(\omega_0)$ and f is continuous on ω_0, then*

$$\delta_{[\max(f,0),\min(f,0)]} = 0.$$

PROOF. Let $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$. Since $f^+f^- = 0$,

$$\delta_{[f^+,f^-]} = \frac{1}{2}\{f^+\sigma_{f^-} + f^-\sigma_{f^+}\}.$$

Let $\omega^+ = \{x \in \omega; f(x)>0\}$ and $\omega^- = \{x \in \omega; f(x)<0\}$. Then ω^+, ω^- are open sets. Hence we see that $\sigma_{f^-}|\omega^+ = 0$ and $\sigma_{f^+}|\omega^- = 0$. Therefore $\delta_{[f^+,f^-]} = 0$.

COROLLARY. *For a continuous $f \in \mathscr{B}_{\mathrm{loc}}(\omega_0)$, $\delta_{|f|} = \delta_f$.*

REMARK 4.3. We shall see later (§ 7) that the above results hold for any $f \in \mathscr{B}_{\mathrm{loc}}(\omega_0)$.

4.3. Dirichlet integrals of locally bounded potentials on a PB-domain

LEMMA 4.3. *Let ω be a PB-domain and let $p=U_\omega^\sigma$ with $\sigma\in\mathscr{M}_E(\omega)$. Suppose $U_\omega^{|\sigma|}$ is locally bounded on ω. Then p is $|\pi|$-square-integrable on ω,*

$$\delta_p(\omega) \leqq \beta_\omega I_\omega(\sigma)$$

and

$$\delta_p(\omega) = I_\omega(\sigma)-\int_\omega p^2 d\pi\,.$$

PROOF. Theorem 1.2 implies that p is $|\pi|$-square-integrable. First, suppose $\sigma\geqq 0$. Let $\{\omega_n\}$ be an exhaustion of ω. For each n, $p_n\equiv R_p^{\omega_n,\omega}$ is a potential on ω, $S_h(p_n)\subset\overline{\omega}_n$ and $p_n=p$ on ω_n by virtue of Lemma 1.7. Since p is bounded on $\overline{\omega}_n$, Lemma 1.4 implies that each p_n is bounded. Hence $\mu_n\equiv\sigma_{p_n}\in\mathscr{M}_B^+(\omega)$. Since $p_n\uparrow p$, we have $I_\omega(\mu_n)\uparrow I_\omega(\sigma)$ and $I_\omega(\mu_n-\sigma)\to 0$ (Corollary 2 to Theorem 1.1). By Proposition 2.2 (cf. (4.4) in the proof of Theorem 4.1), we see that

$$\delta_{p_n}(\omega) = I_\omega(\mu_n)-\int_\omega p_n^2 d\pi\,. \tag{4.7}$$

By Theorem 2.1, $\int_\omega p^2 d\pi^-\leqq(\beta_\omega-1)I_\omega(\sigma)$. Hence

$$\delta_{p_n}(\omega) \leqq I_\omega(\mu_n)+\int_\omega p_n^2 d\pi^- \leqq I_\omega(\sigma)+\int_\omega p^2 d\pi^- \leqq \beta_\omega I_\omega(\sigma)\,.$$

Since $p_n=p$ on ω_n, $\delta_p(\omega_n)=\delta_{p_n}(\omega_n)\leqq\delta_{p_n}(\omega)\leqq\beta_\omega I_\omega(\sigma)$, which implies that $\delta_p(\omega)\leqq\beta_\omega I_\omega(\sigma)$.

Similarly, we see that $\delta_{p_n-p_m}(\omega)\leqq\beta_\omega I_\omega(\mu_n-\mu_m)$, and hence

$$\delta_{p_n-p}(\omega_m) = \delta_{p_n-p_m}(\omega_m) \leqq \beta_\omega I_\omega(\mu_n-\mu_m)\,.$$

Therefore

$$\delta_{p_n-p}(\omega) \leqq \beta_\omega I_\omega(\mu_n-\sigma) \to 0 \qquad (n\to\infty)\,.$$

It follows that $\delta_{p_n}(\omega)\to\delta_p(\omega)$. Since $I_\omega(\mu_n)\to I_\omega(\sigma)$ and $\int_\omega p_n^2\,d\pi\to\int_\omega p^2\,d\pi$, (4.7) implies that

$$\delta_p(\omega) = I_\omega(\sigma)-\int_\omega p^2 d\pi\,.$$

Next, let σ be arbitrary. Applying the above result to $f_1=U_\omega^{\sigma^+}$, $f_2=U_\omega^{\sigma^-}$ and $f_3=U_\omega^{|\sigma|}$, we see that

$$\begin{aligned}\delta_p(\omega) &= 2\delta_{f_1}(\omega)+2\delta_{f_2}(\omega)-\delta_{f_3}(\omega)\\ &= 2I_\omega(\sigma^+)+2I_\omega(\sigma^-)-I_\omega(|\sigma|)-\int_\omega(2f_1^2+2f_2^2-f_3^2)d\pi\\ &= I_\omega(\sigma)-\int_\omega p^2 d\pi.\end{aligned}$$

Finally, applying Theorem 1.2 again, we see that $\delta_p(\omega)\leqq\beta_\omega I_\omega(\sigma)$ in the same way as above.

LEMMA 4.4. *Let ω be a PB-domain and $p=U_\omega^\sigma$ with $\sigma\in\mathscr{M}_E(\omega)$. Let $\{\omega_n\}$ be an exhaustion of ω and let $p_n=U_{\omega_n}^\sigma$. Suppose $U_\omega^{|\sigma|}$ is locally bounded on ω. Then*

$$\delta_{p-p_n}(\omega_n)+\int_{\omega_n}(p-p_n)^2 d|\pi| \to 0 \qquad (n\to\infty).$$

PROOF. We may assume that $\sigma\geqq 0$. Since $\int_\omega p^2 d|\pi|<+\infty$, $0\leqq p_n\leqq p$ on ω_n and $p_n\to p$, Lebesgue's convergence theorem implies that $\int_{\omega_n}(p-p_n)^2 d|\pi|\to 0\ (n\to\infty)$. Thus it remains to show that $\delta_{p-p_n}(\omega_n)\to 0\ (n\to\infty)$. First we remark that $u_n\equiv p-p_n$ belongs to $\mathscr{H}_{BE}(\omega_n)$ by virtue of Lemma 2.8. Since $\sigma|\omega_n\in\mathscr{M}_B^+(\omega_n)$ and $\pi^-(\omega_n)<+\infty$, the definition of $\delta_{[f,g]}$ and Proposition 2.4 yield

$$\begin{aligned}\delta_{[p-p_n,p_n]}(\omega_n) &= \delta_{[u_n,p_n]}(\omega_n)\\ &= \frac{1}{2}\left\{\int_{\omega_n}u_n d\sigma-\sigma_{u_np_n}(\omega_n)-\int_{\omega_n}u_np_n d\pi\right\}\\ &= -\int_{\omega_n}u_np_n d\pi\\ &= -\int_{\omega_n}(p-p_n)p_n d\pi.\end{aligned}$$

On the other hand, by the above lemma,

$$\delta_{p_n}(\omega_n) = I_{\omega_n}(\sigma)-\int_{\omega_n}p_n^2 d\pi$$

and

$$\delta_p(\omega) = I_\omega(\sigma)-\int_\omega p^2 d\pi.$$

Therefore

$$\begin{aligned}\delta_{p-p_n}(\omega_n) &= \delta_p(\omega_n)-\delta_{p_n}(\omega_n)-2\delta_{[p-p_n,p_n]}(\omega_n)\\ &\leqq \delta_p(\omega)-I_{\omega_n}(\sigma)+\int_{\omega_n}p_n^2 d\pi+2\int_{\omega_n}(p-p_n)p_n d\pi\\ &= I_\omega(\sigma)-I_{\omega_n}(\sigma)-\int_{\omega_n}(p-p_n)^2 d\pi-\int_{\omega-\omega_n}p^2 d\pi\\ &\to 0 \quad (n\to\infty).\end{aligned}$$

LEMMA 4.5. *Let ω be a PB-domain, $p=U_\omega^\sigma$ with $\sigma\in\mathscr{M}_E(\omega)$ and $u\in\mathscr{H}(\omega)$ with $\delta_u(\omega)+\int_\omega u^2 d|\pi|<+\infty$. Suppose $U_\omega^{|\sigma|}$ is locally bounded on ω. Then*

$$\delta_{[u,p]}(\omega) = -\int_\omega up\,d\pi.$$

PROOF. By the corollary to Theorem 4.1, we see that $\delta_{[u,p]}(\omega)$ has a definite finite value. Obviously, $\int_\omega up\,d\pi$ is also definite. Let $\{\omega_n\}$ be an exhaustion of ω and let $p_n=U_{\omega_n}^\sigma$. By Proposition 2.4 (cf. the proof of the previous lemma),

$$\delta_{[u,p_n]}(\omega_n) = -\int_{\omega_n}up_n d\pi.$$

By Lebesgue's convergence theorem,

$$\int_{\omega_n}up_n d\pi \to \int_\omega up\,d\pi \qquad (n\to\infty).$$

On the other hand, by the corollary to Theorem 4.1, we have

$$\begin{aligned}&|\delta_{[u,p_n]}(\omega_n)-\delta_{[u,p]}(\omega)|\\ &\leqq |\delta_{[u,p-p_n]}(\omega_n)|+|\delta_{[u,p]}(\omega-\omega_n)|\\ &\leqq \delta_u(\omega)^{1/2}\,\delta_{p-p_n}(\omega_n)^{1/2}+\delta_u(\omega-\omega_n)^{1/2}\,\delta_p(\omega-\omega_n)^{1/2}\\ &\to 0 \quad (n\to\infty),\end{aligned}$$

where we used the previous lemma to conclude the convergence.

§ 5. The spaces of harmonic functions with finite Dirichlet integral and with finite energy

5.1. Lattice structures

Given an open set ω, we consider the following spaces of harmonic functions:

$$\mathscr{H}_D(\omega) = \{u \in \mathscr{H}(\omega); \delta_u(\omega) < +\infty\},$$

$$\mathscr{H}_{D'}(\omega) = \{u \in \mathscr{H}(\omega); \delta_u(\omega) + \int_\omega u^2 d\pi^- < +\infty\},$$

$$\mathscr{H}_E(\omega) = \{u \in \mathscr{H}(\omega); \delta_u(\omega) + \int_\omega u^2 d|\pi| < +\infty\}.$$

Since $(u+v)^2+(u-v)^2=2(u^2+v^2)$ and $\delta_{u+v}+\delta_{u-v}=2(\delta_u+\delta_v)$, we see that these are linear subspaces of $\mathscr{H}(\omega)$. Note that if 1 is superharmonic on ω, then $\mathscr{H}_{D'}(\omega)=\mathscr{H}_D(\omega)$. Let

$$\|u\|_{D,\omega} = \delta_u(\omega)^{1/2},$$

$$\|u\|_{D',\omega} = \{\delta_u(\omega) + \int_\omega u^2 d\pi^-\}^{1/2},$$

$$\|u\|_{E,\omega} = \{\delta_u(\omega) + \int_\omega u^2 d|\pi|\}^{1/2}.$$

These are semi-norms on $\mathscr{H}_D(\omega)$, $\mathscr{H}_{D'}(\omega)$ and $\mathscr{H}_E(\omega)$, respectively, They are norms if and only if $|\pi||\omega' \neq 0$ for every component ω' of ω.

LEMMA 5.1. *Let ω be a PB-domain. Then*

$$I_\omega(\sigma_{|u|}) \leqq 2(\beta_\omega - 1)\|u\|^2_{D',\omega}$$

for any $u \in \mathscr{H}_{D'}(\omega)$.

