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Pseudo-coalescent and Locally Pseudo-coalescent Classes of Lie Algebras

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Introduction

In the recent study of infinite-dimensional Lie algebras, an important role has been played by the concepts of coalescent and locally coalescent classes of Lie algebras introduced in [7, 9]. These classes have been investigated by R. K. Amayo [1–4], R. K. Amayo and I. Stewart [5], S. Tôgô [10, 11], and S. Tôgô and N. Kawamoto [12]. Corresponding to the concept of coalescency, we have introduced that of pseudo-coalescency and shown several analogous results in [8]. We furthermore introduce the new concept, local pseudo-coalescency, which corresponds to local coalescency. We say that a class $\mathfrak{X}$ is locally pseudo-coalescent if and only if whenever H is an $\mathfrak{X}$-subideal and K is an $\mathfrak{X}$-weak ideal of a Lie algebra L, then for every finite subset F of the join $J=\langle H, K\rangle$ there exists an $\mathfrak{X}$-weak ideal X of L such that $F\subseteq X\leq J$. Any pseudo-coalescent class of Lie algebras is obviously locally pseudo-coalescent. We ask whether the results for local coalescency hold analogously for local pseudo-coalescency. The purpose of this paper is to investigate the properties of locally pseudo-coalescent classes and to show that several classes of Lie algebras are pseudo-coalescent.

In Section 2, we shall ask whether the join of a subideal and a weak ideal is a weak ideal. In Section 3, we shall show the pseudo-coalescency of $\mathfrak{C}$, $\mathfrak{G}$ and other classes, which are coalescent [1]. We shall also show that the join of a solvable (resp. nilpotent) subideal and a solvable (resp. nilpotent) weak ideal is a solvable (resp. nilpotent) weak ideal if these are permutable. In Section 4, we shall show that if $\mathfrak{X}$ is a Q-closed and locally pseudo-coalescent subclass of $(\mathrm{E}\mathfrak{A})_{(\omega)}$, then the class $\mathfrak{X}_{(\omega)}$ is locally pseudo-coalescent. We shall also show that for any classes $\mathfrak{X}$ and $\mathfrak{Y}$ such that $\mathfrak{X}\leq\mathfrak{Y}\leq\mathrm{M}\mathfrak{X}$, $\mathfrak{X}$ is locally pseudo-coalescent if and only if so is $\mathfrak{Y}$, which is analogous to Theorem 3.2 in [12]. Using these results we shall show that the following classes are locally pseudo-coalescent over a field of characteristic 0: the class $\mathfrak{N}$ of nilpotent Lie algebras, the class $\mathfrak{N}_{(\omega)}$, the class $\mathfrak{D}$ of Lie algebras L such that every subalgebra of L is a subideal, the class $\mathfrak{F}_t$ of Fitting algebras, the class $\mathfrak{B}$ of Baer algebras.

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§1. Preliminaries

We shall be concerned with Lie algebras over a field Φ which are not necessarily finite-dimensional. Throughout this paper, L will be an arbitrary Lie algebra over a field Φ and $\mathfrak{X}$ will be an arbitrary class of Lie algebras over Φ, that is, an arbitrary collection of Lie algebras over Φ such that $(0) \in \mathfrak{X}$ and if $H \in \mathfrak{X}$ and $H \simeq K$, then $K \in \mathfrak{X}$, unless otherwise specified.

We employ the notation and terminology in [5, 8, 12]. By $H \leq L$, $H \lhd L$, H si L and $H \lhd^m L$ we mean respectively that H is a subalgebra, an ideal, a subideal and an m-step subideal of L. H is a weak ideal of L provided that H is a subalgebra of L and $L(\mathrm{ad}\, H)^n \subseteq H$ for some $n \geq 0$, and we write H wi L, more precisely H n-wi L [8]. A Lie algebra (resp. a subalgebra, an ideal, a subideal, a weak ideal of L) belonging to $\mathfrak{X}$ is called an $\mathfrak{X}$-algebra (resp. an $\mathfrak{X}$-subalgebra, an $\mathfrak{X}$-ideal, an $\mathfrak{X}$-subideal, an $\mathfrak{X}$-weak ideal of L). $\mathfrak{X}$ is pseudo-coalescent provided for any $\mathfrak{X}$-subideal H and $\mathfrak{X}$-weak ideal K of L $\langle H, K \rangle$ is an $\mathfrak{X}$-weak ideal of L [8]. For any class $\mathfrak{X}$, $\mathfrak{X}_{(\omega)}$ (resp. $\mathfrak{X}_\omega$) denotes the class of Lie algebras L such that $L/L^{(\omega)} \in \mathfrak{X}$ (resp. $L/L^\omega \in \mathfrak{X}$), where $L^{(\omega)} = \bigcap_{i=0}^{\infty} L^{(i)}$, $L^\omega = \bigcap_{i=1}^{\infty} L^i$ [10].

$\mathfrak{F}$, $\mathfrak{A}$, $\mathfrak{N}$, $\mathrm{E}\mathfrak{A}$ and $\mathfrak{G}$ denote respectively the classes of finite-dimensional, abelian, nilpotent, solvable and finitely generated Lie algebras.

