HIROSHIMA MATH. J.
8 (1978), 371–400

Algebraic Curves with Non-classical Types of Gap Sequences for Genus Three and Four

Kaname KOMIYA

(Received January 20, 1978)

It is well known as Weierstrass theorem that for each point P on an algebraic curve C of genus $g \geqq 1$ defined over a field of characteristic 0, there are exactly g natural numbers (gaps) $n_1, n_2, \dots, n_g$ such that there is no function on C whose pole divisor is precisely one of n_iP, and moreover, these gaps are 1, 2,..., g for all but a finite number of points. The exceptional points are called Weierstrass points. In his paper [8], F. K. Schmidt tried to generalize the notion of Weierstrass points to the case of characteristic $p>0$. Noting that Weierstrass points are closely related to the ramification divisor of the canonical system of C, he first introduced the notion of Wronskian determinants by means of iterative higher derivations and succeeded in constructing a general theory of Weierstrass points for an algebraic curve defined over a field of characteristic $p>0$. His theory presents a striking contrast to the classical case; namely there appear special curves whose ordinary points may have non-classical types of gap sequences. As an illustration of the general theory, he determined distributions of gaps at ordinary points in case of genus 3 or 4, and gave examples of algebraic curves with non-classical types of gap sequences ([8] § 6).

The purpose of this paper is to determine precisely the family of algebraic curves of genus 3 or 4 whose ordinary points have gaps different from the classical ones.

In § 1, for the later use, we shall summarize some results on iterative higher derivations and Wronskian determinants of the canonical system on C. It was proved in [8] that if ordinary points on C of genus 3 defined over an algebraically closed field k have non-classical types of gap sequences, then the characteristic of k must be 3. In § 2, we shall show that moreover C is birationally equivalent to the plane curve

$$y^3 + y - x^4 = 0.$$

This example was given originally in [8]. In § 3, we shall give a classical fact that every non-hyperelliptic curve of genus 4 has a trigonal linear system, i.e. a linear system of dimension 1 and of degree 3 (Th. 2 and its Cor.). Moreover we shall analyze equations of curves of genus 4 having trigonal linear systems and non-classical types of gap sequences. It was proved in [8] that if ordinary points on C of genus 4 defined over an algebraically closed field k have non-classical types of gap sequences, then the characteristic p of k must be 2 or 5. In §4 and §5, we shall show that moreover (i) in the case of $p=2$, C is birationally equivalent to one of the plane curves

$$x^3y^3 + x^3 + y^3 + \lambda = 0, \qquad \lambda \in k,\ \lambda \neq 0, 1,$$

and the converse is also true, and (ii) in the case of $p=5$, C is birationally equivalent to the plane curve

$$y^5 + y - x^3 = 0.$$

The last example was given originally in [8].

The author expresses his sincere thanks to Professor M. Nishi for his advise to write this paper and other much encouragement, and to Professor H. Yanagihara for his kindly criticism. By them, the present paper could be improved in many parts.

§1. Preliminaries

We generally follow terminologies and notations of Schmidt [8] and Weil [10], [11]. We denote by C a complete non-singular algebraic curve of genus $g \geqq 2$ defined over an algebraically closed field k of characteristic p, and we denote by $k(C)$ the function field of C over k. In the sequel, we understand that points and divisors are always rational over k unless otherwise specified.

In this section, we summarize some results given in [2], [3], [7] and [8] for the sake of later sections. For a point P of C, a positive integer n is called a gap (number) at P, if there is no function x of C such that the pole divisor of x is nP. The gaps of C are related closely with the Hermitian P-invariants of the canonical system of C as follows. Let D be a canonical divisor of C whose components are different from P, and let t be a local parameter of C at P. Then there exists a base $\{x_1, x_2,\dots, x_g\}$ of the function space $L(D)$ over k such that

$$(1)\qquad \begin{aligned} x_1 &= a_1 + a_{11}t + a_{12}t^2 + \cdots \\ x_2 &= a_2t^{h_1} + a_{21}t^{h_1+1} + a_{22}t^{h_1+2} + \cdots \\ &\cdots\cdots\cdots\cdots\cdots \\ x_g &= a_gt^{h_{g-1}} + a_{g1}t^{h_{g-1}+1} + a_{g2}t^{h_{g-1}+2} + \cdots \end{aligned}$$

where $a_i \neq 0$ for $i=1, 2,\dots, g$, and $0<h_1<h_2<\cdots<h_{g-1}$. We call this base a *Hermitian* P-*base* of $L(D)$. If we put

$$(2)\qquad n_i = h_{i-1} + 1, \qquad i = 1, 2,\dots, g,$$

where $h_0=0$, then $\{n_1, n_2,\dots, n_g\}$ is the set of the gaps at P which we call the gap sequence at P ([3], p. 492 and [8], p. 82). This can be verified by Riemann-Roch theorem. From this fact, we know that $\{h_0, h_1,\dots, h_{g-1}\}$ is independent of a canonical divisor D and a local parameter t, and it depends only on P. We call $h_0, h_1,\dots, h_{g-1}$ the *Hermitian* P-*invariants* ([8], p. 81). For every point P on a non-hyperelliptic curve, as easily seen, we have

$$h_1 = 1. \tag{3}$$

Moreover these numbers have the following geometric meaning ([3], Ch. 26). Let Λ be the canonical system of C. Then there exist g divisors $D_0, D_1,\dots, D_{g-1}$ with components different from P such that

$$D_0,\ h_1\mathrm{P} + D_1,\ h_2\mathrm{P} + D_2,\dots,\ h_{g-1}\mathrm{P} + D_{g-1} \tag{4}$$

are contained in Λ. This follows easily from the definition of h_i. Conversely, suppose there exists a divisor D' with components different from P such that $h\mathrm{P}+D'$ is a positive canonical divisor. Then h is equal to one of h_i as easily seen. Therefore the Hermitian P-invariants are characterized in terms of canonical divisors using (4).

