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On the Fixed Points of Elliptic Elements of B-Groups

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Recently there have been many results ([1], [4]) concerning the properties of B-groups, but few informations about the location of the fixed points of their elliptic elements.

In this paper we shall give some properties of fixed points of these elements. Before stating our theorem we shall explain notation.

Let G be a B-group, $\Lambda(G)$ the limit set of G, and $E(G)$ the set of fixed points of elliptic elements in $\Lambda(G)$. Let $E_d(G)$ (resp. $E_e(G)$) be the subset of $E(G)$ consisting of the fixed points of the elements conjugate in G to the elliptic elements in some degenerate (resp. elementary) group in $\{G_1,\dots, G_t\}$ (see section 1).

Our main result is the following theorem.

THEOREM. *Let G be a B-group with a simply connected invariant component Δ_0. Then the following three propositions hold:*

(1) $E(G)=E_d(G)\cup E_e(G)$ *and* $E_d(G)\cap E_e(G)=\emptyset$.

(2) *If* $E_d(G)\neq\emptyset$, *then G is not regular.*

(3) *For any* $z\in E_d(G)$, *its stability subgroup* $G_z=\{E|E(z)=z,\ E\in G\}$ *is an elliptic cyclic group and z can not lie on the boundaries of components except* Δ_0.

1. Let us begin with recalling some notation and definitions.

Let G be a kleinian group. Denote by $\Omega(G)$ the region of discontinuity of G, and $\Omega(G)'$ the set of points z with the property that each z has a neighborhood W such that $V(W)\cap W=\emptyset$ for all $V\in G$, $V\neq 1$. Then $\Omega(G)-\Omega(G)'$ consists of isolated fixed points of elliptic elements of G and the stability group $G_z=\{E|E(z)=z,\ E\in G\}$ for any $z\in\Omega(G)-\Omega(G)'$ is an elliptic cyclic group.

The components of $\Omega(G)$ are called components of G. A component Δ_0 of G is called invariant if $V(\Delta_0)=\Delta_0$ for every $V\in G$. For each component Δ of G, let G_Δ be the subgroup of G which keeps Δ invariant, and set $\Delta'=\Delta\cap\Omega(G)'$. Then $S=\Delta'/G_\Delta$ is a Riemann surface and the canonical projection $\Delta'\to S$ is conformal. If $\{\Delta_0, \Delta_1,\dots\}$ is a complete list of non-conjugate components of G and if $S_i=\Delta_i'/G_{\Delta_i}$, then $\Omega(G)'/G=S_0+S_1+\cdots$. The surfaces $S_0, S_1,\dots$ are called the factors of G.

A finitely generated, non-elementary kleinian group G with a simply connected invariant component Δ_0 is called a B-group. A B-group G is called quasifuchsian if there are a fuchsian group F and a global quasiconformal homeomorphism w: $\hat{\boldsymbol{C}} \to \hat{\boldsymbol{C}}$ so that $G = w \circ F \circ w^{-1}$, where $\hat{\boldsymbol{C}} = \boldsymbol{C} \cup \{\infty\}$. A B-group G for which $\Delta_0 = \Omega(G)$ is called degenerate. If a B-group G satisfies Area$(\Omega(G)'/G)$ $= 2$ Area (Δ_0'/G), then G is called regular, where Area denotes the Poincaré area.

From now on we shall denote by G a B-group with a simply connected invariant component Δ_0. Since there is a conformal mapping w: $U \to \Delta_0$, $F = w^{-1} \circ G \circ w$ is a fuchsian group, where U is the upper half-plane. Such group F is called the fuchsian equivalent of G. In this case the canonical isomorphism φ: $G \to F$ is defined by $\varphi(V) = w^{-1} \circ V \circ w$. Here we note that φ is not uniquely determined by Δ_0, but for every $V \in G$, trace$^2 \varphi(V)$ is determined by Δ_0. If there is a parabolic element $P \in G$ such that $\varphi(P)$ is hyperbolic, then P is called an accidental parabolic transformation. If H is a hyperbolic element of F, then H has an axis A_H in U; this is the non-euclidean line in U joining the fixed points of H. If $\varphi(V)$ is hyperbolic, then the axis A_V of V in Δ_0 is defined by $A_V = w(A_{\varphi(V)})$.