We recall a construction of Hartley [7] p. 265–266. Let L be a Lie algebra over a field Φ of characteristic 0. Let Φ_0 be the field of formal power series

$$a = \sum_{i=m}^{\infty} a_i t^i, \qquad a_i \in \Phi, \quad m \in \boldsymbol{Z},$$

and $L^\dagger$ be the Lie algebra over Φ_0 consisting of all formal power series

$$x = \sum_{i=m}^{\infty} x_i t^i, \qquad x_i \in L, \quad m \in \boldsymbol{Z}.$$

For $H \leq L$ $H^\dagger$ is the set of all elements $x \in L^\dagger$ with $x_i \in H$ for all i. Then $H^\dagger \leq L^\dagger$. For $M \leq L^\dagger$ $M^\downarrow$ is the set of first coefficients of elements of M, together with 0. Then $M^\downarrow \leq L$. We have several properties:

LEMMA 1.1. *Let Φ be a field of characteristic* 0.

(a) *If $H \lhd^m L$, then $H^\dagger \lhd^m L^\dagger$.*

(b) *If H m-*wi *L, then $H^\dagger$ m-*wi *$L^\dagger$.*

(c) *$L \in \mathfrak{N}$ (resp. $\mathrm{E}\mathfrak{A}$) if and only if $L^\dagger \in \mathfrak{N}$ (resp. $\mathrm{E}\mathfrak{A}$).*

(d) *If H is finite-dimensional over Φ, then $H^\dagger$ is finite-dimensional over Φ_0.*

(e) *If $M \lhd^n L^\dagger$, then $M^\downarrow \lhd^n L$.*

(f) *If M n-*wi *$L^\dagger$, then $M^\downarrow$ n-*wi *L.*

(g) *If* $M \in \mathfrak{N}$ *(resp.* $\mathrm{E}\mathfrak{A}$*), then* $M^{\downarrow} \in \mathfrak{N}$ *(resp.* $\mathrm{E}\mathfrak{A}$*).*

PROOF. (a), (c), (d), (e) and (g) have been proved in [3, 5].

(b) Let H m-wi L. Let $x = \sum_{i=n}^{\infty} x_i t^i \in L^{\uparrow}$ and $y_r = \sum_{i(r)=n(r)}^{\infty} y_{i(r)} t^{i(r)} \in H^{\uparrow}$, where $r = 1, 2, \dots, m$. Then we have

$$[x, y_1, \dots, y_m] = \sum z_i t^i, \quad z_i = \sum_{j+j(1)+\cdots+j(m)=i} [x_j, y_{j(1)}, \dots, y_{j(m)}].$$

Since H m-wi L, $z_i \in H$ and so $[x, y_1, \dots, y_m] \in H^{\uparrow}$. Therefore $L^{\uparrow}(\operatorname{ad} H^{\uparrow})^m \subseteq H^{\uparrow}$, i.e., $H^{\uparrow}$ m-wi $L^{\uparrow}$.

(f) Let M n-wi $L^{\uparrow}$. Let x_0 be a non-zero element of L and y_0^i be a non-zero element of $M^{\downarrow}$, where $i = 1, 2, \dots, n$. Then there exist $x \in L^{\uparrow}$ and $y^1, y^2, \dots, y^n \in M$ such that x_0 is the first coefficient of $x = t^r \sum_{i=0}^{\infty} x_i t^i \in L^{\uparrow}$ and y_0^j is the first coefficient of $y^j = t^{r(j)} \sum_{i=0}^{\infty} y_i^j t^i \in M$, where $j = 1, 2, \dots, n$. Since M n-wi $L^{\uparrow}$, $z = [x, y^1, y^2, \dots, y^n] \in M$. Therefore $[x_0, y_0^1, \dots, y_0^n]$ is zero or the first coefficient of z and then it is in $M^{\downarrow}$. Thus $M^{\downarrow}$ n-wi L. The proof is complete.

Let d be a derivation of L. Define a mapping $\exp(td)$ of $L^{\uparrow}$ as follows: for $x = \sum x_i t^i \in L^{\uparrow}$,

$$x^{\exp(td)} = \sum u_r t^r, \quad \text{where} \quad u_r = \sum_{i+j=r} x_i d^j / j!.$$

Then $\exp(td)$ is a Lie automorphism of $L^{\uparrow}$. If A is a subspace of L, we denote by $\exp(tA)$ the group of automorphisms of $L^{\uparrow}$ generated by the elements $\exp(t \operatorname{ad}(x))$ with $x \in A$.

§2. The join of a subideal and a weak ideal

In this section, we shall discuss by what conditions the join of a pair of a subideal and a weak ideal is a weak ideal.

We begin with the following

LEMMA 2.1. *If* H m-wi L, K n-wi L *and* $H \lhd J = \langle H, K \rangle$, *then* J l-wi L, *where* $l = mn$.

This is Lemma 2.3 in [8].

Let $H, K \leq L$. H and K are said to be permutable (or H permutes with K) [1] if $\langle H, K \rangle = H + K$, i.e., the subalgebra generated by H and K equals their vector space sum. Trivially if $H \lhd L$, then H permutes with every subalgebra K of L. An ordered pair (H, K) is a modular pair under L whenever $H \leq M$ si L then $M \cap \langle H, K \rangle = \langle H, M \cap K \rangle$.

Then we have the following

LEMMA 2.2. *Let* H si L, K wi L *and* $J = \langle H, K \rangle$. *If*

(1) *H and K are permutable, or*

(2) *(H, K) forms a modular pair under L,*

then $J \operatorname{wi} L$.