Next we shall give a brief account of an iterative higher derivation of $K=k(\mathrm{C})$ and a Wronskian determinant of the canonical system of C. Doing so, another interpretation of the Hermitian P-invariants, and hence of a gap sequence would be possible. Let x be an element of K such that K is separably algebraic over $k(x)$. We define maps $D_x{}^\nu$: $K\to K$, $\nu=0, 1, 2,\dots$, as follows. Let P and Q be independent generic points of C over k. Then the function field $k(\mathrm{P})(\mathrm{C})$ of C over $k(\mathrm{P})$ may be identified with $k(\mathrm{P}, \mathrm{Q})$. For $f\in K$, represent $f(\mathrm{Q})\in k(\mathrm{P})(\mathrm{C})$ as a power series of a local parameter $x(\mathrm{Q})-x(\mathrm{P})$ at P:

$$f(\mathrm{Q}) = \eta_0 + \eta_1(x(\mathrm{Q}) - x(\mathrm{P})) + \eta_2(x(\mathrm{Q}) - x(\mathrm{P}))^2 + \cdots$$

where η_ν, $\nu=0, 1, 2,\dots$, are elements in $k(\mathrm{P})\simeq k(\mathrm{C})$. Hence we can put $\eta_\nu= f^{(\nu)}(\mathrm{P})$ for some $f^{(\nu)}\in K$. Define

$$D_x^\nu f = f^{(\nu)}, \qquad \nu = 0, 1, 2,\dots.$$

Then $\{D_x^\nu;\ \nu=0, 1, 2,\dots\}$ is called an iterative higher derivation of K over k ([2], p. 217). It satisfies the following formulas: putting $y^{(\nu)}=D_x^\nu y$,

$$(y + z)^{(\nu)} = y^{(\nu)} + z^{(\nu)}, \tag{5}$$

$$(yz)^{(\nu)} = y^{(\nu)}z + y^{(\nu-1)}z' + \cdots + y'z^{(\nu-1)} + yz^{(\nu)}, \tag{6}$$

$$(y^{(\mu)})^{(\nu)} = \binom{\mu+\nu}{\mu} y^{(\mu+\nu)}, \tag{7}$$

(8) $$y^{(0)} = y,$$

(9) $$c^{(\nu)} = 0, \quad \text{for} \quad c \in k, \ \nu \geqq 1.$$

Further it satisfies

(10) $$x' = 1, \ x^{(\nu)} = 0 \qquad \text{for} \quad \nu \geqq 2.$$

It is known that D_x^ν is the unique iterative higher derivation of K over k satisfying (10) ([2], p. 231). From these, we have ([2], p. 232), for a power series expansion $y = \sum_{\mu=\mu_0}^{\infty} c_\mu x^\mu$ of $y \in K$ where μ_0 may be negative

(11) $$y^{(\nu)} = \sum_{\mu=\mu_0}^{\infty} \binom{\mu}{\nu} c_\mu x^{\mu-\nu}.$$

Now the definition of a Weierstrass point depends on the fact that except a finite number of points, gap sequences of points on C coincide with each other. This fact is proved by using a Wronskian determinant as follows ([3], p. 490 and [8], p. 77). Let $\{x_1, x_2,\dots, x_g \in K = k(\mathrm{C})\}$ be a base of a function space $L(D)$ of a canonical divisor D, and let t be a function in K such that K is separably algebraic over $k(t)$. Put

(12) $$\Delta_{t;\nu_1,\nu_2,\dots,\nu_{g-1}}(x_1, x_2,\dots, x_g) = \begin{vmatrix} x_1 & x_2 & \cdots & x_g \\ x_1^{(\nu_1)} & x_2^{(\nu_1)} & \cdots & x_g^{(\nu_1)} \\ \multicolumn{4}{c}{\cdots\cdots\cdots} \\ x_1^{(\nu_{g-1})} & x_2^{(\nu_{g-1})} & \cdots & x_g^{(\nu_{g-1})} \end{vmatrix}$$

where $0 < \nu_1 < \nu_2 < \cdots < \nu_{g-1}$ and $x_i^{(\nu)} = D_t^\nu x_i$. We call $\nu_0 = 0, \nu_1,\dots, \nu_{g-1}$ the *orders* of this determinant. Then the Hermitian P-invariants are also characterized as the minimum system of orders $h_0, h_1,\dots, h_{g-1}$ in lexicographic order among such systems of orders $\nu_0 = 0, \nu_1,\dots, \nu_{g-1}$ that, for a local parameter t at P and a canonical divisor D without P as a component, $\Delta_{t;\nu_1,\dots,\nu_{g-1}}(x_1,\dots, x_g)(\mathrm{P}) \neq 0$. We define a *Wronskian determinant* $\Delta_t(x_1,\dots, x_g)$ of C as such a determinant (12) that it has the minimum system of orders $\mu_0 = 0, \mu_1,\dots, \mu_{g-1}$ in lexicographic order among all non zero determinants (12). We note the minimum system of orders μ_i is independent of a canonical divisor D, a base $\{x_i\}$ of $L(D)$, and a local parameter t, and is determined only by C. Here we have the following formula: for each Hermitian P-invariants $h_0, h_1,\dots, h_{g-1}$,

(13) $$h_0 = \mu_0 = 0, \ h_1 \geqq \mu_1, \ h_2 \geqq \mu_2,\dots, h_{g-1} \geqq \mu_{g-1}.$$

In fact, if we assume $h_s < \mu_s$, then by the definition of orders of a Wronskian determinant

$$\operatorname{rank}\begin{bmatrix} x_1 & x_2 & \cdots & x_g \\ x_1' & x_2' & \cdots & x_g' \\ x_1^{(2)} & x_2^{(2)} & \cdots & x_g^{(2)} \\ & \cdots\cdots\cdots & & \\ x_1^{(\mu_s-1)} & x_2^{(\mu_s-1)} & \cdots & x_g^{(\mu_s-1)} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} x_1 & \cdots & x_g \\ x_1' & \cdots & x_g' \\ x_1^{(2)} & \cdots & x_g^{(2)} \\ & \cdots\cdots & \\ x_1^{(\mu_s-1)} & \cdots & x_g^{(\mu_s-1)} \end{bmatrix} = s.$$

On the other hand, $\operatorname{rank}(x_j^{(i)})_{i=0,h_1,\dots,h_s;\ j=1,2,\dots,g} = s+1$. This is a contradiction.

To determine exceptional points, we consider the following *synthetic ramification divisor* (zusammengesetzte Verzweigungsdivisor) of the canonical system of C ([3], Ch. 26, § 3 and [8], p. 80)

$$V_C = gD + \operatorname{div}(\Delta_t(x_1,\dots,x_g)) + (\mu_1+\mu_2+\cdots+\mu_{g-1})\operatorname{div}(dt). \tag{14}$$