PROOF. For any PC-domain ω' such that $\bar{\omega}' \subset \omega$, $u|\omega' \in \mathscr{H}_{BE}(\omega')$. Hence, by Proposition 2.3, the least harmonic majorant v of $|u|$ on ω' exists. Let $p = -U_{\omega'}^{\sigma|u|}$. Then $p \geqq 0$ and $|u| = v - p$ on ω'. By Lemma 4.5,

$$\delta_{[v,p]}(\omega') + \int_{\omega'} vp\, d\pi = 0.$$

Hence, using Lemma 4.3, we deduce

$$\begin{aligned} I_{\omega'}(\sigma_{|u|}) &= \delta_p(\omega') + \int_{\omega'} p^2 d\pi \\ &= -\delta_{[|u|,p]}(\omega') - \int_{\omega'} |u| p\, d\pi \\ &\leqq -\delta_{[|u|,p]}(\omega') + \int_{\omega'} |u| p\, d\pi^- \\ &\leqq \delta_{|u|}(\omega')^{1/2}\delta_p(\omega')^{1/2} + \left(\int_{\omega'} u^2 d\pi^-\right)^{1/2}\left(\int_{\omega'} p^2 d\pi^-\right)^{1/2}. \end{aligned}$$

By the corollary to Lemma 4.1, $\delta_{|u|}=\delta_u$. By Lemma 4.3,

$$\delta_p(\omega') \leqq \beta_{\omega'} I_{\omega'}(\sigma_{|u|}) \leqq \beta_\omega I_{\omega'}(\sigma_{|u|}).$$

By Theorem 1.2,

$$\int_{\omega'} p^2 d\pi^- \leqq (\beta_{\omega'}-1) I_{\omega'}(\sigma_{|u|}) \leqq (\beta_\omega - 1) I_{\omega'}(\sigma_{|u|}).$$

Hence,

$$I_{\omega'}(\sigma_{|u|}) \leqq \left[\{\beta_\omega \delta_{|u|}(\omega')\}^{1/2} + \left\{(\beta_\omega - 1)\int_{\omega'} u^2 d\pi^- \right\}^{1/2} \right] I_{\omega'}(\sigma_{|u|})^{1/2},$$

so that

$$I_{\omega'}(\sigma_{|u|}) \leqq (2\beta_\omega - 1)\|u\|^2_{D',\omega'}.$$

Letting $\omega' \uparrow \omega$, we obtain the required inequality.

Given $u, v \in \mathscr{H}(\omega)$, if $\max(u, v)$ (resp. $\min(u, v)$) has a harmonic majorant (resp. harmonic minorant) on ω, then its least harmonic majorant (resp. its greatest harmonic minorant) will be denoted by $u \vee_\omega v$ (resp. $u \wedge_\omega v$).

THEOREM 5.1. (cf. [9, Lemma 3.3 and Thoerem 3.1]). *If ω is a PB-domain, then $\mathscr{H}_{D'}(\omega)$ and $\mathscr{H}_E(\omega)$ are vector lattices with respect to the operations $\vee_\omega$ and $\wedge_\omega$. Furthermore, we have the following estimates:*

$$\|u \vee_\omega (-u)\|_{D',\omega} \leqq \{1 + 3(\beta_\omega - 1)\}\|u\|_{D',\omega} \qquad \text{for} \quad u \in \mathscr{H}_{D'}(\omega)$$

and

$$\|u \vee_\omega (-u)\|_{E,\omega} \leqq \{1 + 3(\beta_\omega - 1)\}\|u\|_{E,\omega} \qquad \text{for} \quad u \in \mathscr{H}_E(\omega).$$

PROOF. Let $u \in \mathscr{H}_{D'}(\omega)$ and $\nu = -\sigma_{|u|}\,(\geqq 0)$. By the above lemma, we see that $p = U^\nu_\omega$ is a potential, and hence $v = u \vee_\omega (-u)$ exists; in fact $v = |u| + p$. Since $I_\omega(\nu) < +\infty$ by the above lemma, it follows from Theorem 1.2 and Lemma 4.3 that

$$\delta_p(\omega) + \int_\omega p^2 d|\pi| < +\infty.$$

Therefore $v \in \mathscr{H}_{D'}(\omega)$, and if in particular $u \in \mathscr{H}_E(\omega)$ then $v \in \mathscr{H}_E(\omega)$. Thus, $\mathscr{H}_{D'}(\omega)$ and $\mathscr{H}_E(\omega)$ are vector lattices with respect to $\vee_\omega$ and $\wedge_\omega$.

Now, let $\{\omega_n\}$ be an exhaustion of ω, $p_n = U^\nu_{\omega_n}$ and $u_n = p|\omega_n - p_n$. Then $u_n \in \mathscr{H}_{BE}(\omega_n)$ ($\subset \mathscr{H}_E(\omega_n)$; cf. Remark 4.2), $v = |u| + u_n + p_n$ and $v - u_n \geqq |u|$ on ω_n. By Lemmas 4.3 and 4.5 and the corollary to Lemma 4.2, we deduce

$$\delta_{v-u_n}(\omega_n)+\int_{\omega_n}(v-u_n)^2 d\pi = \delta_u(\omega_n)+\int_{\omega_n}u^2 d\pi - I_{\omega_n}(v).$$

Hence,

$$\begin{aligned}
&\delta_{v-u_n}(\omega_n)+\int_{\omega_n}(v-u_n)^2 d\pi^- \\
&\quad =\delta_u(\omega_n)+\int_{\omega_n}u^2 d\pi^- +\int_{\omega_n}\{u^2-(v-u_n)^2\}d\pi^+ \\
&\qquad +2\int_{\omega_n}\{(v-u_n)^2-u^2\}d\pi^- - I_{\omega_n}(v) \\
&\quad \leqq \delta_u(\omega)+\int_{\omega}u^2 d\pi^- + 2\int_{\omega_n}\{(v-u_n)^2-u^2\}d\pi^- - I_{\omega_n}(v)
\end{aligned}$$

and

$$\begin{aligned}
&\delta_{v-u_n}(\omega_n)+\int_{\omega_n}(v-u_n)^2 d|\pi| \\
&\quad = \delta_u(\omega_n)+\int_{\omega_n}u^2 d|\pi| + 2\int_{\omega_n}\{(v-u_n)^2-u^2\}d\pi^- - I_{\omega_n}(v) \\
&\quad \leqq \delta_u(\omega)+\int_{\omega}u^2 d|\pi| + 2\int_{\omega_n}\{(v-u_n)^2-u^2\}d\pi^- - I_{\omega_n}(v).
\end{aligned}$$

By Lemma 4.4, $\delta_{u_n}(\omega_n)\to 0$ and $\int_{\omega_n}u_n^2 d|\pi|\to 0$ $(n\to\infty)$. Hence

$$\|v\|^2 \leqq \|u\|^2 + 2\int_{\omega}(v^2-u^2)d\pi^- - I_{\omega}(v), \tag{5.1}$$

where $\|u\|=\|u\|_{D',\omega}$ if $u\in\mathscr{H}_{D'}(\omega)$, $=\|u\|_{E,\omega}$ if $u\in\mathscr{H}_E(\omega)$. If $\pi^-=0$, then (5.1) immediately implies the required estimates. If $\pi^-\neq 0$, then $\beta_\omega>1$. Since $v^2-u^2\leqq ku^2+(1+k^{-1})p^2$ for any $k>0$,

$$\begin{aligned}
2\int_{\omega}(v^2-u^2)d\pi^- &\leqq 2k\int_{\omega}u^2 d\pi^- + 2\left(1+\frac{1}{k}\right)\int_{\omega}p^2 d\pi^- \\
&\leqq 2k\|u\|^2 + 2\left(1+\frac{1}{k}\right)(\beta_\omega-1)I_\omega(v).
\end{aligned}$$

Letting $k=2(\beta_\omega-1)$ and using Lemma 5.1, we have from (5.1)

$$\begin{aligned}
\|v\|^2 &\leqq \{1+4(\beta_\omega-1)+2(\beta_\omega-1)(2\beta_\omega-1)\}\|u\|^2 \\
&\leqq \{1+3(\beta_\omega-1)\}^2\|u\|^2.
\end{aligned}$$

COROLLARY (cf. [11, Theorem 2] and [6, Theorem 10 D]). *If* 1 *is super-*

harmonic on a domain ω, *then* $\mathscr{H}_D(\omega)$ *is a vector lattice with respect to* $\vee_\omega$ *and* $\wedge_\omega$ *and*

$$\|u \vee_\omega (-u)\|_{D,\omega} \leqq \|u\|_{D,\omega}.$$

REMARK 5.1. We do not know whether this corollary remains valid in case 1 is not superharmonic.

5.2. Bounded families in $\mathscr{H}_{D'}(\omega)$ and $\mathscr{H}_E(\omega)$

THEOREM 5.2. *If* ω *is a PB-domain such that* $|\pi||\omega \neq 0$, *then the family*

$$\mathscr{H}^1_{D'}(\omega) \equiv \{u \in \mathscr{H}_{D'}(\omega);\ \|u\|_{D',\omega} \leqq 1\}$$

is locally uniformly bounded on ω.

PROOF. First suppose $\pi^-|\omega \neq 0$. Consider the family

$$\mathscr{U} = \{u \in \mathscr{H}_{D'}(\omega);\ u \geqq 0,\ \|u\|_{D',\omega} \leqq 1+3(\beta_\omega - 1)\}.$$

If $u \in \mathscr{H}^1_{D'}(\omega)$, then $|u| \leqq u \vee_\omega (-u)$ and $\|u \vee_\omega (-u)\|_{D',\omega} \leqq 1+3(\beta_\omega - 1)$ by the previous theorem. Hence it is enough to show that $\mathscr{U}$ is locally uniformly bounded. Fix $x_0 \in \omega$. We shall show that $\{u(x_0); u \in \mathscr{U}\}$ is bounded. Supposing the contrary, we would find $u_n \in \mathscr{U}$, $n=1, 2,\dots$, such that $u_n(x_0) \geqq n$. Let $v_n = u_n/u_n(x_0)$. Then, Harnack's principle (cf. [9, § 3.3, (B)]) implies that there is a subsequence $\{v_{n_j}\}$ converging to a $v \in \mathscr{H}(\omega)$ locally uniformly on ω. In particular, $v(x_0)=1$ and $v>0$ on ω. Now,

$$\begin{aligned}\int_\omega v_n^2 d\pi^- &= \frac{1}{u_n(x_0)^2}\int_\omega u_n^2 d\pi^- \\ &\leqq \frac{1}{n^2}\|u_n\|^2_{D',\omega} \\ &\leqq \frac{1}{n^2}\{1+3(\beta_\omega - 1)\} \to 0 \quad (n\to\infty).\end{aligned}$$

Therefore, we may assume that $v_{n_j} \to 0$ π^--a.e. on ω. It follows that $v=0$ π^--a.e. on ω, which is a contradiction. Thus we have seen that $\{u(x_0); u \in \mathscr{U}\}$ is bounded. Then, by Harnack's inequality (cf. [9, § 3.3, (A)]), we conclude that $\mathscr{U}$ is locally uniformly bounded on ω.

Next, suppose $\pi^-|\omega = 0$, i.e., $\pi \geqq 0$ on ω. Let ω' be any PC-domain such that $\bar\omega' \subset \omega$ and $\pi|\omega' \neq 0$. Choose another PC-domain ω^* such that $\bar\omega' \subset \omega^*$ and $\bar\omega^* \subset \omega$. Let $\alpha = \inf_{\omega'} U^\pi_{\omega^*}$. By our assumption, $\alpha > 0$. Given $u \in \mathscr{H}(\omega)$, let $\mu = \sigma_{-u^2}$ $(\geqq 0)$. Then $u^2 = h - U^\mu_{\omega^*}$ on ω^* with $h \in \mathscr{H}_{BE}(\omega^*)$ (cf. [9, Lemma

2.12]). In the proof of [9, Proposition 2.2], we showed that

$$\mu(\omega^*) \geqq \int_{\omega^*} h\, d\pi \geqq \int_{\omega_*} u^2 d\pi .$$

Hence

$$\begin{aligned}
\|u\|_{D,\omega^*}^2 &= \delta_u(\omega^*) \\
&= \frac{1}{2}\left\{\mu(\omega^*) - \int_{\omega^*} u^2 d\pi\right\} \\
&\geqq \frac{1}{2}\int_{\omega^*}(h-u^2)d\pi \\
&= \frac{1}{2}\int_{\omega^*} U_{\omega^*}^{\mu} d\pi \\
&= \frac{1}{2}\int_{\omega_*} U_{\omega_*}^{\pi} d\mu \geqq \frac{\alpha}{2}\,\mu(\omega'),
\end{aligned}$$

so that

$$\begin{aligned}
\|u\|_{E,\omega'}^2 &= \frac{1}{2}\left\{\mu(\omega') + \int_{\omega'} u^2 d\pi\right\} \\
&= \mu(\omega') - \delta_u(\omega') \leqq \mu(\omega') \leqq \frac{2}{\alpha}\|u\|_{D,\omega_*}^2 .
\end{aligned}$$

Hence,

$$\{u\,|\,\omega';\ u \in \mathscr{H}_{D'}^1(\omega)\} \subset \left\{v \in \mathscr{H}_E(\omega');\ \|v\|_{E,\omega'} \leqq \left(\frac{2}{\alpha}\right)^{1/2}\right\}.$$

The family on the right is locally uniformly bounded by virtue of [9, Thoerem 3.2], and hence $\mathscr{H}_{D'}^1(\omega)$ is locally uniformly bounded on ω'. Since ω' can be chosen arbitrarily close to ω, we obtain the theorem.