The proof may be a slight modification of those of Lemma 2.3 in [1] and Lemma 2.34 in [2].

Let $\mathfrak{X}$ be a class of Lie algebras and $H \leq L$. We say that H is an $\mathfrak{X}$-acceptable subalgebra of L [5] if there is an ideal H_0 of H such that $H/H_0 \in \mathfrak{X}$ and $[H_0, L] \subseteq H$. Evidently if $H \leq K \leq L$ and H is an $\mathfrak{X}$-acceptable subalgebra of L, then H is an $\mathfrak{X}$-acceptable subalgebra of K.

Replacing subideals by weak ideals in Lemma 4.3.1 in [5], we have the following

LEMMA 2.3. (a) *If H is an $\mathfrak{X}$-acceptable weak ideal of L and $\mathfrak{X}$ is* Q-*closed, then H is an $\mathfrak{X} \cap \mathfrak{N}$-acceptable weak ideal of L.*

(b) *If L is a Lie algebra over a field of characteristic* 0 *and H is a $\mathfrak{G}$-acceptable weak ideal of L, then $H^{\dagger}$ is a $\mathfrak{G}$-acceptable weak ideal of $L^{\dagger}$.*

By the above lemma, we can show the following

LEMMA 2.4. *Let L be a Lie algebra over a field of characteristic* 0. *Let K be a $\mathfrak{G}$-acceptable weak ideal of L and H be a subspace of L. Then there exist finitely many automorphisms $\alpha_1, \alpha_2, \dots, \alpha_r \in \exp(tK)$ such that $<H, K> = M^{\downarrow} + K$, where $M = <H^{\dagger\alpha_1}, \dots, H^{\dagger\alpha_r}>$.*

The proof of this lemma is similar to that of the statement before Theorem 4.3.3 in [5].

Then we can prove the analogous results of Theorem 4.3.3 in [5].

THEOREM 2.5. *Let L be a Lie algebra over a field of characteristic* 0. *If $H \operatorname{si} L$, $K \operatorname{wi} L$ and H, K are $\mathfrak{G}$-acceptable subalgebras of L, then $J = <H, K>$ is a $\mathfrak{G}$-acceptable weak ideal of L.*

PROOF. Since the join of finitely many $\mathfrak{G}$-acceptable subalgebras of a Lie algebra is a $\mathfrak{G}$-acceptable subalgebra of that algebra (Lemma 4.3.2 in [5]), J is a $\mathfrak{G}$-acceptable subalgebra of L. So it is enough to show that $J \operatorname{wi} L$. By Lemma 2.3 H and K are $\mathfrak{F} \cap \mathfrak{N}$-acceptable subalgebras of L. Let $H \lhd^m L$. If $m \leq 1$, then $J \operatorname{wi} L$ by Lemma 2.1. We may assume that $m > 1$. Let M be the subalgebra of $L^{\dagger}$ as in the above lemma so that $J = M^{\downarrow} + K$. If $\alpha \in \{\alpha_1, \alpha_2, \dots, \alpha_r\}$, then $H^{\dagger\alpha}$ is a $\mathfrak{G}$-acceptable m-step subideal of $L^{\dagger}$ by Lemmas 1.1 and 2.3. From the fact that the join of finitely many $\mathfrak{G}$-acceptable subideals of a Lie algebra is a $\mathfrak{G}$-acceptable subideal of that algebra (Theorem 4.3.3 in [5]), $M = <H^{\dagger\alpha_1}, \dots, H^{\dagger\alpha_r}>$ is a $\mathfrak{G}$-acceptable subideal of $L^{\dagger}$ and so $M^{\downarrow} \operatorname{si} L$. Therefore, by Lemma 2.2 $J = M^{\downarrow} + K \operatorname{wi} L$. This completes the proof.

Let $H \lhd^m L$, $K \operatorname{wi} L$ and $J = <H, K>$. Then we can prove that J is a weak ideal of L for $m=2$, but for $m=3$ J is not necessarily a weak ideal (by the example in [7]).

PROPOSITION 2.6. *Let L be a Lie algebra over Φ and let $H \lhd^m L$. Suppose that*

(1) $m=2$, *or*

(2) $m=3$, $H = H_3 \lhd H_2 \lhd H_1 \lhd L$ *and* H_2/H_3, H_1/H_2 *are at most* 1-*dimensional. Then for any* $K \operatorname{wi} L$, $<H, K> \operatorname{wi} L$.

PROOF. Suppose that $H \lhd M \lhd L$. By Proposition 2.1.10 in [5], we have $<H^K> = H^K \lhd M$ and then $<H^K> \lhd K+M$. Let K n-wi L. Then $(K+M)$ n-wi L by Lemma 2.1 and K n-wi$(K+M)$. Therefore $<H, K> = K + <H^K>$ n-wi$(K+M)$ and so $<H, K>$ $2n$-wi L. Thus the case (1) is proved.