It is known that the divisor V_C is independent of a canonical divisor D, a base $\{x_i\}$ of $L(D)$, and a local parameter t ([8], p. 80). For a point P of C, select a positive canonical divisor D without P as a component, a local parameter t at P, and a Hermitian P-base $x_1=1, x_2,\dots, x_g$ of $L(D)$. Let $h_0=0, h_1,\dots, h_{g-1}$ be the Hermitian P-invariants. Then, the order $\gamma_P(V_C)$ in V_C is equal to $v_P(\Delta_t(x_1,\dots,x_g))$ and we have, by (1) and (11),

$$\Delta_t(x_1,\dots,x_g) = \begin{vmatrix} 1 & a_2t^{h_1}+\cdots, & \cdots & a_gt^{h_{g-1}}+\cdots \\ 0 & \binom{h_1}{\mu_1}a_2t^{h_1-\mu_1}+\cdots, & \cdots & \binom{h_{g-1}}{\mu_1}a_gt^{h_{g-1}-\mu_1}+\cdots \\ & \cdots\cdots & & \\ 0 & \binom{h_1}{\mu_{g-1}}a_2t^{h_1-\mu_{g-1}}+\cdots, & \cdots & \binom{h_{g-1}}{\mu_{g-1}}a_gt^{h_{g-1}-\mu_{g-1}}+\cdots \end{vmatrix}$$

$$= \begin{vmatrix} \binom{h_1}{\mu_1} & \binom{h_2}{\mu_1} & \cdots & \binom{h_{g-1}}{\mu_1} \\ \binom{h_1}{\mu_2} & \binom{h_2}{\mu_2} & \cdots & \binom{h_{g-1}}{\mu_2} \\ & \cdots\cdots & & \\ \binom{h_1}{\mu_{g-1}} & \binom{h_2}{\mu_{g-1}} & \cdots & \binom{h_{g-1}}{\mu_{g-1}} \end{vmatrix} a_2a_3\cdots a_gt^e + B_{e+1}t^{e+1}+\cdots\cdots$$

where $e=\sum_{i=1}^{g-1}(h_i-\mu_i)$ and $B_\nu\in k$ for $\nu\geqq e+1$ ([3], p. 456). This means

$$\gamma_P(V_C)\geqq(h_1-\mu_1)+(h_2-\mu_2)+\cdots+(h_{g-1}-\mu_{g-1}), \quad \text{([8], p. 82)} \tag{15}$$

where the equality holds if and only if

$$\det\left[\binom{h_i}{\mu_j}\right]_{i,j=1,2,\dots,g-1} \neq 0. \tag{16}$$

In particular, P is a component of V_C if and only if the Hermitian P-invariants $h_0, h_1,\dots, h_{g-1}$ are different from the orders $\mu_0, \mu_1,\dots, \mu_{g-1}$ of a Wronskian determinant of C. Put

$$m_1 = \mu_0 + 1,\ m_2 = \mu_1 + 1,\dots, m_g = \mu_{g-1} + 1. \tag{17}$$

Then a point of C with the gap sequence $m_1, m_2,\dots, m_g$ is called an ordinary point of C and a non-ordinary point is called a Weierstrass point. In other words P is a Weierstrass point if and only if P is a component of V_C.

As for the orders $\mu_0=0, \mu_1,\dots, \mu_{g-1}$ of a Wronskian determinant of C, the following facts were proved by F. K. Schmidt in [8].

THEOREM A ([8], Satz 4 and [7], Satz 6). *Let μ_i be one of orders of a Wronskian determinant of* C. *If the characteristic $p=0$, all non-negative integers μ not greater than μ_i are also orders of a Wronskian determinant. In the case of characteristic $p>0$, if $\mu_i=a_0+a_1p+\cdots+a_sp^s$ with $0\leqq a_j\leqq p-1$, then all non-negative integers $\mu=c_0+c_1p+\cdots+c_sp^s$ with $0\leqq c_j\leqq a_j$ are also orders.*

For later applications, we state the following Lemma used in the proof of the above Theorem A.

LEMMA B ([7], Hilfssatz 3). *For natural numbers μ and v, the binomial coefficient $\binom{v}{\mu}$ is not divisible by a prime number p if and only if the p-adic coefficients of μ are respectively not greater than those of v.*

By Th. A in the classical case of characteristic $p=0$, the gap sequence at ordinary points of C is $1, 2,\dots, g$. We also know, in a hyperelliptic curve of any characteristic, ordinary points have the classical type of gap sequence $\{1, 2,\dots, g\}$ ([8], Satz 8). Moreover, we have, as to μ_1 of orders of a Wronskian determinant for any curve ([8], Satz 5),

$$\mu_1 = 1. \tag{18}$$

As for the existence of curves whose ordinary points have non-classical types of gap sequences, F. K. Schmidt gave very nice examples ([8], Satz 9) and he proved by Th. A the following

THEOREM C ([8], Satz 6). *If a curve of genus g defined over an algebraically closed field of characteristic $p>0$ has a non-classical type of gap sequence, then we have*

$$p + 1 \leqq 2g - 2.$$

From these theorems and Riemann-Roch theorem, we obtain the following possibility of non-classical types of gap sequences at ordinary points for curves of genus 3 or 4 ([8], p. 95):

$$g=3,\ p=3:\ 1,\,2,\,4, \tag{19}$$

$$g=4,\ p=2:\ 1,\,2,\,3,\,5, \tag{20}$$

$$g=4,\ p=3:\ 1,\,2,\,4,\,5, \tag{21}$$

$$g=4,\ p=5:\ 1,\,2,\,3,\,6. \tag{22}$$

But (21) does not occur by the following

THEOREM D ([8], Satz 10). *On a curve of genus* 4 *defined over an algebraically closed field of characteristic* 3, *ordinary points have the classical type of gap sequence* $\{1, 2, 3, 4\}$.

For (19) and (22), F. K. Schmidt constructed examples defined by the equations respectively

$$y^3+y-x^4=0 \quad \text{and} \tag{23}$$

$$y^5+y-x^3=0 \tag{24}$$

which are special cases of the above mentioned general examples with non-classical types of gap sequences at ordinary points.

§2. The case of genus 3

In this section, we shall prove that a curve C defined over k of characteristic 3 which is birationally equivalent to the plane curve (23) is essentially the only possible one of genus 3, whose ordinary points have a non-classical type of gap sequence.

Let C be a non-hyperelliptic curve of genus 3 defined over an algebraically closed field k. Suppose ordinary points of C have a non-classical type of gap sequence. Then by (19), the characteristic of k must be 3 and this gap sequence must be 1, 2, 4, and hence the orders of a Wronskian determinant of C are 0, 1, 3 by (17). For every Weierstrass point P, the Hermitian P-invariants must be 0, 1, 4 by (3), (4) and (13). And since we have

$$\begin{vmatrix} \binom{1}{1} & \binom{1}{3} \\ \binom{4}{1} & \binom{4}{3} \end{vmatrix} = 1 \neq 0,$$

$\gamma_{\mathrm{P}}(V_{\mathrm{C}})=1$ by (15) and (16). On the other hand, we have $\deg(V_{\mathrm{C}})=28$ by (14).