Let $S_0 = \Delta_0'/G$, and let $\{P_1, \dots, P_k\}$ be a basis for the accidental parabolic transformations in G. We use $[P_i]$ to denote the conjugacy class of P_i. Let $\{\alpha_1, \dots, \alpha_k\}$ be the homotopically independent set of loops corresponding to $[P_1], \dots, [P_k]$, respectively. The associated 2-complex will be denoted by $K(S_0; \alpha_1, \dots, \alpha_k)$. Denote by $S_1, \dots, S_t$ the factors of $K(S_0; \alpha_1, \dots, \alpha_k)$. For $1 \leqq i \leqq t$, let G_i be the image of $\pi_1(S_i, O_i)$ under the natural homomorphism Φ_i: $\pi_1(S_0, O_i) \to G$, where each O_i is a point on S_i and S_i is regarded as a subset of S_0 (cf. [4, §§6,7]).

2. At first we shall give

PROPOSITION 1. *Let* $1 \leqq i \leqq t$. *Then one and only one of the following three statements holds*:

(a) G_i *is a quasifuchsian group.*

(b) G_i *is a degenerate group without accidental parabolic transformations.*

(c) G_i *is an elementary group with the signature* $(0, 3; 2, 2, \infty)$.

PROOF. The proof of Theorem 5 in [4] gives that each G_i is a finitely generated kleinian group which has a simply connected invariant component and which contains no accidental parabolic transformation. Hence G_i is a quasifuchsian group, a degenerate group without accidental parabolic transformations, or an elementary group.

Assume that (*a*) and (*b*) do not hold. It follows that G_i is elementary. We shall show that its signature must be $(0, 3; 2, 2, \infty)$. Let its signature be $(g, n; \nu_1, \dots, \nu_n)$. Then $A = 2(g-1) + \sum_{i=1}^{n}(1-(1/\nu_i))$ is not positive. By the construction of $K(S_0; \alpha_1 \dots, \alpha_k)$, there is at least one ν_j among $\nu_1, \dots, \nu_n$ such that $\nu_j = \infty$. If $g \neq 0$, then it implies that $A > 0$. Hence g must be 0. If $n \geqq 4$, then A would be positive. Hence we have $n \leqq 3$. Since S_i is the factor of $K(S_0; \alpha_1, \ldots, \alpha_k)$, n is greater than 2. Hence $n=3$. So the signature must be $(0, 3; 2, 2, \infty)$.

REMARK. When G is an elementary group with the signature $(0, 3; 2, 2, \infty)$, G contains an elliptic element with the period 2 and a parabolic element which have a common fixed point.

PROPOSITION 2. *Assume that $G_1, \ldots, G_r$ $(r \leqq t)$ are quasifuchsian. Then non-invariant factors of G are topologically equal to $S_1, \ldots, S_r$.*

PROOF. Let $1 \leqq i \leqq r$. Since G_i is quasifuchsian, it has two invariant components Δ_{i1}, Δ_{i2}. We may assume $\Delta_{i1} \supset \Delta_0$. As $\partial\Delta_0 = \Lambda(G) \supset \Lambda(G_i) = \partial\Delta_{i1}$, it is seen that $\Delta_{i2} \cap \Lambda(G) = \emptyset$. Hence Δ_{i2} is a component of G.

In the same manner as in the proof of Theorem 5 in [4], we can show that $\{\Delta_0, \Delta_{12}, \Delta_{22}, \ldots, \Delta_{r2}\}$ is a complete list of non-conjugate components of G, that $G_{\Delta_{i2}} = G_i$, and that Δ'_{i2}/G_i is topologically equal to S_i. Thus non-invariant factors of G are topologically equal to $S_1, \ldots, S_r$.