Next we assume (2). By the case (1), we have that $H_3 \lhd H_2 \lhd H_1 \cap <H_2, K> \lhd <H_2, K> \operatorname{wi} L$. So it is enough to show that $<H_3, K> \operatorname{wi} <H_2, K>$. Hence we may assume that $<H_2, K> = L$. Since $H_3 \leq H_2 \cap <H_3, K> \leq H_2$ and $\dim(H_2/H_3) \leq 1$, we have $H_2 \leq <H_3, K>$ or $H_3 = H_2 \cap <H_3, K>$. In the first case, we have $<H_3, K> = <H_2, K> = L$. In the second case, we have $H_2 \leq <H_2, H_1 \cap <H_3, K>> \leq H_1$. Since $\dim(H_1/H_2) \leq 1$, we have $H_1 = <H_2, H_1 \cap <H_3, K>>$ or $H_2 = <H_2, H_1 \cap <H_3, K>>$. If $H_1 = <H_2, H_1 \cap <H_3, K>>$, then $H_3 \lhd H_1$ and so $<H_3, K> \operatorname{wi} L$ from the case (1). If $H_2 = <H_2, H_1 \cap <H_3, K>>$, then $H_2 \geq H_1 \cap <H_3, K>$ and so $H_3 \leq H_1 \cap <H_3, K> \leq H_2$. As $\dim(H_2/H_3) \leq 1$, we have $H_2 = H_1 \cap <H_3, K>$ or $H_3 = H_1 \cap <H_3, K>$. In the first case, we have $H_2 \leq <H_3, K>$ and so $L = <H_2, K> \leq <H_3, K>$, i.e., $<H_3, K> = L$. In the second case, we have $[H_3, K] \subseteq H_1 \cap <H_3, K> = H_3$ since $H_1 \lhd L$. Since $H_3 \lhd H_2$, $H_3 \lhd <H_2, K> = L$ and so $<H_3, K> \operatorname{wi} L$ by Lemma 2.1. Thus the case (2) is proved. The proof is complete.

In the above proposition, (1) is an analogue of the last statement of Proposition 2.1.10 in [5] and (2) is a modification of Theorem 3.3 in [6].

LEMMA 2.7. *If $H \lhd^m L$ and K n-wi L, then $<H^m, K^n> \operatorname{wi} L$ for $0 < n \leq m$.*

PROOF. We can prove it in the same way as the statement before Lemma 2.31 in [2].

§3. The pseudo-coalescency of $\mathfrak{C}$ and $\mathfrak{G}$

Let $\mathfrak{C}$ be the class of Lie algebras L such that L^2 has finite codimension in L. It easily follows that if $L \in \mathfrak{C}$, then every L^m of the lower central series of L has finite codimension in L. Clearly $\mathfrak{C}$ contains the classes $\mathfrak{F}$, $\mathfrak{F}_\omega$, $\mathfrak{G}$, Min, Min-$\lhd^n$ $(1 \leq n)$, Min-asc, Max, Max-$\lhd^n$ $(1 \leq n)$, Max-asc, perfect Lie algebras and simple Lie algebras. Furthermore $\mathfrak{C}$ is Q-closed and $\mathfrak{C}_{\omega} = \mathfrak{C}_{(\omega)} = \mathfrak{C}$. By Lemma 2.3, we have the following

LEMMA 3.1. *If* K n-wi$L$ *and* $K \in \mathfrak{C}$, *then* K *is a* $\mathfrak{G} \cap \mathfrak{N}$*-acceptable weak ideal of* L.

The class $\mathfrak{C}$ has been defined by R. K. Amayo [2] to prove the coalescency of $\mathfrak{G}$, which was an open question of I. Stewart [9]. We shall show the following

THEOREM 3.2. *Over fields of characteristic* 0, *the class* $\mathfrak{C}$ *is pseudo-coalescent.*

PROOF. Let H si L, K wi L, H, $K \in \mathfrak{C}$ and $J = \langle H, K \rangle$. Then H and K are $\mathfrak{G}$-acceptable subalgebras of L by Lemma 3.1 and so J is a $\mathfrak{G}$-acceptable weak ideal of L by Theorem 2.5. We have

$$J/J^2 = (H + J^2/J^2) + (K + J^2/J^2).$$

Since H, $K \in \mathfrak{C}$, it follows that $H + J^2/J^2$ is an $\mathfrak{F}$-subideal and $K + J^2/J^2$ is an $\mathfrak{F}$-weak ideal of J/J^2. Therefore, by the pseudo-coalescency of $\mathfrak{F}$ (Theorem 4.4 in [8]) $J/J^2 \in \mathfrak{F}$ and so $J \in \mathfrak{C}$. This completes the proof.

We say that a class $\mathfrak{X}$ is p-persistent if in any Lie algebra the join of an $\mathfrak{X}$-subideal and an $\mathfrak{X}$-weak ideal is always an $\mathfrak{X}$-algebra. Evidently any pseudo-coalescent class is p-persistent and any p-persistent class is persistent.

By the definition we have the following

PROPOSITION 3.3. *Let* $\mathfrak{X}$ *be a p-persistent class. Then for any pseudo-coalescent class* $\mathfrak{Y}$, $\mathfrak{X} \cap \mathfrak{Y}$ *is pseudo-coalescent.*

COROLLARY 3.4. *Over fields of characteristic* 0, *the classes* $\mathfrak{G}$, $\mathfrak{G}_{(\omega)}$ *and* $\mathfrak{G}_{\omega}$ *are pseudo-coalescent.*

PROOF. Obviously, $\mathfrak{G}$ is p-persistent and $\mathfrak{G} < \mathfrak{C}$. Therefore $\mathfrak{G}$ is pseudo-coalescent by the above proposition and so are $\mathfrak{G}_{(\omega)}$, $\mathfrak{G}_{\omega}$ by Theorem 4.1 in [8].