Therefore there are 28 Weierstrass points on C. Hence, let P_1 and P_2 be two Weierstrass points; then $l(3P_i)=2$, $l(4P_i)=3$ for $i=1, 2$ and hence $4P_1$ and $4P_2$ are canonical divisors. Let $P_1+P_2+Q_1+Q_2$ be a canonical divisor. Then there exist functions x and y in $K=k(C)$ such that

$$\text{(25)} \qquad \operatorname{div}(x) = P_2 + Q_1 + Q_2 - 3P_1 \quad \text{and} \quad \operatorname{div}(y) = 4P_2 - 4P_1.$$

Here, since Hermitian invariants of P_1 and of P_2 are 0, 1, 4, we have $P_i \neq Q_j$ for each $i=1, 2$ and $j=1, 2$ by (4). Since $[K: k(x)]=\deg 3P_1=3$ and $[K: k(y)] =\deg 4P_1=4$, we have $K=k(x, y)$. Let $f(x, y)=0$ be an irreducible equation for x, y. Let C$'$ be the projective plane curve defined by $f(x, y)=0$ and H a hyperplane defined by a generic equation $ax+by+c=0$ over k. Then the degree of the intersection $H \cdot C'$ is equal to the degree of the zero divisor of $ax+by+c$ and hence of its pole divisor on C. Therefore $f(x, y)$ must be of degree 4. Since x and y are finite, and hence integral, over $k[y]$ and $k[x]$ respectively, we can put

$$f(x, y) = x^4 + \gamma_1(y)x^3 + \gamma_2(y)x^2 + \gamma_3(y)x + \gamma_4(y) = 0$$

where $\gamma_i(y) \in k[y]$ for $i=1, 2, 3, 4$, $\deg \gamma_3 \leqq 2$, and $\deg \gamma_4=3$. Here, the coefficient of yx^3 must be zero because yx^3 is the only term with least $v_{P_1}(yx^3)$. Moreover, since $x=0$ must be the quadruple root of $f(x, 0)=0$ by (25), we have

$$\text{(26)} \qquad f(x, y) = x^4 + ayx^2 + (b_1y^2 + b_2y)x + c_0y^3 + c_1y^2 + c_2y$$

where a, b_i and c_i are in k. Replacing y by $c_0^{1/3}y$, we may assume $c_0=1$. It is sufficient to show that $a=b_1=b_2=c_1=0$ because the curve $x^4+y^3+cy=0$ is birationally equivalent to the plane curve $y_1^3+y_1-x_1^4=0$ by the transformation $x=(-c)^{3/8}x_1$, $y=c^{1/2}y_1$.

$\{1, x, y\}$ is clearly a base of $L(4P_1)$. Since, in (25), Q_1 and Q_2 do not coincide with P_2, x is a local parameter at P_2. Let D_x^ν be the iterative higher derivation of K with respect to x and put $y^{(\nu)}=D_x^\nu y$. Since the orders of a Wronskian determinant of C are 0, 1, 3, we have

$$\Delta_{x;1,2}(1, x, y) = \begin{vmatrix} 1 & x & y \\ 0 & 1 & y' \\ 0 & 0 & y^{(2)} \end{vmatrix} = y^{(2)} = 0.$$

If we operate D_x^2 on the equation (26), then we have from $y^{(2)}=0$

$$\text{(27)} \qquad a(2y'x + y) + b_1((y')^2x + 2y'y) + b_2y' + c_1(y')^2 = 0.$$

On the other hand we have from (25)

$$\operatorname{div}(dy) = 3P_2 + D - 5P_1$$

where D is a positive divisor of degree 6 and its components are different from P_1 and P_2. If P is a component of D, then, since dy is zero at P, we have

$$\operatorname{div}(y - y(\mathrm{P})) = 2\mathrm{P} + E - 4\mathrm{P}_1$$

where E is a positive divisor of degree 2 and its components are different from P_1. Assume that $x - x(\mathrm{P})$ is not a local parameter at P. Then we have

$$\operatorname{div}(x - x(\mathrm{P})) = 2\mathrm{P} + \mathrm{P}' - 3\mathrm{P}_1.$$

Hence $2\mathrm{P}+\mathrm{P}'+\mathrm{P}_1 \sim 2\mathrm{P}+E$ and so $\mathrm{P}'+\mathrm{P}_1 \sim E$. And hence, since C is non-hyperelliptic, we have $\mathrm{P}'+\mathrm{P}_1 = E$. This is a contradiction. Therefore $x - x(\mathrm{P})$ is a local parameter at P. This implies that we can write $dy = y'dx$, and hence we have $y'(\mathrm{P}) = 0$. Specializing (27) to P, $ay(\mathrm{P}) = 0$, and hence $a = 0$. Expressing y as a power series of a local parameter x at P_2, we have by (25)

$$y = c_4x^4 + c_5x^5 + \cdots$$

where $c_4 \neq 0$, and hence by (11)

$$y' = c_4x^3 + 2c_5x^4 + \cdots.$$

Putting these expansions into (27), we have $b_2 = c_1 = b_1 = 0$. Thus we obtain

THEOREM 1. *If an algebraic curve* C *of genus* 3 *over an algebraically closed field k has a non-classical type of gap sequence at ordinary points, then the characteristic of k must be* 3, C *is birationally equivalent to the plane curve*

$$y^3 + y - x^4 = 0, \tag{28}$$

and C *has the gap sequence* $\{1, 2, 4\}$ *at ordinary points.*

REMARK ([4]). The Riemann surface defined by the equation (28) together with the Riemann surface defined by

$$x^4 + y^4 + 1 + 3(x^2y^2 + x^2 + y^2) = 0$$

have 12 Weierstrass points. This number is least among non-hyperelliptic Riemann surfaces of genus 3. By reduction mod 3, both surfaces coincide with the curve in Th. 1.

§3. Trigonality of curves of genus 4

A non-hyperelliptic curve C of genus $g \geq 3$ over an algebraically closed field k is called *trigonal* if C carries a fixed point free, linear system g_3^1 of degree 3 and of dimension 1 ([1], p. 308). Since C is non-hyperelliptic, g_3^1 must be complete. This definition is equivalent to say that there exists a function x in $K=k(\mathrm{C})$ such that $[K: k(x)]=3$. We call g_3^1 a trigonal linear system. A point P such that $2\mathrm{P}+\mathrm{P}'$ is in g_3^1 is called g_3^1-special ([6]). Let D be a divisor in g_3^1. If $\{1, x\}$ is a base of $L(D)$ over k, then g_3^1-special points are nothing but ramification points of the covering $\pi: \mathrm{C}\to\mathbf{P}^1$ defined by $\pi(\mathrm{Q})=(1, x(\mathrm{Q}))$ where $\mathbf{P}^1$ is the projective line over k with homogeneous coordinates (X_0, X_1).

First we state the following classical fact ([3], p. 527) and give a proof for it.