PROPOSITION 3. *G is regular if and only if each G_i $(1 \leqq i \leqq t)$ is quasifuchsian or elementary.*

PROOF. We denote by $\mathrm{Area}(g, n; \nu_1, \ldots, \nu_n)$ the area of the surface with the signature $(g, n; \nu_1, \ldots, \nu_n)$, namely $\mathrm{Area}(g, n; \nu_1, \ldots, \nu_n) = 2\pi\{2(g-1) + \sum_{i=1}^{n}(1-(1/\nu_i))\}$. We easily obtain the following three relations:

(1) $\mathrm{Area}(g, n; \nu_1, \ldots, \nu_n) = \mathrm{Area}(g-1, n+2; \nu_1, \ldots, \nu_n, \infty, \infty)$.

(2) $\mathrm{Area}(g, n; \nu_1, \ldots, \nu_n) = \mathrm{Area}(g_1, m+1; \nu_1, \ldots, \nu_m, \infty)$
$+\mathrm{Area}(g_2, n-m+1; \nu_{m+1}, \ldots, \nu_n, \infty)$, where $g_1 + g_2 = g$.

(3) $\mathrm{Area}(g, n; 2, 2, \nu_1, \ldots, \nu_{n-2}) = \mathrm{Area}(g, n-1; \nu_1, \ldots, \nu_{n-2}, \infty)$.

Using the above (1), (2), (3) and Proposition 2, we see that induction on k, the number of elements in the basis for the accidental parabolic transformations in G, leads to our conclusion.

Considering the fuchsian equivalent of G, we can easily show the next proposition.

PROPOSITION 4. *Any elliptic element in G has one of its fixed points in Δ_0. In particular, if G is degenerate, then another fixed point of this element lies in $\Lambda(G)$.*

PROPOSITION 5. *Any elliptic element in G is conjugate in G to an elliptic element in only one of G_i's.*

PROOF. We note that G can be built up in k steps from $G_1, G_2, \ldots, G_t$

(if necessary, we consider the conjugate group instead of G_j) by using the Combination Theorems I, II (cf. [4, § 9]). Taking an elliptic element E in G, we derive from Proposition 4 that one of the fixed points of E, say z, is in Δ_0. There is some factor S_i with the removed point corresponding to z in $K(S_0; \alpha_1, \ldots, \alpha_k)$. Hence E is conjugate in G to some element of G_i corresponding to S_i.

3. As the proof of our theorem is complicated, we shall divide it into five lemmas.

LEMMA 1. *An elliptic element E in G does not have a common fixed point with a loxodromic element V in G.*

PROOF. Let z and w be the fixed points of E. Then one of them, say z, lies in Δ_0 by Proposition 4. Hence V does not fix z. If V has the fixed point w, then the subgroup of G generated by E and V is not discrete (cf. [3, Chapter III]). This is a contradiction.

LEMMA 2. *Let P_1 and P_2 be two parabolic elements in G which have a common fixed point. Then there exist non-zero integers m and n such that $P_1^m = P_2^n$.*

PROOF. By a suitable conjugation in $SL'(2, \boldsymbol{C})$, we may assume that the common fixed point of P_1 and P_2 is at ∞. If our conclusion is not true, then we may consider that $P_1(z)$ and $P_2(z)$ have the forms $z+1$, and $z+\tau$ with $\operatorname{Im}\tau>0$, respectively. Let S be a parallelogram whose vertices are 0, 1, τ and $1+\tau$. If $S\subset\Delta_0$, then it is obvious that $\bigcup_j\bigcup_k P_1^j P_2^k(S)=\boldsymbol{C}\subset\Delta_0$. This contradicts the assumption that G is non-elementary. Hence S is not contained in Δ_0. If $S\subset\Delta_0^c$, then $\bigcup_j\bigcup_k P_1^j P_2^k(S)=\boldsymbol{C}\subset\Delta_0^c$, which is also a contradiction. If $S\cap\Delta_0\neq\emptyset$, then Δ_0 can not be simply connected. However, this contradicts our assumption that G is a B-group. Thus there exist non-zero integers m and n such that $P_1^m = P_2^n$.