COROLLARY 1 (cf. [9, Theorem 3.2]). *If ω is a PB-domain such that $|\pi||\omega \neq 0$, then the family*

$$\mathscr{H}_E^1(\omega) = \{u \in \mathscr{H}_E(\omega);\ \|u\|_{E,\omega} \leqq 1\}$$

is locally uniformly bounded on ω.

COROLLARY 2. *If ω is a PB-domain and 1 is superharmonic on ω, but not harmonic on ω, then the family*

$$\mathscr{H}_D^1(\omega) = \{u \in \mathscr{H}_D(\omega);\ \|u\|_{D,\omega} \leqq 1\}$$

is locally uniformly bounded on ω.

COROLLARY 3 (cf. [9, Corollary to Theorem 3.2]). *Let* ω *be a PB-domain such that* $|\pi||\omega\neq 0$. *If* $u_n\in\mathscr{H}_{D'}(\omega)$ *and* $\|u_n\|_{D',\omega}\to 0$ *(in particular,* $u_n\in\mathscr{H}_E(\omega)$ *and* $\|u_n\|_{E,\omega}\to 0$*), then* $u_n\to 0$ *and* $u_n\vee_\omega(-u_n)\to 0$ *both locally uniformly on* ω.

REMARK 5.2. In Theorem 5.2 and its three corollaries given above, the condition that $|\pi||\omega\neq 0$ cannot be omitted; though we obtain the same assertions if we normalize functions (see [9, § 3.1 and § 3.3]).

COROLLARY 4. *Let* ω *be a PB-domain and let* ω' *be a PC-domain such that* $\bar{\omega}'\subset\omega$. *Then there is a constant* $M>0$ *such that*

$$\|u\|_{E,\omega'}\leqq M\|u\|_{D',\omega}$$

for all $u\in\mathscr{H}_{D'}(\omega)$.

PROOF. If $|\pi||\omega=0$, then $\|u\|_{E,\omega'}=\|u\|_{D',\omega'}\leqq\|u\|_{D',\omega}$. Suppose $|\pi||\omega\neq 0$. Then, by the theorem, $|u|\leqq M'$ on ω' for all $u\in\mathscr{H}^1_{D'}(\omega)$ for some $M'>0$. Hence

$$\int_{\omega'}u^2d\pi^+\leqq M'^2\|u\|^2_{D',\omega}\pi^+(\omega'),$$

so that

$$\|u\|^2_{E,\omega'}=\|u\|^2_{D',\omega'}+\int_{\omega'}u^2d\pi^+\leqq\{1+M'^2\pi^+(\omega')\}\|u\|^2_{D',\omega}.$$

For a PB-domain ω and $u\in\mathscr{H}_E(\omega)$, $U^{\delta_u}_\omega$ and $U^{u^2|\pi|}_\omega$ are potentials on ω by virtue of Lemma 1.6. Since $\sigma_{u^2}=-2\delta_u-u^2\pi$,

$$h^\omega_u\equiv u^2+2U^{\delta_u}_\omega+U^{u^2\pi}_\omega\in\mathscr{H}(\omega).$$

Since $u^2\geqq 0$, it follows that $h^\omega_u\geqq 0$.

LEMMA 5.2 (cf. [9, Lemma 3.5]). *If* ω *is a PB-domain such that* $|\pi||\omega\neq 0$, *then the family* $\{h^\omega_u;u\in\mathscr{H}^1_E(\omega)\}$ *is locally uniformly bounded on* ω.

PROOF. Let K be any compact set in ω such that $|\pi|(K)>0$. By the above Corollary 1, there is $M>0$ such that $|u(x)|\leqq M$ for all $u\in\mathscr{H}^1_E(\omega)$ and $x\in K$. Since $h^\omega_u\geqq 0$, Harnack's inequality implies

$$\begin{aligned}\sup_{x\in K}h^\omega_u(x)&\leqq\alpha\inf_{x\in K}h^\omega_u(x)\\&\leqq\alpha\{M^2+\inf_K(2U^{\delta_u}_\omega+U^{u^2\pi^+}_\omega)\}\end{aligned}$$

for some $\alpha>0$ which is independent of u. Now,

$$\begin{aligned}
&\inf_K (2U_\omega^{\delta_u}+U_\omega^{u^2\pi^+}) \\
&\leqq \frac{1}{|\pi|(K)}\int_\omega (2U_\omega^{\delta_u}+U_\omega^{u^2\pi^+})d|\pi| \\
&= \frac{1}{|\pi|(K)}\int_\omega U_\omega^{|\pi|}d(2\delta_u+u^2\pi^+) \\
&\leqq \frac{2\beta_\omega-1}{|\pi|(K)}\left(2\delta_u(\omega)+\int_\omega u^2 d\pi^+\right) \leqq \frac{2(2\beta_\omega-1)}{|\pi|(K)}
\end{aligned}$$

for $u\in\mathscr{H}_E^1(\omega)$. Hence

$$\sup_{x\in K} h_u^\omega(x) \leqq \alpha\left\{M^2+\frac{2(2\beta_\omega-1)}{|\pi|(K)}\right\}$$

for all $u\in\mathscr{H}_E^1(\omega)$.

5.3. Completeness of the spaces $\mathscr{H}_{D'}(\omega)$ and $\mathscr{H}_E(\omega)$.

LEMMA 5.3. *Let ω be a PB-domain. If $u_n\in\mathscr{H}_E(\omega)$, $n=1, 2,\dots$, $\{\|u_n\|_{E,\omega}\}$ is bounded and $u_n\to u$ locally uniformly on ω, then $u\in\mathscr{H}_E(\omega)$ and*

$$\|u\|_{E,\omega} \leqq \beta_\omega^{1/2}\liminf_{n\to\infty}\|u_n\|_{E,\omega}.$$

PROOF. The case $\pi|\omega\geqq 0$ is given in [9, Proposition 3.3]. Thus we shall prove the case $\pi^-|\omega\neq 0$. Taking a subsequence, we may assume that $\lim_{n\to\infty}\|u_n\|_{E,\omega}$ exists. Let ω' be any PC-domain such that $\bar\omega'\subset\omega$ and $\pi^-|\omega'\neq 0$. Since $u_n\to u$ uniformly on ω', u is bounded on ω' and $|\pi|(\omega')<+\infty$, we see that $\int_{\omega'}u_n^2 d|\pi|\to\int_{\omega'}u^2 d|\pi|$ and $U_{\omega'}^{u_n^2\pi}\to U_{\omega'}^{u^2\pi}$ uniformly on ω'. Consider the sequence $\{h_{u_n}^{\omega'}\}$ in the notation in §5.2. By Lemma 5.2, it is locally uniformly bounded on ω'. Hence, by Axiom 3, we can choose a subsequence $\{v_j\}$ of $\{u_n\}$ such that $\{h_{v_j}^{\omega'}\}$ converges locally uniformly on ω'. For simplicity, let $\delta_j\equiv\delta_{v_j}$ and $h_j\equiv h_{v_j}^{\omega'}$. Obviously, $h^*\equiv\lim_{j\to\infty}h_j$ is harmonic on ω'. Consider the function

$$v = h^* - u^2 - U_{\omega'}^{u^2\pi}.$$

Since $\sigma_v=-\sigma_{u^2}-u^2\pi=2\delta_u\geqq 0$, v is superharmonic on ω'. Furthermore,

$$v = \lim_{j\to\infty}\{h_j - v_j^2 - U_{\omega'}^{v_j^2\pi}\} = 2\lim_{j\to\infty}U_{\omega'}^{\delta_j} \geqq 0. \tag{5.2}$$

It then follows that

$$2U_{\omega'}^{\delta_u} = U_{\omega'}^{\sigma_v} \leqq v = 2\lim_{j\to\infty} U_{\omega'}^{\delta_j}.$$

Given any open set ω'' such that $\bar{\omega}''\subset\omega'$, let $\lambda\equiv\lambda(\omega'';\omega')$ in the notation in Lemma 2.4. Since $S(\lambda)\subset\bar{\omega}''$ and the convergence in (5.2) is uniform on ω'', we deduce

$$\begin{aligned}\delta_u(\omega'') &\leqq \int_{\omega'} U_{\omega'}^{\lambda} d\delta_u \\ &= \int_{\omega'} U_{\omega'}^{\delta_u} d\lambda^+ - \int_{\omega'} U_{\omega'}^{\delta_u} d\lambda^- \\ &\leqq \lim_{j\to\infty}\int_{\omega'} U_{\omega'}^{\delta_j} d\lambda^+ \\ &= \lim_{j\to\infty}\int_{\omega'} U_{\omega'}^{\lambda^+} d\delta_j \leqq \beta_{\omega'} \liminf_{j\to\infty} \delta_j(\omega').\end{aligned}$$

Letting $\omega''\uparrow\omega'$, we have

$$\delta_u(\omega') \leqq \beta_\omega \liminf_{j\to\infty} \delta_j(\omega').$$

Hence,

$$\begin{aligned}\|u\|_{E,\omega'}^2 &\leqq \beta_\omega \liminf_{j\to\infty} \delta_j(\omega') + \int_{\omega'} u^2 d|\pi| \\ &\leqq \beta_\omega \liminf_{j\to\infty}\left(\delta_j(\omega') + \int_{\omega'} v_j^2 d|\pi|\right) \\ &= \beta_\omega \lim_{n\to\infty} \|u_n\|_{E,\omega}^2.\end{aligned}$$

Since we can choose ω' arbitrarily close to ω, we obtain the required inequality.

THEOREM 5.3 (cf. [9, Theorem 3.3]). *If ω is an open set such that $|\pi||\omega_1\neq 0$ for every component ω_1 of ω, then $\mathscr{H}_E(\omega)$ is a Hilbert space with respect to the norm $\|\cdot\|_{E,\omega}$.*

PROOF. Obviously,

$$(u, v)_{E,\omega} = \delta_{[u,v]}(\omega) + \int_\omega uv\, d|\pi|$$

is well-defined for any $u, v\in\mathscr{H}_E(\omega)$ and is an inner product in $\mathscr{H}_E(\omega)$ such that $(u, u)_{E,\omega} = \|u\|_{E,\omega}^2$. To prove the completeness of $\mathscr{H}_E(\omega)$, let $\{u_n\}$ be a Cauchy sequence in $\mathscr{H}_E(\omega)$, i.e., $\|u_n - u_m\|_{E,\omega}\to 0$ $(n, m\to\infty)$. Let ω_1 be any component of ω and consider the set

$$A = \{x \in \omega_1; \lim_{n\to\infty} u_n(x) \text{ exists}\}.$$

If ω' is a PB-domain such that $\omega' \subset \omega_1$ and $|\pi||\omega' \neq 0$, then, by Corollary 1 to Theorem 5.2, u_n converges to a $u \in \mathscr{H}(\omega')$ locally uniformly on ω', so that $\omega' \subset A$. Furthermore, using the previous lemma, we see that $u \in \mathscr{H}_E(\omega')$ and $\|u_n - u\|_{E,\omega'} \to 0$ $(n\to\infty)$ (cf. the proof of [9, Theorem 3.3]). If ω' is a subdomain of ω_1 such that $|\pi||\omega' = 0$, then by [9, Theorem 3.2], $\{u_n - u_n(x_0)\}$ is convergent locally uniformly on ω' for a fixed $x_0 \in \omega'$, and hence either $\omega' \subset A$ or $\omega' \subset \omega_1 - A$. If $\omega' \subset A$, then, by [9, Theorem 3.3], $u = \lim_{n\to\infty} u_n \in \mathscr{H}_E(\omega')$ and $\|u_n - u\|_{E,\omega'} \to 0$ $(n\to\infty)$. Since PB-domains form a base of open sets, the above results show that A and $\omega_1 - A$ are both open. Since $|\pi||\omega_1 \neq 0$, it follows that $A = \omega_1$. Therefore, $u = \lim_{n\to\infty} u_n$ exists on ω_1 and $\|u - u_n\|_{E,\omega'} \to 0$ $(n\to\infty)$ for any PB-domain ω' contained in ω_1.