Theorem 2. *A non-hyperelliptic curve* C *of genus* 4 *over an algebraically closed field k of any characteristic is trigonal and* C *has at most two trigonal linear systems.*

Proof. First we shall prove that C is trigonal. Let P be a point of C such that $l(3\mathrm{P})=1$. By Riemann-Roch theorem or (4), there exists a positive divisor E of degree 3 such that $3\mathrm{P}+E$ is canonical. If the linear system $|\mathrm{P}+E|$ has a fixed component, then C is trigonal. Hence we may assume $|\mathrm{P}+E|$ is free from fixed components. Therefore, there exists a function x in $K=k(\mathrm{C})$ such that

$$\operatorname{div}(x) = A - (\mathrm{P} + E)$$

where A is a positive divisor whose components are different from those of $\mathrm{P}+E$. By Riemann-Roch theorem, there exist functions y and z in K such that

$$\operatorname{div}(y) = A' - (2\mathrm{P} + E) \quad \text{and} \quad \operatorname{div}(z) = A'' - (3\mathrm{P} + E)$$

where A' and A'' are positive divisors whose components are different from P and those of E. From $[K: k(x)]=4$ and $[K: k(y)]=5$ it follows $K=k(x, y)$. Let $f(x, y)=0$ be a defining irreducible equation of x, y and C$'$ be the affine plane curve defined by this equation. We show that C$'$ must have multiple points. In the space $L(4\mathrm{P}+2E)$,

$$1,\ x,\ y,\ z,\ x^2,\ xy \text{ and } y^2$$

form a base over k as easily seen. Since $xz\in L(4\mathrm{P}+E)$, we may put

$$(x - c)z = c_0 + c_1x + c_2y + c_3x^2 + c_4xy + c_5y^2$$

where $c_5\neq 0$. If z is in $k[x, y]$, then, since $[K: k(x)]=4$ and y is finite over $k[x]$, we can write z uniquely in the form

$$z = A_0(x) + A_1(x)y + A_2(x)y^2 + A_3(x)y^3$$

where $A_i(x)\in k[x]$ for $i=0, 1, 2, 3$, and hence we have $(x-c)A_2(x)=c_5$. This contradicts $c_5\neq 0$. Therefore $z\notin k[x, y]$. On the other hand, z is finite over $k[x, y]$. This implies C$'$ must be not normal and hence singular ([5], p. 122).

Let $\rho: \mathrm{C}\to\mathrm{C}'$ be the birational map defined by $\rho(\mathrm{Q})=(x(\mathrm{Q}), y(\mathrm{Q}))$, and let Q$'$ be a multiple point on C$'$. Put $\{\mathrm{Q}_1, \mathrm{Q}_2,\dots, \mathrm{Q}_r\}=\rho^{-1}(\mathrm{Q}')$. If we put

$\alpha=x(\mathrm{Q}')$, $\beta=y(\mathrm{Q}')$, then

$$\operatorname{div}(x-\alpha)=\sum_{i=1}^{r} n_i \mathrm{Q}_i+B-(\mathrm{P}+E),$$

$$\operatorname{div}(y-\beta)=\sum_{i=1}^{r} m_i \mathrm{Q}_i+B'-(2\mathrm{P}+E)$$

where n_i, $m_i \geqq 1$ for $i=1, 2,\dots, r$ and B, $B' \geqq 0$. Since we have

$$\mathrm{P}+\sum_{i=1}^{r}(n_i-1)\mathrm{Q}_i+B \sim \sum_{i=1}^{r}(m_i-1)\mathrm{Q}_i+B' \tag{29}$$

where P is different from $\mathrm{Q}_1, \mathrm{Q}_2,\dots, \mathrm{Q}_r$ and components of B', and $\deg(\mathrm{P}+\sum(n_i-1)\mathrm{Q}_i+B)=5-r$, we have clearly $1 \leqq r \leqq 3$. If $r=3$, then (29) implies C is hyperelliptic. This is a contradiction. If $r=2$, then (29) implies C is trigonal. If $r=1$, expressing $x-\alpha$, $y-\beta$ by power series in a local parameter t at Q_1:

$$x-\alpha=a_1t+a_2t^2+\cdots \quad \text{and} \quad y-\beta=b_1t+b_2t^2+\cdots,$$

we have $a_1=b_1=0$ by non-regularity of the local ring at Q′ ([5], Ch. 8, 3). Hence n_1, $m_1 \geqq 2$ and so $\mathrm{P}+(n_1-2)\mathrm{Q}_1+B \sim (m_1-2)\mathrm{Q}_1+B'$. Therefore C is trigonal.

Now assume there exist three distinct trigonal systems on C:

$$g_3^1=|D|,\ h_3^1=|D'| \quad \text{and} \quad k_3^1=|D''|.$$

Let $\{1, x\}$ and $\{1, y\}$ be bases of $L(D)$ and $L(D')$ respectively. Then $1, x, y, xy$ are clearly linearly independent elements of $L(D+D')$, and hence by Riemann-Roch theorem $D+D'$ is a canonical divisor. By the same reason, $D+D''$ is canonical, and hence $D' \sim D''$. This is a contradiction. Thus Theorem 2 is proved.

Corollary. *Let* C *be a curve of genus* 4. *If* C *has two trigonal linear systems* $g_3^1=|D|$ *and* $h_3^1=|D'|$, *then* $D+D'$ *is a canonical divisor, and there is no divisor* $\mathrm{P}_1+\mathrm{P}_2$ *such that* $\mathrm{P}_1+\mathrm{P}_2+\mathrm{P}\in g_3^1$ *and* $\mathrm{P}_1+\mathrm{P}_2+\mathrm{Q}\in h_3^1$. *If* C *has a unique trigonal linear system* $g_3^1=|D|$, *then* g_3^1 *is half canonical, namely* $2D$ *is canonical.*

Proof. In former case, we know $D+D'$ is canonical by the proof of Th. 2. Assume $\mathrm{P}_1+\mathrm{P}_2+\mathrm{P}\in g_3^1$ and $\mathrm{P}_1+\mathrm{P}_2+\mathrm{Q}\in h_3^1$. Then $D+\mathrm{P}_1+\mathrm{P}_2+\mathrm{Q} \sim \mathrm{P}_1+\mathrm{P}_2+\mathrm{P}+D'$, and hence $D+\mathrm{Q}\sim \mathrm{P}+D'$. By Riemann-Roch theorem, we know $l(D+\mathrm{Q})=2$, and hence $l(D+\mathrm{Q})=l(D)$. This implies Q must be a fixed component of $|D+\mathrm{Q}|$. Since we can assume the components of D' are different from Q, this leads to a contradiction. In latter case, from Riemann-Roch theorem it follows that $i(D)=l(D)=2$, and hence there exists a positive divisor E such that $D+E$ is canonical. Then, also by Riemann-Roch theorem we know $l(E)=l(D)=2$. Hence by uniqueness of g_3^1 it follows that $E\sim D$, and so $2D$ is canonical.