LEMMA 3. *Let E and P be elliptic and parabolic elements in G, respectively. If the period m of E is not 2, then they do not have a common fixed point.*

PROOF. Assume that this conclusion is not true. Without loss of generality, we may assume that the common fixed point of E and P is at ∞. Then P and E have the following forms:

$$P=\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}, \alpha\neq 0,$$

$$E=\begin{bmatrix} \exp\dfrac{i\pi}{m} & \beta \\ 0 & \exp\dfrac{-i\pi}{m} \end{bmatrix}.$$

The commutator C of P and E is of the form:

$$C = PEP^{-1}E^{-1} = \begin{bmatrix} 1 & \alpha\left(1 - \exp\dfrac{2i\pi}{m}\right) \\ 0 & 1 \end{bmatrix}.$$

It is evident that C also fixes ∞ and is parabolic. If m is not equal to 2, then we can show that $C^r \neq P^n$ for any non-zero integers r and n. This contradicts Lemma 2.

COROLLARY. *Let P be an accidental parabolic transformation with a fixed point z in G. If an elliptic element E in G also fixes z, then the period of E is 2.*

LEMMA 4. *Let E_1 and E_2 be elliptic elements in G which have one and only one fixed point in common. Then both periods are 2.*

PROOF. Without loss of generality, we may assume that the common fixed point is at ∞. Then E_1 and E_2 may be written in the following forms:

$$E_1 = \begin{bmatrix} \exp\dfrac{i\pi}{r_1} & 0 \\ 0 & \exp\dfrac{-i\pi}{r_1} \end{bmatrix},$$

$$E_2 = \begin{bmatrix} \exp\dfrac{i\pi}{r_2} & a \\ 0 & \exp\dfrac{-i\pi}{r_2} \end{bmatrix}, \; a \neq 0,$$

where r_i is the period of $E_i\,(i=1, 2)$. It is easily seen that the commutator C of E_1 and E_2 is of the form:

$$C = E_1E_2E_1^{-1}E_2^{-1} = \begin{bmatrix} 1 & -a\left(1 - \exp\dfrac{2i\pi}{r_1}\right)\exp\dfrac{i\pi}{r_2} \\ 0 & 1 \end{bmatrix}.$$

Hence C is parabolic and fixes ∞. It follows from this fact that E_i $(i=1, 2)$ and C have a common fixed point. Thus Lemma 3 leads immediately to our conclusion.

LEMMA 5. *Let P and E be parabolic and elliptic elements in G, respectively. If P and E have a common fixed point, then P is an accidental parabolic transformation and E is contained in some elementary group conjugate in G*

to some group in $\{G_1,\dots, G_t\}$.

PROOF. Let F be the fuchsian equivalent of G. Then there is a canonical isomorphism φ: $G\to F$. It is well known that if E is an elliptic element in G, then $\varphi(E)$ is elliptic and of the same period with E; and also that if P is parabolic in G, then $\varphi(P)$ is parabolic or hyperbolic.

Now we derive from Lemma 3 that the period of E is 2. We shall show that EP is an elliptic element with the period 2. Without loss of generality, we may assume that

$$E=\begin{bmatrix} i & \alpha \\ 0 & -i \end{bmatrix},\ P=\begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix},\ \beta\neq 0.$$

It is easily seen that

$$EP=\begin{bmatrix} i & i\beta+\alpha \\ 0 & -i \end{bmatrix}.$$

Hence EP is an elliptic element with the period 2.

Set $\tilde{E}=\varphi(E)$ and $\tilde{P}=\varphi(P)$. Since φ is the canonical isomorphism, $\tilde{E}\tilde{P}$ is also an elliptic element with the period 2. Since F is fuchsian, we may write

$$\tilde{E}=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ \tilde{P}=\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where $ad-bc=1$ and $a, b, c, d\in \boldsymbol{R}$. Therefore $\tilde{E}\tilde{P}$ is of the form:

$$\tilde{E}\tilde{P}=\begin{bmatrix} -c & -d \\ a & b \end{bmatrix}.$$

It follows that $b=c$, since the period of $\tilde{E}\tilde{P}$ is 2.

Let us assume that $\tilde{P}$ is parabolic. Substituting $b=c$ into $ad-bc=1$ and using $a+d=\pm 2$, we obtain $a=d=\pm 1$ and $b=c=0$. Hence $\tilde{P}$ is the identity and this contradicts the hypothesis that $\tilde{P}$ is parabolic. Therefore $\tilde{P}$ must be hyperbolic. It follows from this fact that P is an accidental parabolic transformation.