For any compact set K in ω, the above result implies that

$$\delta_{u_n - u}(K) + \int_K (u_n - u)^2 d|\pi| \to 0.$$

Hence

$$\delta_u(K) + \int_K u^2 d|\pi| = \lim_{n\to\infty} \left\{\delta_{u_n}(K) + \int_K u_n^2 d|\pi|\right\}$$
$$\leqq \lim_{n\to\infty} \|u_n\|_{E,\omega} < +\infty.$$

Thus, $u \in \mathscr{H}_E(\omega)$. Furthermore, for each m,

$$\delta_{u-u_m}(K) + \int_K (u - u_m)^2 d|\pi| = \lim_{n\to\infty} \left\{\delta_{u_n - u_m}(K) + \int_K (u_n - u_m)^2 d|\pi|\right\}$$
$$\leqq \lim_{n\to\infty} \|u_n - u_m\|_{E,\omega} \to 0 \quad (m\to\infty).$$

Hence $\|u - u_m\|_{E,\omega} \to 0$. Thus, $\mathscr{H}_E(\omega)$ is complete.

THEOREM 5.4. *If ω is an open set such that $|\pi||\omega_1 \neq 0$ for every component ω_1 of ω, then $\mathscr{H}_{D'}(\omega)$ is a Hilbert space with respect to the norm $\|\cdot\|_{D',\omega}$.*

PROOF. For $u, v \in \mathscr{H}_{D'}(\omega)$.

$$(u, v)_{D',\omega} = \delta_{[u,v]}(\omega) + \int_\omega uv\, d\pi^-$$

is well-defined and is an inner product in $\mathscr{H}_{D'}(\omega)$ such that $(u, u)_{D',\omega} = \|u\|^2_{D',\omega}$. Let $\{u_n\}$ be a Cauchy sequence in $\mathscr{H}_{D'}(\omega)$. If ω' is a PB-domain contained in ω and ω'' is a PC-domain such that $\bar{\omega}'' \subset \omega'$, then Corollary 4 to Theorem 5.2

implies that

$$\|u_n - u_m\|_{E,\omega''} \leqq M\|u_n - u_m\|_{D',\omega'} \to 0 \qquad (n, m \to \infty)$$

for some constant $M>0$. Hence, by the previous theorem, there is $u \in \mathscr{H}_E(\omega'')$ such that $\|u_n - u\|_{E,\omega''} \to 0$ $(n \to \infty)$ and $u_n \to u$ locally uniformly on ω''. Since such ω'''s cover ω, an argument similar to the last part of the proof of the previous theorem shows that $u = \lim_{n\to\infty} u_n \in \mathscr{H}_{D'}(\omega)$ and $\|u_n - u\|_{D',\omega} \to 0$ $(n \to \infty)$.

COROLLARY (cf. [11, Theorems 3 and 4]). *If* 1 *is superharmonic on* ω *and is not harmonic on any component of* ω, *then* $\mathscr{H}_D(\omega)$ *is a Hilbert space with respect to the norm* $\|\cdot\|_{D,\omega}$.

REMARK 5.3. If $\pi = 0$ on some component of ω, then $\|\cdot\|_{E,\omega}$ and $\|\cdot\|_{D',\omega}$ fail to be norms; though $\mathscr{H}_E(\omega)$ and $\mathscr{H}_{D'}(\omega)$ are still complete with respect to these semi-norms respectively (see [9, Theorem 3.3]).

REMARK 5.4. The above corollary may remain valid in case 1 is not superharmonic on ω. In fact, if the harmonic space is given by solutions of $\Delta u = Pu$ on a Euclidean domain, then we can show that the space of Dirichlet-finite solutions is complete with respect to the Dirichlet norm.

§ 6. Dirichlet potentials and Dirichlet functions on a PB-domain

6.1. Quasi-continuous functions

Let ω be a PB-domain. We consider the capacity $\hat{C}_\omega$ on ω relative to the kernel

$$\hat{G}_\omega(x, y) = \frac{G_\omega(x, y)}{s_\omega(x)s_\omega(y)} \qquad (s_\omega \equiv 1 + U_\omega^{\pi^-}),$$

i.e.,

$$\hat{C}_\omega(K) = \sup\left\{\mu(K);\ \mu \in \mathscr{M}_B^+(\omega),\ \int_\omega \hat{G}_\omega(x, y)d\mu(y) \leqq 1 \text{ for all } x \in \omega\right\}$$

$$= \sup\left\{\int_K s_\omega\, d\nu;\ \nu \in \mathscr{M}_B^+(\omega),\ U_\omega^\nu \leqq s_\omega \text{ on } \omega\right\}$$

for every compact set K in ω. $\hat{C}_\omega$ defines a Choquet capacity on ω (cf. [9, Proposition 5.2]). By [9, Lemma 5.5], we see

LEMMA 6.1. *A set* $e \subset \Omega$ *is polar if and only if* $\hat{G}_\omega(e \cap \omega) = 0$ *for every PB-domain* ω.

Next we prove

LEMMA 6.2. *Let ω and ω' be two PB-domains such that $\omega'\subset\omega$ and let K_0 be a compact set in ω'. Then there are constants $c_1=c_1(\omega,\omega')\geqq 1$ and $c_2=c_2(\omega,\omega',K_0)\geqq 1$ such that*

$$\hat{C}_\omega(A)\leqq c_1\hat{C}_{\omega'}(A)$$

for all Borel sets A in ω' and

$$\hat{C}_{\omega'}(A)\leqq c_2\hat{C}_\omega(A)$$

for all Borel sets A contained in K_0.

PROOF. It is enough to prove the inequalities for compact sets A. If $U_\omega^\nu\leqq s_\omega$ on A with $\nu\in\mathscr{M}_B^+(\omega)$, then $U_{\omega'}^\nu\leqq U_\omega^\nu\leqq s_\omega\leqq\beta_\omega s_{\omega'}$ on A. Hence

$$\hat{C}_{\omega'}(A)\geqq\frac{1}{\beta_\omega}\int_A s_{\omega'}d\nu\geqq\frac{1}{\beta_\omega^2}\int_A s_\omega\,d\nu\,.$$

Thus,

$$\hat{C}_{\omega'}(A)\geqq\frac{1}{\beta_\omega^2}\hat{C}_\omega(A)\,.$$

Next, suppose $A\subset K_0$. Let $G_\omega(x,y)=G_{\omega'}(x,y)+h(x,y)$ for $x,y\in\omega'$. Then, $h(x,y)$ is positive and continuous on $\omega\times\omega$. Put $M=\sup_{x\in K_0,y\in K_0}h(x,y)$ and $m=\inf_{x\in K_0,y\in K_0}G_{\omega'}(x,y)$. Then $0<M<+\infty$ and $0<m<+\infty$. Let $c_2=1+M/m$. Then $G_\omega(x,y)\leqq c_2G_{\omega'}(x,y)$ for all $x,y\in K_0$. Thus, if $\nu\in\mathscr{M}_B^+(\omega)$ and $S(\nu)\subset K_0$, then $U_\omega^\nu\leqq c_2U_{\omega'}^\nu$ on K_0. Let $\nu\in\mathscr{M}_B^+(\omega)$, $S(\nu)\subset A$ and $U_{\omega'}^\nu\leqq s_{\omega'}$ on A. Then $U_\omega^\nu\leqq c_2s_\omega$ on A, so that

$$\hat{C}_\omega(A)\geqq\frac{1}{c_2}\int_A s_\omega\,d\nu\geqq\frac{1}{c_2}\int_A s_{\omega'}d\nu\,.$$

It then follows that

$$\hat{C}_\omega(A)\geqq\frac{1}{c_2}\hat{C}_{\omega'}(A)\,.$$

An extended real valued function f on an open set ω_0 is said to be *quasi-continuous* there if, for any PB-domain ω contained in ω_0, $f|\omega$ is quasi-continuous with respect to the capacity $\hat{C}_\omega$. By virtue of the above lemma, a function on a PB-domain ω_0 is quasi-continuous in the above sense if and only if it is quasi-continuous with respect to $\hat{C}_{\omega_0}$. By Lemma 6.1, a quasi-continuous function is finite q.e.; if f is quasi-continuous and $g=f$ q.e., then g is quasi-continuous.

LEMMA 6.3. *Let ω_0 be an open set and f be a quasi-continuous function on ω_0. Then f is μ-measurable for any non-negative measure μ on ω_0 such that $\mu|\omega \in \mathscr{M}_E(\omega)$ for each PC-domain ω with $\bar{\omega} \subset \omega_0$; in particular, f is $|\pi|$-measurable.*

This lemma is easily verified by the definition of quasi-continuity and Lemmas 1.3 and 6.1 (cf. [4, p. 52]).

LEMMA 6.4. *Let ω_0 be an open set and let f be a quasi-continuous function on ω_0. If f is μ-integrable and $\int_\omega f d\mu = 0$ for any $\mu \in \mathscr{M}_B^+(\omega)$ with a PC-domain ω such that $\bar{\omega} \subset \omega_0$, then $f=0$ q.e. on ω_0.*

PROOF. Let ω' be any PB-domain contained in ω_0. *If* $\mu \in \mathscr{M}_B^+(\omega')$ and $S(\mu)$ is compact in ω', then f is μ-integrable and $\int_{\omega'} f d\mu = 0$ by assumption. Hence, [9, Corollary to Lemma 5.7] implies that $f=0$ q.e. on ω' with respect to the capacity $\hat{C}_{\omega'}$. This means that $f=0$ q.e. on ω_0.

REMARK 6.1. Similarly, we also see that [9, Lemma 5.7] is valid in the present case.

6.2. Dirichlet potentials

Let ω be a PB-domain and consider the classes

$$\mathscr{M}_{BC}(\omega) = \{\sigma \in \mathscr{M}_B(\omega);\, U_\omega^{|\sigma|} \text{ is continuous}\},$$

$$\mathscr{P}_{BC}(\omega) = \{U_\omega^\sigma;\, \sigma \in \mathscr{M}_{BC}(\omega)\}.$$

Every function in $\mathscr{P}_{BC}(\omega)$ is bounded continuous on ω. $\mathscr{P}_{BC}(\omega)$ is a normed space with respect to the norm

$$\|U_\omega^\sigma\|_{I,\omega} = I_\omega(\sigma)^{1/2} \qquad (\text{i.e., } \|f\|_{I,\omega} = I_\omega(\sigma_f)^{1/2}).$$

THEOREM 6.1. *Let ω be a PB-domain and let*

$$\mathscr{D}_0(\omega) = \left\{f;\ \begin{array}{l} \text{there is a sequence } \{f_n\} \text{ in } \mathscr{P}_{BC}(\omega) \text{ such that} \\ f_n \to f \text{ q.e. on } \omega \text{ and } \|f_n - f_m\|_{I,\omega} \to 0 \quad (n, m \to \infty) \end{array}\right\}.$$

Then $\mathscr{D}_0(\omega)$ has the following properties:

(a) *If $f \in \mathscr{D}_0(\omega)$ and f_1 is a function on ω such that $f_1 = f$ q.e. on ω, then $f_1 \in \mathscr{D}_0(\omega)$.*

(b) *Any function in $\mathscr{D}_0(\omega)$ is quasi-continuous on ω.*

(c) *For $f \in \mathscr{D}_0(\omega)$, if $\{f_n\}$ is a sequence in $\mathscr{P}_{BC}(\omega)$ such that $f_n \to f$ q.e.*

on ω and $\|f_n - f_m\|_{I,\omega} \to 0$ $(n, m \to \infty)$, then

$$\|f\|_{I,\omega} \equiv \lim_{n\to\infty} \|f_n\|_{I,\omega}$$

exists and is independent of the choice of $\{f_n\}$.