Returning to our case, here we prove the following

LEMMA 1. *Assume that an ordinary point of a curve* C *of genus* 4 *defined over an algebraically closed field k has a non-classical type of gap sequence. If* C *has two trigonal linear systems, then the characteristic of k must be* 2. *Moreover if* $2P+P'$ *belongs to one of trigonal linear systems,* P *must be* P′.

PROOF. Assume C has two trigonal linear systems g_3^1 and h_3^1. Let P be a point such that $2P+P'\in g_3^1$ and let D and D' be divisors of g_3^1 and h_3^1 respectively such that their components are different from P. Then P is not h_3^1-special by Cor. to Th. 2. Thus if $P+P_1+P_2$ is a divisor in h_3^1, $P\neq P_1$ and $P\neq P_2$. Hence we have 4 canonical divisors:

$$(30)\qquad D+D',\ P+P_1+P_2+D,\ 2P+P'+D' \quad\text{and}\quad 3P+P'+P_1+P_2.$$

If $P\neq P'$, then Hermitian P-invariants must be 0, 1, 2, 3 by (4). This contradicts the assumption that C has a non-classical type of gap sequence. Therefore $P=P'$, and hence $3P\in g_3^1$. A similar argument is valid for h_3^1. As for characteristic p of k, by Th. C and Th. D in § 1 we have $p=2$ or 5. Let $\{1, x\}$ be a base of the space $L(D)$ over k. If Q is a component of the zero divisor of dx, then Q is g_3^1-special, and so by the above assertion, $3Q\in g_3^1$. Hence by (30) where Q takes the place of P, we know that the Hermitian Q-invariants are 0, 1, 3, 4. Assume characteristic $p=5$. Then by (22), the gap sequence at ordinary points must be 1, 2, 3, 6, and hence the orders of a Wronskian determinant of C are 0, 1, 2, 5. This contradicts $h_3\geqq\mu_3$ in (13). Therefore $p=2$. Thus Lemma 1 is proved.

Now we determine curves of genus 4 with two different trigonal linear systems.

PROPOSITION 1. *Let* C *be a curve of genus* 4 *defined over an algebraically closed field k of characteristic* $p\neq 3$ *which has two trigonal linear systems* g_3^1 *and* h_3^1. *If* $3P\in g_3^1$ *for each* g_3^1*-special point* P, *then* C *is birationally equivalent to a plane curve defined by an equation*

$$f(x, y)=\gamma_0(x)y^3+\gamma_1(x)=0$$

where $\gamma_0(x)$ *and* $\gamma_1(x)$ *are polynomials in x of degree* 3.

In particular, if we have also $3Q\in h_3^1$ *for each* h_3^1*-special point* Q, *then* C *is birationally equivalent to a plane curve*

$$(31)\qquad f(x, y)=x^3y^3+x^3+y^3+\lambda=0, \qquad \lambda\neq 0, 1.$$

Conversely, a curve over an algebraically closed field k of characteristic $p\neq 3$ *which is birationally equivalent to a plane curve* (31) *is of genus* 4 *and has* *two trigonal linear systems such that if* $2P+P'$ *belongs to one of them*, $P=P'$.

PROOF. Let

$$D = Q_1 + Q_2 + Q_3 \quad \text{and} \quad D' = Q_1' + Q_2' + Q_3',$$

where $Q_1, Q_2, Q_3, Q_1', Q_2', Q_3'$ are distinct each other, be divisors in g_3^1 and h_3^1 respectively. Let x and y be non constant functions in $K=k(C)$ such that

$$\text{div}(x) = P_1 + P_2 + P_3 - D \text{ and } \text{div}(y) = P_1' + P_2' + P_3' - D' \tag{32}$$

where P_i, P_i', Q_i, Q_i' for all $i=1, 2, 3$ are distinct each other. Then, by assumptions for g_3^1-special points and D, we have

$$\text{div}(dx) = 2\textstyle\sum_{i=1}^{6} S_i - 2D$$

where $\{S_1, S_2,..., S_6\}$ is the set of all g_3^1-special points. It is clear that $\{S_1, S_2,..., S_6\}$ is disjoint from $\{Q_1, Q_2, Q_3\}$. Now we may assume $P_1'=S_1$ and $Q_1'=S_4$ by Cor. to Th. 2. As $k(x)\neq k(y)$, we have $K=k(x, y)$. 16 functions $x^i y^j$ for $i, j=0, 1, 2, 3$ clearly belong to the space $L(3(D+D'))$. As $l(3(D+D'))=15$, these functions are linearly dependent over k. Hence C is birationally equivalent to a plane curve

$$f(x, y) = \gamma_0(x)y^3 + \gamma_1(x)y^2 + \gamma_2(x)y + \gamma_3(x) = 0 \tag{33}$$

where $\gamma_i(x)$ for $i=0, 1, 2, 3$ are polynomials of degree at most 3. Put $\alpha_i=x(S_i)$ and $\beta_i=y(S_i)$ for $i=1, 2,..., 6$. Then we have

$$\text{div}(x-\alpha_i) = 3S_i - D, \qquad i = 1, 2,..., 6. \tag{34}$$

If β_i is finite, then by (34), $y=\beta_i$ is a common root of $f(\alpha, y)=0$,

$$\partial_y f(\alpha, y) = 3\gamma_0(\alpha)y^2 + 2\gamma_1(\alpha)y + \gamma_2(\alpha) = 0 \quad \text{and}$$

$$\partial_y^2 f(\alpha, y) = 3\gamma_0(\alpha)y + \gamma_1(\alpha) = 0$$

where $\alpha=\alpha_i$ and ∂_y^ν: $k[x, y]\rightarrow k[x, y]$ for $\nu=0, 1, 2,...$ are the partial derivatives defined by

$$\partial_y^\nu\left(\sum a_{mn}x^m y^n\right) = \sum \binom{n}{\nu} a_{mn}x^m y^{n-\nu}. \tag{35}$$

Hence β_i is also a common root of

$$\partial_y f(\alpha, y) - y\partial_y^2 f(\alpha, y) = \gamma_1(\alpha)y + \gamma_2(\alpha) = 0 \quad \text{and}$$

$$3f(\alpha, y) - 2y\partial_y f(\alpha, y) + y^2\partial_y^2 f(\alpha, y) = \gamma_2(\alpha)y + 3\gamma_3(\alpha) = 0.$$

Therefore, $x=\alpha_i$ must be a common root of

(36) $$\gamma_1(x)\gamma_2(x) - 9\gamma_0(x)\gamma_3(x) = 0 \text{ and } \gamma_2^2(x) - 3\gamma_1(x)\gamma_3(x) = 0.$$