In the same manner as in the proof of Theorem 7 in [4], we can show that an elliptic element with a fixed point on the axis of an accidental parabolic transformation is contained in some elementary group conjugate in G to some group in $\{G_1,\dots, G_t\}$. Thus it suffices to prove that one of the fixed points of $\tilde{E}$ lies on the axis of $\tilde{P}$. One sees that the axis of $\tilde{P}$ is

$$\left|z-\frac{a-d}{2b}\right|=\frac{\sqrt{(a+d)^2-4}}{2|b|}.$$

This equation is satisfied by i which is a fixed point of $\tilde{E}$. Our lemma is now completely proved.

4. Now we are ready to prove our theorem.

PROOF OF (1). Take a point $z \in E(G)$. Let E be an elliptic element with z as a fixed point. It follows from Proposition 5 that E is conjugate in G to an element in some group in $\{G_1,\dots, G_t\}$. Assume that E is conjugate in G to an element E' in some quasifuchsian group G_j. It is found from Proposition 2 that there exists a component Δ of G such that $G_\Delta = G_j$. Then the fixed points of E' are in $\Omega(G)$. This is a contradiction. Hence E is conjugate in G to an element in some elementary group or some degenerate group in $\{G_1,\dots, G_t\}$. Thus z is contained in $E_d(G) \cup E_e(G)$. Since it is obvious that $E(G) \supset E_d(G) \cup E_e(G)$, it follows that $E(G) = E_d(G) \cup E_e(G)$.

Next assume that $E_d(G) \cap E_e(G) \neq \emptyset$, and take a point w in $E_d(G) \cap E_e(G)$. Since w is contained in $E_d(G)$ (resp. $E_e(G)$), there exists an elliptic element E_1 (resp. E_2) with w as a fixed point in some degenerate group (resp. some elementary group) which is conjugate in G to some group in $\{G_1,\dots, G_t\}$. From Remark given after Proposition 1, it follows that there exists a parabolic element in G which has a common fixed point with E_2. Lemma 5 implies that E_1 is contained in some elementary group. This contradicts Proposition 5. Thus $E_d(G) \cap E_e(G)$ is empty.

PROOF OF (2). If $E_d(G) \neq \emptyset$, then there is at least one degenerate group in $\{G_1,\dots, G_t\}$. It follows from Proposition 3 that $\mathrm{Area}\,(\Omega(G)'/G) < 2\,\mathrm{Area}\,(\Delta_0'/G)$. Hence G is not regular.

PROOF OF (3). At first we derive from Lemma 1 that no loxodromic element belongs to G_z. Moreover we know from Lemma 5 that any parabolic element in G does not fix z. Since it is found from Lemma 4 and the above fact that all elliptic elements in G_z have two common fixed points, G_z is an elliptic cyclic group.

Next assume that $z \in E_d(G)$ lies on the boundary of some component $\Delta (\neq \Delta_0)$. Let E be an elliptic element which fixes z. Since $G_\Delta = \{T \mid T(\Delta) = \Delta,\ T \in G\}$ is a quasifuchsian group, we know that the fixed points of the elliptic elements in G_Δ lie in $\Omega(G_\Delta)$. Hence E does not belong to G_Δ, and so $E(\Delta) \neq \Delta$. On the other hand $E(\Delta)$ is also a component of G and its boundary contains z. It is known that, if $\partial E(\Delta) \cap \partial\Delta \neq \emptyset$, then $\partial E(\Delta) \cap \partial\Delta$ consists of only one point which is a fixed point of a parabolic cyclic subgroup of G (cf. [5, § 5]). This fact leads to a contradiction. Therefore the points in $E_d(G)$ can not lie on the boundaries of components except Δ_0.

Thus our theorem is completely proved.

REMARK. Let G be a finitely generated function group and let z be a limit point which is fixed by an elliptic element with the period $r\ (r\geqq 3)$ in G. Then we can show that z does not lie on the boundaries of components except an invariant component.

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