(d) *If we identify functions which are equal q.e. on ω, then $\mathscr{D}_0(\omega)$ is a Hilbert space with respect to the above norm $\|\cdot\|_{I,\omega}$ and contains $\mathscr{P}_{BC}(\omega)$ as a dense subspace.*

(e) *If $f_n, f \in \mathscr{D}_0(\omega)$, $f_n \to f$ q.e. on ω and $\|f_n - f_m\|_{I,\omega} \to 0$ $(n, m \to \infty)$, then $\|f_n - f\|_{I,\omega} \to 0$ $(n \to \infty)$.*

(f) *If $f_n, f \in \mathscr{D}_0(\omega)$ and $\|f_n - f\|_{I,\omega} \to 0$, then there is a subsequence of $\{f_n\}$ converging to f q.e. on ω.*

(g) *For any $f \in \mathscr{D}_0(\omega)$, there is a potential p on ω such that $|f| \leqq p$ on ω.*

PROOF. For $\sigma \in \mathscr{M}_B(\omega)$, let

$$\hat{U}^\sigma_\omega(x) \equiv \int_\omega \hat{G}_\omega(x, y) d\sigma(y) = \frac{1}{s_\omega(x)} \int_\omega \frac{G_\omega(x, y)}{s_\omega(y)} d\sigma(y).$$

Since ω is a PB-domain, we see that $\sigma \in \mathscr{M}_{BC}(\omega)$ if and only if $\hat{U}^{|\sigma|}_\omega(x)$ is bounded and continuous. Let

$$\hat{\mathscr{P}}_{BC}(\omega) = \{\hat{U}^\sigma_\omega;\ \sigma \in \mathscr{M}_{BC}(\omega)\},$$

$$\|\hat{U}^\sigma_\omega\|_{\hat{E},\omega} = I_\omega(s_\omega^{-1}\sigma)^{1/2}$$

and

$$\hat{\mathscr{D}}_0(\omega) = \left\{ g;\ \begin{array}{l} \text{there is a sequence } \{g_n\} \text{ in } \hat{\mathscr{P}}_{BC}(\omega) \text{ such that} \\ g_n \to g \text{ q.e. on } \omega \text{ and } \|g_n - g_m\|_{\hat{E},\omega} \to 0\ (n, m \to \infty) \end{array} \right\}.$$

Since $\mathscr{P}_{BC}(\omega) = \{s_\omega g;\ g \in \hat{\mathscr{P}}_{BC}(\omega)\}$ and $\|s_\omega g\|_{I,\omega} = \|g\|_{\hat{E},\omega}$ for $g \in \hat{\mathscr{P}}_{BC}(\omega)$, we see that $\mathscr{D}_0(\omega) = \{s_\omega g;\ g \in \hat{\mathscr{D}}_0(\omega)\}$. Now, applying [9, Theorem 5.1 and Propositions 5.3 and 5.4] to the harmonic structure $\mathfrak{H}_\omega / s_\omega$ and noting that s_ω is positive continuous, we obtain the required results.

REMARK 6.2. In case 1 is superharmonic on ω, the space $\mathscr{D}_0(\omega)$ is the same as $\mathscr{E}_0(\omega)$ given in [9].

PROPOSITION 6.1. *If ω is a PB-domain and $\sigma \in \mathscr{M}_E(\omega)$, then $f \equiv U^\sigma_\omega \in \mathscr{D}_0(\omega)$ and $\|f\|^2_{I,\omega} = I_\omega(\sigma)$.*

PROOF. By Lemma 1.5, we can choose $\sigma_n \in \mathscr{M}_{BC}(\omega)$, $n = 1, 2, \dots$, such that

$U_\omega^{\sigma_n} \to f$ q.e. on ω and $I_\omega(\sigma - \sigma_n) \to 0$ $(n \to \infty)$. Hence $f \in \mathscr{D}_0(\omega)$ and $\|f\|_{I,\omega}^2 = \lim_{n \to \infty} I_\omega(\sigma_n) = I_\omega(\sigma)$.

The following three lemmas will be used in the next section.

LEMMA 6.5. *Let ω be a PB-domain. If $f \in \mathscr{P}_{BC}(\omega)$, then $|f| \in \mathscr{P}_{BC}(\omega)$ and $\| |f| \|_{I,\omega} = \|f\|_{I,\omega}$.*

PROOF. If $f = U_\omega^\sigma$ with $\sigma \in \mathscr{M}_{BC}(\omega)$, then $|f| = U_\omega^{|\sigma|} - 2\min(U_\omega^{\sigma^+}, U_\omega^{\sigma^-})$. It follows that $|f| \in \mathscr{P}_{BC}(\omega)$. By the corollary to Lemma 4.2, $\delta_{|f|} = \delta_f$. Hence, by Lemma 4.3, we have

$$\| |f| \|_{I,\omega}^2 = \delta_{|f|}(\omega) + \int_\omega |f|^2 d\pi = \delta_f(\omega) + \int_\omega f^2 d\pi = \|f\|_{I,\omega}^2 .$$

LEMMA 6.6. *Let ω be a PB-domain. Then, for any $\mu \in \mathscr{M}_E^+(\omega)$ and $f \in \mathscr{D}_0(\omega)$,*

$$\int_\omega |f| d\mu \leqq \|f\|_{I,\omega} I_\omega(\mu)^{1/2} .$$

PROOF. Let $\{f_n\}$ be a sequence in $\mathscr{P}_{BC}(\omega)$ such that $f_n \to f$ q.e. on ω and $\|f - f_n\|_{I,\omega} \to 0$ $(n \to \infty)$. Let $\sigma_n = \sigma_{|f_n|}$. By the above lemma, $\sigma_n \in \mathscr{M}_{BC}(\omega)$ and $I_\omega(\sigma_n) = \|f_n\|_{I,\omega}^2$. Hence

$$\int_\omega |f_n| d\mu = \int_\omega U_\omega^{\sigma_n} d\mu \leqq I(\sigma_n)^{1/2} I_\omega(\mu)^{1/2} = \|f_n\|_{I,\omega} I_\omega(\mu)^{1/2} .$$

By Lemma 1.3, $\mu(e) = 0$ for a polar set e. Hence, Fatou's lemma implies

$$\begin{aligned} \int_\omega |f| d\mu &\leqq \liminf_{n \to \infty} \int_\omega |f_n| d\mu \\ &\leqq \{\lim_{n \to \infty} \|f_n\|_{I,\omega}\} I_\omega(\mu)^{1/2} = \|f\|_{I,\omega} I_\omega(\mu)^{1/2} . \end{aligned}$$

LEMMA 6.7. *Let ω be a PB-domain and ω' be a PC-domain such that $\bar{\omega}' \subset \omega$. If $f \in \mathscr{D}_0(\omega')$, then*

$$f^\# = \begin{cases} f & on \quad \omega' \\ 0 & on \quad \omega - \omega' \end{cases}$$

is an element of $\mathscr{D}_0(\omega)$.

PROOF. Let $\{f_n\}$ be a sequence in $\mathscr{P}_{BC}(\omega')$ such that $f_n \to f$ q.e. on ω' and $\|f_n - f_m\|_{I,\omega'} \to 0$ $(n, m \to \infty)$. By virtue of Lemma 1.5, we may assume that $S(\sigma_{f_n})$ is compact in ω' for each n. Let $\sigma_n \equiv \sigma_{f_n}$ for simplicity. Each σ_n can be regarded as a measure on ω. Using Lemma 2.2, we see that $p_n \equiv U_\omega^{\sigma_n^+}$ and $q_n \equiv U_\omega^{\sigma_n^-}$ are bounded on ω, so that $\sigma_n \in \mathscr{M}_B(\omega)$. By Lemma 1.7,

$$\tilde{p}_n \equiv \hat{R}_{p_n}^{\omega-\omega',\omega} \quad \text{and} \quad \tilde{q}_n \equiv \hat{R}_{q_n}^{\omega-\omega',\omega}$$

are bounded potentials on ω and $p_n - q_n = \tilde{p}_n - \tilde{q}_n$ q.e. on $\omega - \omega'$. Let μ_n and ν_n be the associated measures of $\tilde{p}_n$ and $\tilde{q}_n$ respectively, and let $\tau_n = \mu_n - \nu_n$. Since $\tilde{p}_n|\omega - \bar{\omega}' = p_n|\omega - \bar{\omega}'$ and $\tilde{q}_n|\omega - \bar{\omega}' = q_n|\omega - \bar{\omega}'$ and they are harmonic on $\omega - \bar{\omega}'$, we see that $S(\mu_n) \subset \partial\omega'$ and $S(\nu_n) \subset \partial\omega'$. Therefore $\tau_n \in \mathscr{M}_B(\omega)$ for each n. Let $g_n \equiv p_n - q_n - \tilde{p}_n + \tilde{q}_n = U_\omega^{\sigma_n - \tau_n}$. Then $g_n \in \mathscr{D}_0(\omega)$ by Proposition 6.1. Furthermore, $g_n = 0$ q.e. on $\omega - \omega'$. On the other hand, by Axiom D (see Corollary 1 to Theorem 1.1), we see that $p_n - \tilde{p}_n = U_{\omega'}^{\sigma_n^+}$ and $q_n - \tilde{q}_n = U_{\omega'}^{\sigma_n^-}$ on ω' (see, e.g., [3, p. 129] or [5, p. 225]). Hence $g_n = f_n$ on ω'. It then follows that $g_n \to f^{\#}$ q.e. on ω. Furthermore, using the fact that $S(\tau_n) \subset \partial\omega'$, Lemma 1.3 and Proposition 6.1, we deduce

$$\|g_n - g_m\|_{I,\omega} = \int_\omega (g_n - g_m)\, d(\sigma_n - \tau_n - \sigma_m + \tau_m)$$

$$= \int_{\omega'} (f_n - f_m)\, d(\sigma_n - \sigma_m) = \|f_n - f_m\|_{I,\omega'} \to 0$$

$(n, m \to \infty)$. Thus, it follows from Theorem 6.1 that $f^{\#} \in \mathscr{D}_0(\omega)$.

6.3. Dirichlet functions and gradient measures

For a PB-domain ω, let

$$\mathscr{D}(\omega) \equiv \mathscr{H}_D(\omega) + \mathscr{D}_0(\omega) = \{u + f_0;\, u \in \mathscr{H}_D(\omega), f_0 \in \mathscr{D}_0(\omega)\}.$$

This is a linear space consisting of quasi-continuous functions on ω.

THEOREM 6.2. *Let ω be a PB-domain. For each $f \in \mathscr{D}(\omega)$, there is a unique non-negative measure δ_f^ω on ω having the following property: if $f = u + f_0$ with $u \in \mathscr{H}_D(\omega)$ and $g \in \mathscr{D}_0(\omega)$ and if $\{f_n\}$ is a sequence in $\mathscr{P}_{BC}(\omega)$ such that $f_n \to f_0$ q.e. on ω and $\|f_n - f_m\|_{I,\omega} \to 0$ $(n, m \to \infty)$, then $\delta_{u+f_n}(A) \to \delta_f^\omega(A)$ for any Borel set A in ω.*

PROOF. Let $\{f_n\}$ be a sequence in $\mathscr{P}_{BC}(\omega)$ as described in the theorem. By Lemma 4.3,

$$\delta_{f_n}(\omega) \leqq \beta_\omega \|f_n\|_{I,\omega}^2, \qquad n = 1, 2, \ldots$$

and

$$\delta_{f_n - f_m}(\omega) \leqq \beta_\omega \|f_n - f_m\|_{I,\omega}^2, \qquad n, m = 1, 2, \ldots.$$

Since $\delta_{u+f_n} \leqq 2(\delta_u+\delta_{f_n})$, it follows that $\{\delta_{u+f_n}(A)\}$ is bounded for any Borel set A in ω. Furthermore,

$$\begin{aligned}&|\delta_{u+f_n}(A)^{1/2}-\delta_{u+f_m}(A)^{1/2}|\\&\leqq \delta_{f_n-f_m}(A)^{1/2} \leqq \delta_{f_n-f_m}(\omega)^{1/2} \leqq \beta_\omega^{1/2}\|f_n-f_m\|_{I,\omega}\\&\qquad\to 0 \ (n, m\to\infty).\end{aligned}$$

Therefore, $\{\delta_{u+f_n}(A)\}$ is a Cauchy sequence, so that

$$\delta_f^\omega(A) \equiv \lim_{n\to\infty}\delta_{u+f_n}(A)$$

exists. The uniform convergence with respect to A implies that δ_f^ω is also a measure on ω. Obviously $\delta_f^\omega \geqq 0$. If $\{f_n^*\}$ is another sequence in $\mathscr{P}_{BC}(\omega)$ such that $f_n^*\to f_0$ q.e. on ω and $\|f_n^*-f_m^*\|_{I,\omega}\to 0$ $(n, m\to\infty)$, then by Theorem 6.1, we see that $\|f_n-f_n^*\|_{I,\omega}\to 0$ $(n\to\infty)$. Then, by an argument similar to the above, we see that $\delta_{u+f_n}(A)-\delta_{u+f_n^*}(A)\to 0$ $(n\to\infty)$. Thus δ_f^ω is uniquely determined by f.