If β_i is infinite, then by (34), $y^{-1}=0$ is a triple root of

$$\gamma_0(\alpha_i) + \gamma_1(\alpha_i)y^{-1} + \gamma_2(\alpha_i)y^{-2} + \gamma_3(\alpha_i)y^{-3} = 0,$$

and hence

$$\gamma_0(\alpha_i) = \gamma_1(\alpha_i) = \gamma_2(\alpha_i) = 0,$$

and so $x=\alpha_i$ is also a common root of (36). Since $S_1=P_1'$ and $S_4=Q_1'$, $\beta_1=0$ and β_4 is infinite, and hence $\gamma_i(x)$ for $i=1, 2, 3$ are divisible by $x-\alpha_1$, and $\gamma_i(x)$ for $i=0, 1, 2$ are divisible by $x-\alpha_4$. Hence we can put

$$\gamma_2^2(x) - 3\gamma_1(x)\gamma_3(x) = (x-\alpha_1)^2(x-\alpha_4)\gamma(x)$$

where $\gamma(x)$ is a polynomial of degree at most 3. But the equation $\gamma(x)=0$ must have four distinct roots α_2, α_3, α_5 and α_6. Hence the polynomial $\gamma(x)$ must be equal to zero. Therefore

(37) $$\gamma_2^2(x) = 3\gamma_1(x)\gamma_3(x).$$

On the other hand, $\gamma_3(x)$ must be squarefree and of degree 3. In fact, put $\alpha'=x(P_2')$ and $\alpha''=x(P_3')$. Then $\gamma_3(x)$ is divisible by $x-\alpha_1$, $x-\alpha'$ and $x-\alpha''$ by (32). If $\alpha_1=\alpha'$, then $\operatorname{div}(x-\alpha_1)=P_1'+P_2'+P'-D$ for some point P', and hence $P_1'+P_2'+P'\in g_3^1$. On the other hand, by (32) $P_1'+P_2'+P_3'\in h_3^1$. But by Cor. to Th. 2, this is impossible. Therefore $\alpha_1\neq\alpha'$. Similarly we have $\alpha_1\neq\alpha''$ and $\alpha'\neq\alpha''$. Therefore by (37),

$$\gamma_1(x) = b\gamma_3(x) \quad \text{and} \quad \gamma_2(x) = c\gamma_3(x).$$

Since $\gamma_0(\alpha_4)=\gamma_1(\alpha_4)=\gamma_2(\alpha_4)=0$, we have $\gamma_3(\alpha_4)\neq 0$ by (33), and hence $b=c=0$. Hence the polynomials $\gamma_1(x)$ and $\gamma_2(x)$ are equal to zero. Therefore

$$f(x, y) = \gamma_0(x)y^3 + \gamma_3(x).$$

Here we note that the covering $\pi\colon C\to\mathbf{P}^1$ defined by $\pi(Q)=(1, x(Q))$ is ramified over $x=\alpha_i$ for $i=1, 2,\ldots, 6$ which are roots of $\gamma_0(x)\gamma_3(x)=0$ by (36). Therefore $\deg\gamma_0(x)=\deg\gamma_3(x)=3$. We also note that $K/k(x)$ is a cyclic extension of degree 3 and $y^3\in k(x)$.

Moreover assume that $3P\in h_3^1$ for each h_3^1-special point P. Then if we replace x by $(x-\alpha)/(x-\beta)$ for some α and β in k, we may put

$$f(x, y) = (x^3+a)y^3 + c(x^3+b) = x^3y^3 + cx^3 + ay^3 + cb$$

as seen in the same way as the above. Here, since $(x^3+a)(x^3+b)=0$ must have six distinct roots, $ab\neq 0$ and $a\neq b$. Therefore replacing x by $a^{1/3}x$ and y by

$c^{1/3}y$, C is birationally equivalent to a plane curve defined by an equation (31).

Conversely, let C be a curve over k of characteristic $p \neq 3$ which is birationally equivalent to a plane curve defined by (31). Since $[k(C): k(x)] = [k(C): k(y)] = 3$ and $k(C) = k(x, y)$, C has two distinct trigonal linear systems and we can put

$$\operatorname{div}(x) = \sum_{i=1}^{3} P_i - \sum_{i=1}^{3} Q_i \quad \text{and} \quad \operatorname{div}(y) = \sum_{i=1}^{3} P'_i - \sum_{i=1}^{3} Q'_i.$$

Here since $f(0, y) = y^3 + \lambda = 0$, we know that $P_i \neq P_j$ for $i \neq j$. And from the equation $y^3 + 1 + x^{-3}y^3 + \lambda x^{-3} = 0$, we know that $Q_i \neq Q_j$ for $i \neq j$. This implies also that the covering $\pi\colon C \to \mathbf{P}^1$ defined by $\pi(P) = (1, x(P))$ is not ramified over $x = \infty$. Let C′ be the affine plane curve defined by (31). From

$$f_x(x, y) = 3x^2(y^3 + 1) \quad \text{and} \quad f_y(x, y) = 3y^2(x^3 + 1)$$

it follows that C′ is non-singular. Therefore $\pi\colon C \to \mathbf{P}^1$ is ramified exactly over the values of x such that $f_y(x, y) = 0$, namely over the roots of $(x^3 + \lambda)(x^3 + 1) = 0$. Hence from (31), we know that π is completely ramified exactly at P'_1, P'_2, P'_3, Q'_1, Q'_2 and Q'_3. Therefore we have

$$\operatorname{div}(dx) = 2\sum_{i=1}^{3} P'_i + 2\sum_{i=1}^{3} Q'_i - 2\sum_{i=1}^{3} Q_i.$$

From $\deg(dx) = 6$, it follows that the genus of C is 4. By symmetry of the equation (31) with respect to x and y, we know that the covering $\pi'\colon C \to \mathbf{P}^1$ defined by $\pi'(P) = (1, y(P))$ is also completely ramified at P_i, Q_i for $i = 1, 2, 3$. Thus our Proposition is proved.