Let $\mathfrak{X}$ be {I, Q, E}-closed. Then $\mathfrak{X}$ is coalescent if and only if $\mathfrak{X} \cap \text{E}\mathfrak{A}$ is coalescent (Theorem E in [4]). We shall consider a necessary condition for $\mathfrak{X} \cap \text{E}\mathfrak{A}$ to be pseudo-coalescent. First we show the following lemma which is an analogue of Theorem (3) in [4].

LEMMA 3.5. *Let* $\mathfrak{X}$ *be* {I, Q, E}*-closed. Let* H *be an* $\mathfrak{X}$*-subideal and* K *be an* $\mathfrak{X}$*-weak ideal of* L. *If* H, K *are permutable or the ordered pair* (H, K) *forms a modular pair under* L, *then* $J = \langle H, K \rangle$ *is an* $\mathfrak{X}$*-weak ideal of* L.

PROOF. By Lemma 2.2, for both cases J is a weak ideal of L and so it is enough to show that $J \in \mathfrak{X}$.

Suppose that H and K are permutable. Let $H \lhd^m L$ and we induct on m. If $m=0$, the result is trivial. If $m=1$, then $H \lhd L$ and $J/H \simeq K/H \cap K \in \mathrm{Q}\mathfrak{X}=\mathfrak{X}$. Since $\mathfrak{X}=\mathrm{E}\mathfrak{X}$, $J \in \mathfrak{X}$. Assume that $m>1$ and the result is true for $m-1$. Let $H_1=\langle H^L \rangle$. Then $H_1 \cap J = H + H_1 \cap K$, which implies H permutes with $H_1 \cap K$. Since $H_1 \cap K \lhd K \in \mathfrak{X}$, $H_1 \cap K \in \mathrm{I}\mathfrak{X}=\mathfrak{X}$ and $H_1 \cap K$ wi H_1. Now $H \lhd^{m-1} H_1$. Hence, by the inductive assumption $H_1 \cap J \in \mathfrak{X}$. Since $H_1 \cap J \lhd J$, $J = H_1 \cap J + K$ and $J/H_1 \cap J \simeq K/K \cap H_1 \in \mathrm{Q}\mathfrak{X}=\mathfrak{X}$. As $\mathfrak{X}=\mathrm{E}\mathfrak{X}$, we have $J \in \mathfrak{X}$.

The case which (H, K) forms a modular pair under L is similarly proved.

The following classes are $\{\mathrm{I}, \mathrm{Q}, \mathrm{E}\}$-closed:

$$\mathfrak{F},\ \mathrm{E}\mathfrak{A},\ \text{Min},\ \text{Min-si},\ \text{Min-asc},\ \text{Max},\ \text{Max-si},\ \text{Max-asc}.$$

It follows that the join of an $\mathrm{E}\mathfrak{A}$-subideal and an $\mathrm{E}\mathfrak{A}$-weak ideal, which are permutable, is an $\mathrm{E}\mathfrak{A}$-weak ideal. The class $\mathfrak{N}$ is $\{\mathrm{I}, \mathrm{Q}\}$-closed but not E-closed. However we can prove the statement similar to Lemma 3.5 by using Lemma 3.3 in [8]: If an $\mathfrak{N}$-subideal H and an $\mathfrak{N}$-weak ideal K of L are permutable, then the join $\langle H, K \rangle$ is an $\mathfrak{N}$-weak ideal of L.

THEOREM 3.6. *Let Φ be of characteristic* 0. *If $\mathfrak{X}$ is a p-persistent subclass of $\mathfrak{C}$, then $\mathfrak{X} \cap \mathrm{E}\mathfrak{A}$ is pseudo-coalescent.*

PROOF. Let H si L, K wi L and $H, K \in \mathfrak{X} \cap \mathrm{E}\mathfrak{A}$. Put $J=\langle H, K \rangle$. Since $\mathfrak{X}$ is p-persistent, $J \in \mathfrak{X}$ and so it is enough to show that J is a solvable weak ideal of L. Since $\mathfrak{X} \leq \mathfrak{C}$, H and K are $\mathfrak{G}$-acceptable subalgebras of L by Lemma 3.1. By Lemma 2.4, there exist finitely many automorphisms $\alpha_1, \alpha_2, \dots, \alpha_r \in \exp(tK)$ such that $J=M^{\downarrow}+K$, where $M=\langle H^{\uparrow\alpha_1}, \dots, H^{\uparrow\alpha_r} \rangle$. By Lemmas 1.1 and 2.3, $H^{\uparrow}$ is a solvable $\mathfrak{G}$-acceptable subideal of $L^{\uparrow}$ and so is $H^{\uparrow\alpha_i}$ for any i. By the derived join theorem (Theorem 3.3 in [1]) and Theorem 4.3.3 in [5], M is a solvable $\mathfrak{G}$-acceptable subideal of $L^{\uparrow}$ and hence $M^{\downarrow}$ is a solvable subideal of L by Lemma 1.1. Now $M^{\downarrow}$ and K are permutable. Put $\mathfrak{X}=\mathrm{E}\mathfrak{A}$ in Lemma 3.5. Then $J=M^{\downarrow}+K$ is a solvable weak ideal of L. Therefore $\mathfrak{X} \cap \mathrm{E}\mathfrak{A}$ is pseudo-coalescent. Thus the proof is completed.