For $f, g\in\mathscr{D}(\omega)$, let

$$\delta_{[f,g]}^\omega = \frac{1}{2}(\delta_{f+g}^\omega-\delta_f^\omega-\delta_g^\omega).$$

We can easily see that the mapping $(f, g)\to\delta_{[f,g]}^\omega$ is symmetric and bilinear on $\mathscr{D}(\omega)\times\mathscr{D}(\omega)$.

Note that if $f\in\mathscr{P}_{BC}(\omega)$, then $\delta_f^\omega=\delta_f$; and hence if $f, g\in\mathscr{P}_{BC}(\omega)$, then $\delta_{[f,g]}^\omega=\delta_{[f,g]}$.

THEOREM 6.3. *Let ω be a PB-domain and let $f\in\mathscr{D}_0(\omega)$. Then,*

$$\int_\omega f^2 d|\pi| \leqq (2\beta_\omega-1)\|f\|_{I,\omega}^2, \tag{6.1}$$

$$\int_\omega f^2 d\pi^- \leqq (\beta_\omega-1)\|f\|_{I,\omega}^2, \tag{6.2}$$

$$\delta_f^\omega(\omega) \leqq \beta_\omega\|f\|_{I,\omega}^2, \tag{6.3}$$

$$\delta_f^\omega(\omega)+\int_\omega f^2 d\pi = \|f\|_{I,\omega}^2 \tag{6.4}$$

and

$$\delta_{[u,f]}^\omega(\omega)+\int_\omega uf\,d\pi = 0 \tag{6.5}$$

for $u \in \mathscr{H}_E(\omega)$.

PROOF. Let $\{f_n\}$ be a sequence in $\mathscr{P}_{BC}(\omega)$ such that $f_n \to f$ q.e. on ω and $\|f_n - f_m\|_{I,\omega} \to 0$ $(n, m \to \infty)$. By Theorem 1.2,

$$\int_\omega f_n^2 d|\pi| \leqq (2\beta_\omega - 1)\|f_n\|_{I,\omega}^2,$$

$$\int_\omega f_n^2 d\pi^- \leqq (\beta_\omega - 1)\|f_n\|_{I,\omega}^2$$

and

$$\int_\omega (f_n - f_m)^2 d|\pi| \leqq (2\beta_\omega - 1)\|f_n - f_m\|_{I,\omega}^2 \to 0 \qquad (n, m \to \infty).$$

Since $f_n \to f$ q.e. on ω and $|\pi|(e) = 0$ for a polar set e, Fatou's lemma implies (6.1) and (6.2), and furthermore,

$$\int_\omega (f_n - f)^2 d|\pi| \to 0 \qquad (n \to \infty).$$

Then (6.4) is easily seen by Lemma 4.3. The inequality (6.3) immediately follows from (6.2) and (6.4). Finally, if $u \in \mathscr{H}_E(\omega)$, then, by Lemma 4.5,

$$\delta_{[u, f_n]}(\omega) + \int_\omega u f_n d\pi = 0, \qquad n = 1, 2, \ldots.$$

By the definition of $\delta_{[u,f]}^{\omega}$, we see that $\delta_{[u,f_n]}(\omega) \to \delta_{[u,f]}^{\omega}(\omega)$ $(n \to \infty)$. By the above result, we also see that $\int_\omega u f_n d\pi \to \int_\omega u f\, d\pi$ $(n \to \infty)$. Hence we obtain (6.5).

THEOREM 6.4. *Let* $f \in \mathscr{H}_E(\omega) + \mathscr{D}_0(\omega)$. *If*

$$\delta_{[f,g]}^{\omega}(\omega) + \int_\omega fg\, d\pi = 0$$

for all $g \in \mathscr{D}_0(\omega)$, *then* $f = u$ *q.e. on* ω *with* $u \in \mathscr{H}_E(\omega)$.

PROOF. Let $f = u + f_0$ with $u \in \mathscr{H}_E(\omega)$ and $f_0 \in \mathscr{D}_0(\omega)$. By assumption

$$\delta_{[f, f_0]}^{\omega}(\omega) + \int_\omega f f_0 d\pi = 0$$

and by the above theorem

$$\delta_{[u, f_0]}^{\omega}(\omega) + \int_\omega u f_0 d\pi = 0.$$

Hence

$$\|f_0\|_{I,\omega}^2 = \delta_{f_0}^{\omega}(\omega)+\int_{\omega} f_0^2 d\pi = 0\,,$$

and hence $f_0=0$ q.e. on ω by Theorem 6.1.

§7. Locally Dirichlet-finite functions

7.1. Preliminary lemmas

LEMMA 7.1. *Let ω be a PB-domain and ω' be a PC-domain such that $\overline{\omega}' \subset \omega$. Then, for any $\sigma \in \mathscr{M}_E(\omega)$ such that $U_{\omega}^{|\sigma|}$ is locally bounded on ω,*

$$I_{\omega'}(\sigma) \leqq (2\beta_\omega - 1)^2 I_\omega(\sigma)\,.$$

PROOF. Put $p=U_{\omega}^{\sigma}$, $p'=U_{\omega'}^{\sigma}$ and $u=p|\omega'-p'$. By Lemma 2.8, $u \in \mathscr{H}_{BE}(\omega')$. By Lemmas 4.3 and 4.5,

$$\delta_{p'}(\omega') = \int_{\omega'} p'^2 d\pi = I_{\omega'}(\sigma)\,, \tag{7.1}$$

$$\delta_{[u,p']}(\omega')+\int_{\omega'} up' d\pi = 0\,. \tag{7.2}$$

Hence

$$\begin{aligned} I_{\omega'}(\sigma) &= \delta_{[p,p']}(\omega')+\int_{\omega'} pp' d\pi \\ &\leqq \left\{\delta_p(\omega')+\int_{\omega'} p^2 d\pi^+\right\}^{1/2}\left\{\delta_{p'}(\omega')+\int_{\omega'} p'^2 d\pi^+\right\}^{1/2} \\ &\qquad +\left\{\int_{\omega'} p^2 d\pi^-\right\}^{1/2}\left\{\int_{\omega'} p'^2 d\pi^-\right\}^{1/2} \\ &\leqq \left\{I_\omega(\sigma)+\int_{\omega} p^2 d\pi^-\right\}^{1/2}\left\{I_{\omega'}(\sigma)+\int_{\omega'} p'^2 d\pi^-\right\}^{1/2} \\ &\qquad +\left\{\int_{\omega} p^2 d\pi^-\right\}^{1/2}\left\{\int_{\omega'} p'^2 d\pi^-\right\}^{1/2}. \end{aligned}$$

Since $\int_{\omega'} p'^2 d\pi^- \leqq (\beta_\omega - 1) I_{\omega'}(\sigma)$ and $\int_{\omega} p^2 d\pi^- \leqq (\beta_\omega - 1) I_\omega(\sigma)$ (Theorem 1.2), we deduce that

$$\begin{aligned} I_{\omega'}(\sigma) &\leqq \beta_\omega I_\omega(\sigma)^{1/2} I_{\omega'}(\sigma)^{1/2} + (\beta_\omega - 1) I_\omega(\sigma)^{1/2} I_{\omega'}(\sigma)^{1/2} \\ &= (2\beta_\omega - 1) I_\omega(\sigma)^{1/2} I_{\omega'}(\sigma)^{1/2}\,, \end{aligned}$$

from which the required inequality follows.

LEMMA 7.2. *Let ω, ω' and σ be as in the previous lemma. Then, for* $u=U^{\sigma}_{\omega}|\omega'-U^{\sigma}_{\omega'}$,

$$\delta_u(\omega')+\int_{\omega'}u^2d|\pi| \leqq (2\beta_{\omega}-1)^3 I_{\omega}(\sigma).$$

PROOF. With the same notation as in the above proof, (7.1) and (7.2) imply

$$\delta_u(\omega')+\int_{\omega'}u^2d\pi = \delta_p(\omega')+\int_{\omega'}p^2d\pi-I_{\omega'}(\sigma).$$

Hence, using Lemma 4.3, we have

$$\begin{aligned}
&\delta_u(\omega')+\int_{\omega'}u^2d|\pi| \\
&\leqq \delta_p(\omega')+\int_{\omega'}p^2d\pi^+-\int_{\omega'}p^2d\pi^-+2\int_{\omega'}u^2d\pi^--I_{\omega'}(\sigma) \\
&\leqq I_{\omega}(\sigma)+\int_{\omega}p^2d\pi^--\int_{\omega'}p^2d\pi^-+2\int_{\omega'}(p-p')^2d\pi^--I_{\omega'}(\sigma) \\
&\leqq I_{\omega}(\sigma)+2\int_{\omega}p^2d\pi^--4\int_{\omega'}pp'd\pi^-+2\int_{\omega'}p'^2d\pi^--I_{\omega'}(\sigma).
\end{aligned}$$

If $\pi^-|\omega=0$, then the required inequality is now obvious. If $\pi^-|\omega\neq 0$, then $\beta_{\omega}>1$. Noting that

$$-2pp' \leqq 2(\beta_{\omega}-1)p^2+[2(\beta_{\omega}-1)]^{-1}p'^2$$

and using Theorem 1.2, we have

$$\begin{aligned}
&\delta_u(\omega')+\int_{\omega'}u^2d|\pi| \\
&\leqq I_{\omega}(\sigma)+(4\beta_{\omega}-2)\int_{\omega}p^2d\pi^-+\left(\frac{1}{\beta_{\omega}-1}+2\right)\int_{\omega'}p'^2d\pi^--I_{\omega'}(\sigma) \\
&\leqq \{1+(\beta_{\omega}-1)(4\beta_{\omega}-2)\}I_{\omega}(\sigma)+\{1+(2\beta_{\omega}-1)-1\}I_{\omega'}(\sigma) \\
&\leqq (2\beta_{\omega}-1)^2 I_{\omega}(\sigma)+2(\beta_{\omega}-1)I_{\omega'}(\sigma).
\end{aligned}$$

Then the required inequality follows from the previous lemma.