Applying Theorem 3.6 for $\mathfrak{F}$ and $\mathfrak{G}$, we have the following

COROLLARY 3.7. *Over fields of characteristic* 0 *the following classes are pseudo-coalescent*:

$$\mathrm{E}\mathfrak{A} \cap \mathfrak{F} = \mathrm{E}\mathfrak{A} \cap \mathfrak{F}_{(\omega)},\ \mathrm{E}\mathfrak{A} \cap \mathfrak{F}_{\omega},\ \mathrm{E}\mathfrak{A} \cap \mathfrak{G} = \mathrm{E}\mathfrak{A} \cap \mathfrak{G}_{(\omega)},\ \mathrm{E}\mathfrak{A} \cap \mathfrak{G}_{\omega},\ \mathrm{E}\mathfrak{A} \cap \mathfrak{C}.$$

REMARK. Let the basic field Φ be of characteristic $p>0$. By the Hartley's example in [7], any class containing $\mathfrak{A} \cap \mathfrak{F}$ is not pseudo-coalescent (see [8]). Thus all the classes in Theorem 3.2, Corollaries 3.4 and 3.7 are not pseudo-coalescent.

Next we shall show that $\mathfrak{C}$ is different from both $\mathfrak{F}_{\omega}$ and $\mathfrak{N}_{\omega}$. Let $L=A$

$+\Phi x$ be the Lie algebra over a field Φ of characteristic 0 defined in [7] as follows: A is an abelian subalgebra with basis $e_0, e_1, e_2, \dots$ and $[e_i, x]=e_{i+1}$ for all i. Obviously we have $A=L^2+\Phi e_0$ and then $L \in \mathfrak{C}$. Let $A_n=\sum_{i\geq n} \Phi e_i$. Then $[A_n, x]=A_{n+1}$ and $L^n=A_{n-1}$ $(n\geq 2)$ by induction. Therefore we have $L^\omega=\bigcap_{n=1}^{\infty} L^n=\bigcap_{n=1}^{\infty} A_n=(0)$. Since $L^n=A_{n-1}\neq(0)$ for any n, $L\not\in\mathfrak{N}$. Thus L does not belong to both $\mathfrak{F}_\omega$ and $\mathfrak{N}_\omega$.

§4. Locally pseudo-coalescent classes

In this section we shall investigate the concept corresponding to local coalescency.

Let $\mathfrak{X}$ be a class of Lie algebras. L$\mathfrak{X}$ is the class of Lie algebras L such that any finite subset of L lies inside an $\mathfrak{X}$-subalgebra of L. M$\mathfrak{X}$ is the class of Lie algebras L such that any finite subset of L lies inside an $\mathfrak{X}$-subideal of L. L and M are closure operations. Furthermore, we denote by $\tilde{\text{M}}\mathfrak{X}$ the class of Lie algebras L such that any finite subset of L lies inside an $\mathfrak{X}$-weak ideal of L. Then it is easy to verify that $\tilde{\text{M}}$ is a closure operation and $\mathfrak{X}\leq \text{M}\mathfrak{X}\leq \tilde{\text{M}}\mathfrak{X}\leq \text{L}\mathfrak{X}$. We say that a class $\mathfrak{X}$ is locally pseudo-coalescent if and only if whenever H is an $\mathfrak{X}$-subideal and K is an $\mathfrak{X}$-weak ideal of a Lie algebra L, then every finite subset F of $J=\langle H, K\rangle$ is contained in some $\mathfrak{X}$-weak ideal X of L such that $F\subseteq X\leq J$. Evidently any pseudo-coalescent class is locally pseudo-coalescent.

We begin with the following

Proposition 4.1. (a) *If $\mathfrak{X}$ is locally pseudo-coalescent, then $\mathfrak{X}\cap\mathfrak{G}$ is pseudo-coalescent.*

(b) *If $\mathfrak{X}$ and $\mathfrak{Y}$ are* s*-closed and locally pseudo-coalescent, then $\mathfrak{X}\cap\mathfrak{Y}$ is locally pseudo-coalescent.*

(c) *If $\mathfrak{X}$ is $\tilde{\text{M}}$-closed and locally pseudo-coalescent and $\mathfrak{Y}$ is pseudo-coalescent, then $\mathfrak{X}\cap\mathfrak{Y}$ is pseudo-coalescent.*

Proof. (a) The proof is immediate.

(b) Let H (resp. K) be an $\mathfrak{X}\cap\mathfrak{Y}$-subideal (resp. an $\mathfrak{X}\cap\mathfrak{Y}$-weak ideal) of a Lie algebra L. Put $J=\langle H, K\rangle$ and let F be any finite subset of J. Then there exist an $\mathfrak{X}$-weak ideal X and a $\mathfrak{Y}$-weak ideal Y of L such that $F\subseteq X\leq J$ and $F\subseteq Y\leq J$. Since $\mathfrak{X}$ and $\mathfrak{Y}$ are s-closed, it follows that $X\cap Y$ is an $\mathfrak{X}\cap\mathfrak{Y}$-weak ideal of L. Hence $\mathfrak{X}\cap\mathfrak{Y}$ is locally pseudo-coalescent.

(c) Let H (resp. K) be an $\mathfrak{X}\cap\mathfrak{Y}$-subideal (resp. an $\mathfrak{X}\cap\mathfrak{Y}$-weak ideal) of a Lie algebra L and put $J=\langle H, K\rangle$. Since $\mathfrak{Y}$ is pseudo-coalescent, J is a $\mathfrak{Y}$-weak ideal of L. For any finite subset F of J, there exists an $\mathfrak{X}$-weak ideal X of L such that $F\subseteq X\leq J$. Since X is an $\mathfrak{X}$-weak ideal of J, this implies that $J\in\tilde{\text{M}}\mathfrak{X}=\mathfrak{X}$. Therefore $\mathfrak{X}\cap\mathfrak{Y}$ is pseudo-coalescent.

We shall show the following statements which are analogous to Theorems 4.1 and 4.2 in [8].

THEOREM 4.2. (a) *Let* $\mathfrak{X}$ *be a* Q-*closed subclass of* $(\mathrm{E}\mathfrak{A})_{(\omega)}$. *If* $\mathfrak{X}$ *is locally pseudo-coalescent, then so is* $\mathfrak{X}_{(\omega)}$.

(b) *Let* $\mathfrak{X}$ *be an* {S, Q}-*closed subclass of* $\mathfrak{N}_{\omega}$. *If* $\mathfrak{X}$ *and* $\mathfrak{N} \cap \mathfrak{X}$ *are locally pseudo-coalescent, then so is* $\mathrm{E}\mathfrak{A} \cap \mathfrak{X}$.

PROOF. (a) Let H (resp. K) be an $\mathfrak{X}_{(\omega)}$-subideal (resp. an $\mathfrak{X}_{(\omega)}$-weak ideal) of a Lie algebra L. Put $I = H^{(\omega)} + K^{(\omega)}$. Then $I \lhd L$. Therefore $H+I/I$ (resp. $K+I/I$) is an $\mathfrak{X}$-subideal (resp. an $\mathfrak{X}$-weak ideal) of L/I since $\mathfrak{X}$ is Q-closed. Put $J = <H, K>$ and let F be any finite subset of J. If we denote $\bar{F}$ the image of F under the natural homomorphism of L onto L/I, $\bar{F}$ is a finite subset of J/I. Since $\mathfrak{X}$ is locally pseudo-coalescent, there exists a subalgebra X of L such that $\bar{F} \subseteq X/I \leq J/I$ and X/I is an $\mathfrak{X}$-weak ideal of L/I. Therefore $F \subseteq X \leq J$ and $X \text{ wi } L$. Now we have $\mathfrak{X}_{(\omega)} \leq (\mathrm{E}\mathfrak{A})_{(\omega)}$ since $\mathfrak{X} \leq (\mathrm{E}\mathfrak{A})_{(\omega)}$. Hence $H/H^{(\omega)}$ and $K/K^{(\omega)}$ are solvable and therefore $H^{(\omega)} = H^{(n)}$ and $K^{(\omega)} = K^{(n)}$ for some integer n. It follows that $I^{(\omega)} = I$ and so $X^{(\omega)} \geq I$. Therefore $X/X^{(\omega)} \in \mathrm{Q}\mathfrak{X} = \mathfrak{X}$ and hence $X \in \mathfrak{X}_{(\omega)}$. Thus $\mathfrak{X}_{(\omega)}$ is locally pseudo-coalescent.

(b) Let H (resp. K) be an $\mathrm{E}\mathfrak{A} \cap \mathfrak{X}$-subideal (resp. an $\mathrm{E}\mathfrak{A} \cap \mathfrak{X}$-weak ideal) of a Lie algebra L. Put $J = <H, K>$ and let F be any finite subset of J. Since $\mathfrak{X}$ is locally pseudo-coalescent, there exists an $\mathfrak{X}$-weak ideal X of L such that $F \subseteq X \leq J$. If we put $I = H^{\omega} + K^{\omega}$, then I is an ideal of L. Since $\mathfrak{X} = \mathrm{Q}\mathfrak{X}$ and $\mathfrak{X} \leq \mathfrak{N}_{\omega}$, $H+I/I$ (resp. $K+I/I$) is an $\mathfrak{N} \cap \mathfrak{X}$-subideal (resp. an $\mathfrak{N} \cap \mathfrak{X}$-weak ideal) of L/I. If we denote by $\bar{F}$ the image of F under the natural homomorphism of L onto L/I, $\bar{F}$ is a finite subset of J/I. Since $\mathfrak{N} \cap \mathfrak{X}$ is locally pseudo-coalescent, there exists a subalgebra Y of L such that $\bar{F} \subseteq Y/I \leq J/I$ and Y/I is an $\mathfrak{N} \cap \mathfrak{X}$-weak ideal of L/I. Therefore $F \subseteq Y \leq J$ and $Y \text{ wi } L$. Now I is solvable and so is Y. Since $\mathfrak{X} = \mathrm{S}\mathfrak{X}$, $X \cap Y$ is an $\mathrm{E}\mathfrak{A} \cap \mathfrak{X}$-weak ideal of L and $F \subseteq X \cap Y \leq J$. Thus $\mathrm{E}\mathfrak{A} \cap \mathfrak{X}$ is locally pseudo-coalescent. The proof is complete.

Furthermore, we can show the following theorem which corresponds to Theorem 3.2 in [12].