LEMMA 7.3. *Let ω be a PB-domain and ω' be a PC-domain such that $\bar{\omega}'\subset\omega$. Then, for any $f\in\mathscr{D}(\omega)$, $f|\omega'\in\mathscr{H}_E(\omega')+\mathscr{D}_0(\omega')$ $(\subset\mathscr{D}(\omega'))$ and $\delta^{\omega'}_{f|\omega'}=\delta^{\omega}_f|\omega'$.*

PROOF. Let $f=u+f_0$ with $u\in\mathscr{H}_D(\omega)$ and $f_0\in\mathscr{D}_0(\omega)$. Choose $f_n\in\mathscr{P}_{BC}(\omega)$

such that $f_n \to f_0$ q.e. on ω and $\|f_n - f_m\|_{I,\omega} \to 0$ $(n, m \to \infty)$. Put $\sigma_n = \sigma_{f_n}$, $g_n = U_{\omega'}^{\sigma_n}$ and $u_n = f_n|\omega' - g_n$ $(\in \mathscr{H}_{BE}(\omega'))$. By the previous two lemmas, we have

$$\|g_n - g_m\|_{I,\omega'} \leqq (2\beta_\omega - 1)\|f_n - f_m\|_{I,\omega} \to 0 \qquad (n, m \to \infty)$$

and

$$\|u_n - u_m\|_{E,\omega'} \leqq (2\beta_\omega - 1)^{3/2}\|f_n - f_m\|_{I,\omega} \to 0 \qquad (n, m \to \infty).$$

First assume $|\pi||\omega' \neq 0$. Then $\mathscr{H}_E(\omega')$ is complete by Theorem 5.3. Hence, $u^* = \lim_{n\to\infty} u_n$ exists, $u^* \in \mathscr{H}_E(\omega')$ and $\|u_n - u^*\|_{E,\omega'} \to 0$ $(n \to \infty)$. Then $g_n \to g^* \equiv f_0|\omega' - u^*$ q.e. on ω'. By definition, $g^* \in \mathscr{D}_0(\omega')$. Therefore, $f|\omega' = u|\omega' + u^* + g^* \in \mathscr{H}_E(\omega') + \mathscr{D}_0(\omega')$. If $|\pi||\omega' = 0$, then we first choose $g^* \in \mathscr{D}_0(\omega')$ such that $\|g_n - g^*\|_{I,\omega'} \to 0$ $(n \to \infty)$, which exists by Theorem 6.1 (or [9, Theorem 5.1]). By the same theorem, we see that there is a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ such that $g_{n_k} \to g^*$ q.e. on ω' $(k \to \infty)$. It follows that $\{u_{n_k}(x_0)\}$ is convergent for some $x_0 \in \omega'$. Hence, by [9, Theorem 3.3], there is $u^* \in \mathscr{H}_E(\omega')$ such that $\|u_{n_k} - u^*\|_{E,\omega'} \to 0$ $(k \to \infty)$ and $u_{n_k} \to u$ (locally uniformly) on ω'. Hence,

$$f|\omega' = u|\omega' + u^* + g^* \in \mathscr{H}_E(\omega') + \mathscr{D}_0(\omega').$$

From Theorem 6.2, it follows that

$$\delta_{f|\omega'}^{\omega'}(A) = \lim_{n\to\infty} \delta_{u+u^*+g_n}(A) = \lim_{n\to\infty} \delta_{(u+f_n)+(u^*-u_n)}(A)$$

for any Borel set A in ω'. Since

$$|\delta_{(u+f_n)+(u^*-u_n)}(A)^{1/2} - \delta_{u+f_n}(A)^{1/2}|$$
$$\leqq \delta_{u^*-u_n}(A)^{1/2} \leqq \|u^* - u_n\|_{E,\omega'} \to 0 \qquad (n \to \infty),$$

we see that

$$\delta_{f|\omega'}^{\omega'}(A) = \lim_{n\to\infty} \delta_{u+f_n}(A) = \delta_f^{\omega}(A).$$

Therefore $\delta_{f|\omega'}^{\omega'} = \delta_f^{\omega}|\omega'$.

7.2. Locally Dirichlet-finite functions and their gradient measures

For an open set ω, we define

$$\mathscr{D}_{\mathrm{loc}}(\omega) = \{f;\ \text{for any PC-domain } \omega' \text{ such that } \bar{\omega}' \subset \omega, f|\omega' \in \mathscr{D}(\omega')\}.$$

By virtue of Lemma 7.3, the space $\mathscr{D}(\omega')$ in the above definition may be replaced by $\mathscr{H}_E(\omega') + \mathscr{D}_0(\omega')$. Thus, in case 1 is superharmonic on ω, $\mathscr{D}_{\mathrm{loc}}(\omega)$ coincides with the space $\mathscr{E}_{\mathrm{loc}}(\omega)$ introduced in [9, §6.2]. Also, Lemma 7.3 asserts that $\mathscr{D}(\omega)\subset\mathscr{D}_{\mathrm{loc}}(\omega)$ in case ω is a PB-domain, and furthermore it implies the following

THEOREM 7.1. *For any $f\in\mathscr{D}_{\mathrm{loc}}(\omega)$, there is a unique non-negative measure δ_f such that $\delta_f|\omega'=\delta_f^{\omega'}$ for any PC-domain ω' such that $\bar{\omega}'\subset\omega$.*

The measure δ_f may be called *the gradient measure of $f\in\mathscr{D}_{\mathrm{loc}}(\omega)$*. For $f, g\in\mathscr{D}_{\mathrm{loc}}(\omega)$, their mutual gradient measure is defined by

$$\delta_{[f,g]}=\frac{1}{2}(\delta_{f+g}-\delta_f-\delta_g),$$

which is a signed measue on ω. Obviously, $\mathscr{B}_{\mathrm{loc}}(\omega)\subset\mathscr{D}_{\mathrm{loc}}(\omega)$ and the above definitions of δ_f and $\delta_{[f,g]}$ are compatible with those for $f, g\in\mathscr{B}_{\mathrm{loc}}(\omega)$. We can easily verify that the mapping $(f, g)\to\delta_{[f,g]}$ is symmetric and bilinear on $\mathscr{D}_{\mathrm{loc}}(\omega)\times\mathscr{D}_{\mathrm{loc}}(\omega)$ and the same inequalities as in the corollary to Theorem 4.1 hold for $f, g\in\mathscr{D}_{\mathrm{loc}}(\omega_0)$.

From Theorem 6.3, we obtain

PROPOSITION 7.1. *Every $f\in\mathscr{D}_{\mathrm{loc}}(\omega)$ is locally $|\pi|$-square-integrable on ω.*

Next we prove

PROPOSITION 7.2. *If ω is a PB-domain, then*

$$\left\{f\in\mathscr{D}_{\mathrm{loc}}(\omega);\,\delta_f(\omega)+\int_\omega f^2d|\pi|<+\infty\right\}=\mathscr{H}_E(\omega)+\mathscr{D}_0(\omega).$$

PROOF. Let

$$\mathscr{D}_E(\omega)=\{f\in\mathscr{D}_{\mathrm{loc}}(\omega);\,\delta_f(\omega)+\int_\omega f^2d|\pi|<+\infty\}.$$

By Lemma 7.3 and Theorem 6.3, we see that $\mathscr{H}_E(\omega)+\mathscr{D}_0(\omega)\subset\mathscr{D}_E(\omega)$. Now, let $f\in\mathscr{D}_E(\omega)$ be given. Consider the linear form

$$l(g)=\delta_{[f,g]}(\omega)+\int_\omega fg\,d\pi$$

defined on $\mathscr{D}_0(\omega)$. It is continuous in view of Theorem 6.4. Hence, by Theorem 6.1 (d), there is $f_0\in\mathscr{D}_0(\omega)$ such that

$$l(g)=\delta_{[f_0,g]}(\omega)+\int_\omega f_0g\,d\pi$$

for all $g\in\mathscr{D}_0(\omega)$. Then

$$\delta_{[f-f_0,g]}(\omega)+\int_\omega (f-f_0)g\, d\pi = 0$$

for all $g\in\mathscr{D}_0(\omega)$. Now, using Lemma 6.7, we see that for any PC-domain ω' such that $\bar{\omega}'\subset\omega$ and for any $g\in\mathscr{D}_0(\omega')$

$$\delta_{[f-f_0,g]}(\omega')+\int_{\omega'} (f-f_0)g\, d\pi = 0.$$

By Lemma 7.3, $(f-f_0)|\omega'\in\mathscr{H}_E(\omega')+\mathscr{D}_0(\omega')$. Hence Theorem 6.4 asserts that $f-f_0=u$ q.e. on ω' for some $u\in\mathscr{H}_E(\omega')$. It follows that there is $u\in\mathscr{H}(\omega)$ such that $f-f_0=u$ q.e. on ω. By modifying the values of f_0 on a polar set, we have $f=u+f_0$ on ω. Since $\delta_u\leqq 2(\delta_f+\delta_{f_0})$ and $u^2\leqq 2(f^2+f_0^2)$, we see that $\delta_u(\omega)+\int_\omega u^2 d|\pi|<+\infty$, i.e., $u\in\mathscr{H}_E(\omega)$. Thus $f\in\mathscr{H}_E(\omega)+\mathscr{D}_0(\omega)$, and hence $\mathscr{D}_E(\omega)\subset\mathscr{H}_E(\omega)+\mathscr{D}_0(\omega)$.

REMARK 7.1. It is clear that $\mathscr{D}(\omega)\subset\{f\in\mathscr{D}_{loc}(\omega);\delta_f(\omega)<+\infty\}$; but it is not clear if these spaces coincide.

PROPOSITION 7.3. *If ω is a P-domain and σ is a signed measure on ω such that $U_\omega^{|\sigma|}$ is a potential and $\sigma|\omega'\in\mathscr{M}_E(\omega')$ for each PC-domain ω' with $\bar{\omega}'\subset\omega$, then $U_\omega^\sigma\in\mathscr{D}_{loc}(\omega)$.*

PROOF. By Proposition 6.1, $U_{\omega'}^\sigma\in\mathscr{D}_0(\omega')$ for any PC-domain ω' such that $\bar{\omega}'\subset\omega$. Hence $U_\omega^\sigma\in\mathscr{D}_0(\omega')+\mathscr{H}(\omega')\subset\mathscr{D}_{loc}(\omega')$ for such ω'. It then follows that $U_\omega^\sigma\in\mathscr{D}_{loc}(\omega)$.

7.3. The space $\mathscr{S}_{E,loc}(\omega)$ and its lattice structure

For a PB-domain ω, we consider the spaces

$$\mathscr{P}_E(\omega)=\{f;f=U_\omega^\sigma \text{ q.e. on } \omega \text{ with } \sigma\in\mathscr{M}_E(\omega)\}$$

and

$$\mathscr{S}_E(\omega)=\mathscr{H}_E(\omega)+\mathscr{P}_E(\omega)$$

(cf. [9, §6.4], where $\mathscr{P}_E$ is denoted by $\mathbf{Q}_E$). $\mathscr{P}_E(\omega)$ is a subspace of $\mathscr{D}_0(\omega)$ (Proposition 6.1), and hence $\mathscr{S}_E(\omega)$ is a subspace of $\mathscr{D}(\omega)$. For an open set ω in Ω, let

$$\mathscr{S}_{E,loc}(\omega)=\left\{f;\begin{array}{l}\text{for any PC-domain } \omega' \text{ such that } \bar{\omega}'\subset\omega,\\ f|\omega'\in\mathscr{S}_E(\omega')\end{array}\right\}.$$

Obviously, $\mathscr{B}_{loc}(\omega)\subset\mathscr{S}_{E,loc}(\omega)\subset\mathscr{D}_{loc}(\omega)$. Furthermore, by using Proposition

6.1 and Lemma 7.3, we can show that $\mathscr{S}_E(\omega)\subset\mathscr{S}_{E,\mathrm{loc}}(\omega)$ for a PB-domain ω (cf. the proof of Proposition 7.3).

THEOREM 7.2 (cf. [9, Theorem 6.3 and its corollary]). *The spaces $\mathscr{P}_E(\omega)$ and $\mathscr{S}_E(\omega)$ for a PB-domain ω and $\mathscr{S}_{E,\mathrm{loc}}(\omega)$ for an open set ω are vector lattices with respect to the max. and min. operations and*

$$\delta_{|f|} = \delta_f$$

for any $f\in\mathscr{S}_{E,\mathrm{loc}}(\omega)$.

PROOF. Let ω be a PB-domain and $f\in\mathscr{S}_E(\omega)$. By definition, $f=u+f_0$ with $u\in\mathscr{H}_E(\omega)$ and $f_0\in\mathscr{P}_E(\omega)$. By Theorem 5.1, $u_1\equiv u\vee_\omega 0$ and $u_2\equiv(-u)\vee_\omega 0$ exist and belong to $\mathscr{H}_E(\omega)$. Let $\tau=\sigma_{u_1-\max(u,0)}$. By Lemma 5.1, we see that $\tau\in\mathscr{M}_E^+(\omega)$. Note that $u_1=\max(u,0)+U_\omega^\tau$ and $u_2=\max(-u,0)+U_\omega^\tau$. Put

$$p=\min(U_\omega^{\sigma^+}+u_1,\ U_\omega^{\sigma^-}+u_2),$$

where $\sigma\equiv\sigma_{f_0}=\sigma_f$. Then, p is non-negative superharmonic on ω and $p\leqq U_\omega^{|\sigma|}+U_\omega^\tau$, so that p is a potential on ω. Since $|\sigma|$, $\tau\in\mathscr{M}_E^+(\omega)$, it follows that $p\in\mathscr{P}_E(\omega)$. Hence

$$|f|=u_1+u_2+U_\omega^{|\sigma|}-2p\in\mathscr{S}_E(\omega).$$

If, in particular, $f\in\mathscr{P}_E(\omega)$, then $u=0$, so that $|f|=U_\omega^{|\sigma|}-2p\in\mathscr{P}_E(\omega)$. Thus, $\mathscr{P}_E(\omega)$ and $\mathscr{S}_E(\omega)$ are vector lattices.