THEOREM 4.3. *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be classes such that* $\mathfrak{X} \leq \mathfrak{Y} \leq \mathrm{M}\mathfrak{X}$. *Then* $\mathfrak{X}$ *is locally pseudo-coalescent if and only if* $\mathfrak{Y}$ *is locally pseudo-coalescent.*

PROOF. Assume that $\mathfrak{X}$ is locally pseudo-coalescent. Let H (resp. K) be a $\mathfrak{Y}$-subideal (resp. a $\mathfrak{Y}$-weak ideal) of a Lie algebra L. If F is a finite subset of $J = <H, K>$, then there exist finite sets $A \subseteq H$ and $B \subseteq K$ such that $F \subseteq <A, B> \leq J$. Since H is an $\mathrm{M}\mathfrak{X}$-subalgebra, there exists an $\mathfrak{X}$-subideal M of H containing A. Similarly there exists an $\mathfrak{X}$-subideal N of K containing B. Then $F \subseteq <M, N>$. Now M (resp. N) is an $\mathfrak{X}$-subideal (resp. an $\mathfrak{X}$-weak ideal) of L and $\mathfrak{X}$ is locally pseudo-coalescent. Therefore there exists an $\mathfrak{X}$-weak ideal X of L such that $F\subseteq X\leq \langle M, N\rangle$. Clearly X belongs to $\mathfrak{Y}$ and $F\subseteq X\leq J$. Therefore $\mathfrak{Y}$ is locally pseudo-coalescent.

Conversely, assume that $\mathfrak{Y}$ is locally pseudo-coalescent. Let H (resp. K) be an $\mathfrak{X}$-subideal (resp. an $\mathfrak{X}$-weak ideal) of a Lie algebra L. Then H (resp. K) is a $\mathfrak{Y}$-subideal (resp. a $\mathfrak{Y}$-weak ideal) of L. Put $J=\langle H, K\rangle$ and let F be any finite subset of J. Then there exists a $\mathfrak{Y}$-weak ideal Y of L such that $F\subseteq Y\leq J$. Since $\mathfrak{Y}\leq \mathrm{M}\mathfrak{X}$, there exists an $\mathfrak{X}$-subideal X of Y such that $F\subseteq X\leq Y$. Clearly X is an $\mathfrak{X}$-weak ideal of L and $F\subseteq X\leq J$. Therefore $\mathfrak{X}$ is locally pseudo-coalescent. This proof is complete.

We recall the following classes:

$\mathfrak{D}$: The class of Lie algebras L such that every subalgebra of L is a subideal of L.

$\mathfrak{D}^*$: The class of Lie algebras L such that each element of L generates a subideal of L.

Then it is known that $\mathfrak{N}\leq\mathfrak{D}\leq\mathfrak{D}^*\leq \mathrm{L}\mathfrak{N}$ [2, 9].

Let a field Φ be of characteristic 0.

$\mathfrak{F}_t$: The class of Fitting algebras, that is, the class of Lie algebras which are generated by their nilpotent ideals.

$\mathfrak{B}$: The class of Baer algebras, that is, the class of Lie algebras which are generated by their nilpotent subideals.

By the coealescency of $\mathfrak{N}\cap\mathfrak{F}$ and Theorem 6.2.1 in [5], we have

$$\mathfrak{D}^* = \mathfrak{B} = \mathrm{M}(\mathfrak{N}\cap\mathfrak{F}),\ \mathfrak{N}\leq\mathfrak{F}_t\leq\mathfrak{B},\ \mathfrak{N}\leq\mathfrak{D}\leq\mathfrak{B}.$$

Applying the above theorem for $\mathfrak{N}\cap\mathfrak{F}$ and other pseudo-coalescent classes, we have the following

Corollary 4.4. *Over fields of characteristic* 0 *the following classes are locally pseudo-coalescent*:

(1) $\mathfrak{N}$, $\mathfrak{D}$, $\mathfrak{F}_t$, $\mathfrak{B}$, $\mathfrak{N}_{(\omega)}$,

(2) $\mathrm{M}(\mathrm{E}\mathfrak{A}\cap\mathfrak{F})$, $\mathrm{M}(\mathrm{E}\mathfrak{A}\cap\mathfrak{G})$, $\mathrm{M}\mathfrak{F}$, $\mathrm{M}\mathfrak{G}$, $\mathrm{M}\mathfrak{C}$, $\mathrm{M}\mathfrak{F}_{(\omega)}$, $\mathrm{M}\mathfrak{F}_{\omega}$, $\mathrm{M}\mathfrak{G}_{(\omega)}$, $\mathrm{M}\mathfrak{G}_{\omega}$, $\mathrm{M}\mathfrak{N}_{(\omega)}$.

We generalize the 'only if' part in Theorem 4.3 in the following

Proposition 4.5. *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be classes such that* $\mathfrak{X}\leq\mathfrak{Y}\leq\tilde{\mathrm{M}}\mathfrak{X}$. *If* $\mathfrak{Y}$ *is locally pseudo-coalescent, then so is* $\mathfrak{X}$.

Proof. Let H (resp. K) be an $\mathfrak{X}$-subideal (resp. an $\mathfrak{X}$-weak ideal) of a Lie algebra L. Put $J=\langle H, K\rangle$ and let F be any finite subset of J. Then there exists a $\mathfrak{Y}$-weak ideal Y of L such that $F\subseteq Y\leq J$. Since $\mathfrak{Y}\leq\tilde{\mathrm{M}}\mathfrak{X}$, there exists an $\mathfrak{X}$-weak ideal X of Y such that $F\subseteq X\leq Y$. Therefore X is an $\mathfrak{X}$-weak ideal of

L and $F \subseteq X \leq J$. Thus $\mathfrak{X}$ is locally pseudo-coalescent.

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