Now, for the above f and $\sigma=\sigma_f$, choose $\{\mu_n\}$ and $\{\nu_n\}$ in $\mathscr{M}_{BC}^+(\omega)$ such that $U_\omega^{\mu_n}\uparrow U_\omega^{\sigma^+}$ and $U_\omega^{\nu_n}\uparrow U_\omega^{\sigma^-}$ (cf. Lemma 1.5). Put $f_n=u+U_\omega^{\mu_n-\nu_n}$ and $p_n=\min(U_\omega^{\mu_n}+u_1,U_\omega^{\nu_n}+u_2)$, $n=1,2,\dots$. As above, each p_n is a potential and $p_n\uparrow p$. Since

$$|f_n|=u_1+u_2+U_\omega^{\mu_n+\nu_n},$$

we have

$$|f|-|f_n|=(U_\omega^{\sigma^+}-U_\omega^{\mu_n})+(U_\omega^{\sigma^-}-U_\omega^{\nu_n})-2(p-p_n)$$

and

$$f-f_n=(U_\omega^{\sigma^+}-U_\omega^{\mu_n})-(U_\omega^{\sigma^-}-U_\omega^{\nu_n}).$$

By Corollary 2 to Theorem 1.1, $I_\omega(\sigma^+-\mu_n)\to 0$, $I_\omega(\sigma^--\nu_n)\to 0$ and $I_\omega(\sigma_p-\sigma_{p_n})\to 0$ $(n\to\infty)$. Thus, Proposition 6.1 and Theorem 6.3 imply that $\delta_{|f|-|f_n|}(\omega)\to 0$ and $\delta_{f-f_n}(\omega)\to 0$ $(n\to\infty)$. Since $f_n\in\mathscr{B}_{\mathrm{loc}}(\omega)$ and f_n is continuous, $\delta_{|f_n|}=\delta_{f_n}$ by the corollary to Lemma 4.2. Hence we conclude that $\delta_{|f|}=\delta_f$ on ω.

Now the assertions for $f\in\mathscr{S}_{E,\mathrm{loc}}(\omega)$ are easily verified.

REMARK 7.2. The above proof shows that $\mathscr{H}_{D'}(\omega)+\mathscr{P}_E(\omega)$ is also a vector lattice for a PB-domain ω.

COROLLARY. *If $f, g\in\mathscr{S}_{E,\mathrm{loc}}(\omega)$, then*

$$\delta_{\max(f,g)}+\delta_{\min(f,g)} = \delta_f+\delta_g;$$

in particular, if c is a constant, then

$$\delta_{\max(f,c)}+\delta_{\min(f,c)} = \delta_f .$$

As an application of Theorem 7.2 (or its corollary), we here prove

THEOREM 7.3. *Let ω be any domain in Ω. For $f\in\mathscr{D}_{\mathrm{loc}}(\omega)$, $\delta_f=0$ if and only if $f\equiv$ const. q.e. on ω.*

PROOF. The "if" part is trivial (cf. Theorem 4.1). We shall show the "only if" part. Let ω' be any PC-domain such that $\bar{\omega}'\subset\omega$. By Proposition 7.1, f is $|\pi|$-square-integrable on ω'. Hence, Lemma 1.10 implies that $f\pi\in\mathscr{M}_E(\omega')$, so that $p_0\equiv U_{\omega'}^{f\pi}$ belongs to $\mathscr{P}_E(\omega')\subset\mathscr{D}_0(\omega')$. It follows from Theorem 6.3 that

$$\delta_{[p_0,p]}(\omega')+\int_{\omega'}p_0 p\,d\pi = \int_{\omega'}pf\,d\pi$$

for any $p\in\mathscr{D}_0(\omega')$. Since $\delta_f=0$ by assumption, $\delta_{[f,p]}(\omega')=0$. Hence we have

$$\delta_{[p_0-f,p]}(\omega')+\int_{\omega'}(p_0-f)p\,d\pi = 0$$

for all $p\in\mathscr{D}_0(\omega')$. Then, Theorem 6.4 implies that $f-p_0=u$ q.e. on ω' with $u\in\mathscr{H}_E(\omega')$, i.e., $f|\omega'\in\mathscr{S}_E(\omega')$. Therefore $f\in\mathscr{S}_{E,\mathrm{loc}}(\omega)$. For $\alpha>0$, put $f_\alpha^+=\min(\max(f,\alpha),0)$ and $f_\alpha^-=\min(\max(-f,\alpha),0)$. By the above corollary, we see that $\delta_{f_\alpha^+}=0$ and $\delta_{f_\alpha^-}=0$ for each $\alpha>0$. Since $f\in\mathscr{S}_{E,\mathrm{loc}}(\omega)$, we see that f_α^+ and f_α^- are equal q.e. to functions in $\mathscr{B}_{\mathrm{loc}}(\omega)$. Hence, Theorem 4.1 implies that $f_\alpha^+\equiv$const. q.e. and $f_\alpha^-\equiv$const. q.e. on ω for each $\alpha>0$. This is possible only when $f\equiv$const. q.e. on ω.

7.4. Lattice structure of $\mathscr{D}_{\mathrm{loc}}(\omega)$

Finally, we study the lattice structure of $\mathscr{D}_{\mathrm{loc}}(\omega)$.

THEOREM 7.4 (cf. [9, Theorem 6.4 and its corollary]). *The spaces $\mathscr{D}_0(\omega)$ and $\mathscr{H}_E(\omega)+\mathscr{D}_0(\omega)$ for a PB-domain ω and $\mathscr{D}_{\mathrm{loc}}(\omega)$ for an open set ω are vector* *lattices with respect to the max. and min. operations and*

$$\delta_{|f|} \leqq \delta_f$$

for any $f \in \mathscr{D}_{\mathrm{loc}}(\omega)$.

PROOF. Let ω be a PB-domain and $f = u + f_0$ with $u \in \mathscr{H}_E(\omega)$ and $f_0 \in \mathscr{D}_0(\omega)$. There is a sequence $\{f_n\}$ in $\mathscr{P}_{BC}(\omega)$ such that $f_n \to f_0$ q.e. on ω and $\|f_n - f_0\|_{I,\omega} \to 0$ $(n \to \infty)$. If μ is a measure in $\mathscr{M}_E^+(\omega)$ and $S(\mu)$ is compact in ω, then by Lemma 6.6,

$$\int_\omega |f_0 - f_n| d\mu \leqq \|f_n - f_0\|_{I,\omega} \cdot I_\omega(\mu)^{1/2} \to 0 \quad (n \to \infty).$$

Hence, u being μ-integrable,

$$\begin{aligned} \left| \int_\omega \{|f| - |u + f_n|\} d\mu \right| &\leqq \int_\omega |f - (u + f_n)| d\mu \\ &= \int_\omega |f_0 - f_n| d\mu \to 0 \qquad (n \to \infty). \end{aligned}$$

Therefore,

$$\int_\omega |u + f_n| d\mu \to \int_\omega |f| d\mu \qquad (n \to \infty). \tag{7.3}$$

Put $v = u \vee_\omega (-u)$ and $g_n = |u + f_n| - v$ $(n = 1, 2, \dots)$. Since $u + f_n \in \mathscr{S}_E(\omega)$, $|u + f_n| \in \mathscr{S}_E(\omega)$ and $\delta_{|u+f_n|} = \delta_{u+f_n}$ by Theorem 7.2. Hence

$$\begin{aligned} \delta_{g_n}(\omega) &\leqq 2\{\delta_{|u+f_n|}(\omega) + \delta_v(\omega)\} \\ &= 2\{\delta_{u+f_n}(\omega) + \delta_v(\omega)\} \\ &\leqq 4\delta_{f_n}(\omega) + 4\delta_u(\omega) + 2\delta_v(\omega). \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_\omega g_n^2 d|\pi| &\leqq 2\left\{\int_\omega (u + f_n)^2 d|\pi| + \int_\omega v^2 d|\pi|\right\} \\ &\leqq 4\int_\omega f_n^2 d|\pi| + 4\int_\omega u^2 d|\pi| + 2\int_\omega v^2 d|\pi|. \end{aligned}$$

Hence, using Lemma 4.3 (or Theorem 6.3) and Theorem 5.1, we obtain

$$\delta_{g_n}(\omega) + \int_\omega g_n^2 d|\pi| \leqq 4(2\beta_\omega - 1)\|f_n\|_{I,\omega}^2 + 6\beta_\omega \|u\|_{E,\omega}^2. \tag{7.4}$$

Since $g_n \in \mathscr{S}_E(\omega)$ and $|g_n| \leqq |f_n| + (v - |u|)$, we see that $g_n \in \mathscr{P}_E(\omega)$ $(\subset \mathscr{D}_0(\omega))$. $\{\|g_n\|_{I,\omega}\}$ is bounded by virtue of (7.4). Hence, we can choose a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ converging to a $g \in \mathscr{D}_0(\omega)$ weakly in $\mathscr{D}_0(\omega)$ as a Hilbert space. By Lemma 6.6, the linear functional $f \to \int_\omega f d\mu$ is continuous on $\mathscr{D}_0(\omega)$. Therefore

$$\int_\omega g_{n_k} d\mu \to \int_\omega g d\mu \qquad (k \to \infty).$$

This, together with (7.3), implies that

$$\int_\omega |f| d\mu = \int_\omega (g+v) d\mu.$$

Both $|f|$ and $g+v$ are quasi-continuous on ω. Therefore, applying Lemma 6.4, we conclude that

$$|f| = g+v \quad \text{q.e. on } \omega,$$

which means that $|f| \in \mathscr{H}_E(\omega) + \mathscr{D}_0(\omega)$. If in particular $|f| \in \mathscr{D}_0(\omega)$, then $v=0$, and hence $|f| \in \mathscr{D}_0(\omega)$. Thus, $\mathscr{D}_0(\omega)$ and $\mathscr{H}_E(\omega) + \mathscr{D}_0(\omega)$ are vector lattices with respect to the max. and min. operations.

Furthermore, since $g_{n_k} \to g$ weakly in $\mathscr{D}_0(\omega)$,

$$\|g\|_{I,\omega} \leqq \liminf_{k \to \infty} \|g_{n_k}\|_{I,\omega}.$$

Then, it follows from Theorem 6.3 that

$$\begin{aligned} \delta_{|f|}(\omega) + \int_\omega f^2 d\pi &= \|g\|_{I,\omega}^2 + \delta_v(\omega) + \int_\omega v^2 d\pi \\ &\leqq \liminf_{k \to \infty} \left\{ \|g_{n_k}\|_{I,\omega}^2 + \delta_v(\omega) + \int_\omega v^2 d\pi \right\} \\ &= \liminf_{k \to \infty} \left\{ \delta_{|u+p_k|}(\omega) + \int_\omega (u+p_k)^2 d\pi \right\} \\ &= \liminf_{k \to \infty} \left\{ \delta_{u+p_k}(\omega) + \int_\omega (u+p_k)^2 d\pi \right\}, \end{aligned}$$

where $p_k \equiv f_{n_k}$. Theorem 6.3 also implies that $\delta_{u+p_k}(\omega) \to \delta_{u+f_0}(\omega) = \delta_f(\omega)$ and $\int_\omega (u+p_k)^2 d\pi \to \int_\omega (u+f_0)^2 d\pi = \int_\omega f^2 d\pi$. Therefore,

$$\delta_{|f|}(\omega) + \int_\omega f^2 d\pi \leqq \delta_f(\omega) + \int_\omega f^2 d\pi,$$

that is $\delta_{|f|}(\omega) \leqq \delta_f(\omega)$. Now the last assertion of the theorem is easily verified (cf. the last part of the proof of Proposition 3.7).

REMARK 7.3. The above proof and Remark 7.2 show that $\mathscr{H}_{D'}(\omega)+\mathscr{D}_0(\omega)$ is also a vector lattice for a PB-domain ω.

REMARK 7.4. In the classical case, $\delta_{|f|}=\delta_f$ holds for every $f\in\mathscr{D}_{\mathrm{loc}}(\omega)$. We fail to verify it in our general situation.

COROLLARY. *If f, $g\in\mathscr{D}_{\mathrm{loc}}(\omega)$, then*

$$\delta_{\max(f,g)}+\delta_{\min(f,g)}\leqq\delta_f+\delta_g.$$

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