REMARK 1. *Let C_λ be a curve over an algebraically closed field k of characteristic $p \neq 3$ which is birationally equivalent to a plane curve* (31). *Then $C_\lambda \simeq C_{\lambda'}$ if and only if $\lambda' = \lambda$ or λ^{-1}.* In fact, we may suppose $C_{\lambda'}$ is birationally equivalent to the plane curve $x_1^3 y_1^3 + x_1^3 + y_1^3 + \lambda' = 0$. Suppose $C_\lambda \simeq C_{\lambda'}$. Let z, w in $k(C_\lambda)$ be images of x_1, y_1 in $k(C_{\lambda'})$ by an isomorphism $k(C_{\lambda'}) \to k(C_\lambda)$. Then we have

$$z^3 w^3 + z^3 + w^3 + \lambda' = 0.$$

Since $[k(C_\lambda): k(z)] = [k(C_\lambda): k(w)] = 3$, we have $k(z) = k(x)$ and $k(w) = k(y)$, and hence we can put

$$z = \frac{a_1 x + b_1}{c_1 x + d_1} \quad \text{and} \quad w = \frac{a_2 y + b_2}{c_2 y + d_2}.$$

From the proof of Prop. 1, we know that the coverings $\pi\colon C_\lambda \to \mathbf{P}^1$ and $\pi'\colon C_\lambda \to \mathbf{P}^1$ defined by $\pi(P) = (1, x(P))$ and $\pi'(P) = (1, z(P))$ are completely ramified over the roots of $(x^3 + \lambda)(x^3 + 1) = 0$ and over the roots of $(z^3 + \lambda')(z^3 + 1) = 0$ respectively, and that the set of ramification points of π is equal to that of π'. Hence, if P is a ramification point of π, then $(x(\mathrm{P})^3+\lambda)(x(\mathrm{P})^3+1)=0$ and $(z(\mathrm{P})^3+\lambda')(z(\mathrm{P})^3+1)=0$. If $x(\mathrm{P})^3+\lambda=0$ and $z(\mathrm{P})^3+\lambda'=0$, then $y(\mathrm{P})=w(\mathrm{P})=0$ and hence $b_2=0$. In this case, since there exists another ramification point Q of π such that $x(\mathrm{Q})^3+1=0$ and $z(\mathrm{Q})^3+1=0$, we have $y(\mathrm{Q})=w(\mathrm{Q})=\infty$, and hence $c_2=0$. Therefore we may put $w=by$. By the same way, in conclusion, we may assume that

$$z = ax,\ w = by \quad \text{or} \quad z = ax,\ w = by^{-1}.$$

Thus putting these in (31), we obtain $\lambda'=\lambda$ and $\lambda'=\lambda^{-1}$ respectively. Conversely if $\lambda'=\lambda^{-1}$, then by the transformation $x_1=z^{-1}$ and $y_1=\lambda^{-1/3}w$, we get $z^3w^3+z^3+w^3+\lambda=0$, and hence $\mathrm{C}_\lambda\simeq\mathrm{C}_{\lambda'}$.

REMARK 2. The curves which are birationally equivalent to the plane curves (31) for limiting values $\lambda=1$ and $\lambda=0$ become respectively reducible and elliptic.

Next we seek an equation of a curve of genus 4 with a unique trigonal linear system.

PROPOSITION 2. *Let* C *be a curve of genus* 4 *over an algebraically closed field of characteristic* $p\neq 3$ *which has a unique trigonal linear system* g_3^1. *Let*

$$D = 2\mathrm{P} + \mathrm{P}'$$

be a divisor in g_3^1. *Then* C *is birationally equivalent to a plane curve*

$$f(x, y) = y^3 + \gamma(x)y + (x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_s)^{m_s} = 0$$

satisfying the following conditions:

(i) $\alpha_i\neq\alpha_j$ *for* $i\neq j$, *and* $\deg\gamma(x)\leq 4$.

(ii) *there exist* P_i, P'_i *and* P''_i *such that*

$$\mathrm{div_C}(x-\alpha_i) = \mathrm{P}_i + \mathrm{P}'_i + \mathrm{P}''_i - D, \qquad i = 1, 2,\ldots, s,$$
$$\mathrm{div_C}(y) = m\mathrm{P}' + \textstyle\sum_{i=1}^s m_i\mathrm{P}'_i - E$$

where $4\mathrm{P}\leqq E\leqq 2D$ *and* m *is a non-negative integer which is* 0 *unless* $E=4\mathrm{P}$. *Moreover, if* $\mathrm{P}_i=\mathrm{P}'_i$ *for some* i, *then* $m_i=1$, $\gamma(x)$ *is divisible by* $x-\alpha_i$ *and* $\mathrm{P}_i+\mathrm{P}'_i+\mathrm{P}''_i=3\mathrm{P}_i$.

PROOF. Since $l(2D)=4$, there exist functions x and z on C, rational over k, such that

$$\text{(38)} \qquad \mathrm{div}(x) = A - D \quad \text{and} \quad \mathrm{div}(z) = B - 2D$$

where A and B are positive divisors whose components are different from P, P′, and such that $1, x, x^2, z$ are a base of $L(2D)$. Then $k(\mathrm{C})=k(x, z)$. Indeed, if we assume $k(\mathrm{C})\neq k(x, z)$, then, since $[k(\mathrm{C})\colon k(x)]=3$, $k(x, z)=k(x)$, and hence $z\in k[x]$ by integrality of z over $k[x]$ by (38). So we may put $z=a+bx+cx^2$

because the pole divisor of the right side must be equal to $2D$. This contradicts linear independence of $1, x, x^2, z$. Let $f_1(x, z)=0$ be an irreducible equation for x and z. Since intersection number of the plane curve $f_1(x, z)=0$ and a generic line $ax+bz+c=0$ over k is equal to the degree of the zero divisor of the function $ax+bz+c$ on C, we know that $\deg f_1(x, z)=\deg 2D=6$. Hence, by integrality of z over $k[x]$, we can write

$$f_1(x, z) = z^3 + \gamma_1(x)z^2 + \gamma_2(x)z + \gamma_3(x)$$

where $\deg\gamma_1(x)\leqq 4$, $\deg\gamma_2(x)\leqq 5$ and $\deg\gamma_3(x)=6$. Here the coefficient of x^4z^2 is zero because x^4z^2 is the only term with the least $v_{\mathrm{P}}(x^4z^2)$. Also the respective coefficients a and b of x^5z and x^3z^2 are zero. In fact, if we assume a or b are not zero, then we must have $\operatorname{div}(ax^5z+bx^3z^2)\geqq -6D$, and hence $\operatorname{div}(ax^2+bz)\geqq -D$, so it contradicts independence of $1, x, x^2, z$. Therefore $\deg\gamma_1(x)\leqq 2$ and $\deg\gamma_2(x)\leqq 4$. Put $y=z+3^{-1}\gamma_1(x)$. Then we know clearly that $\{1, x, x^2, y\}$ is a base of $L(2D)$ and C is birationally equivalent to a plane curve

$$f(x, y) = y^3 + \gamma(x)y + \delta(x) = 0 \tag{39}$$

where $\deg\gamma(x)\leqq 4$ and $\deg\delta(x)\leqq 6$. Replacing x by kx if necessary, we may assume

$$\delta(x) = (x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_s)^{m_s}$$

where $m_i\geqq 1$ and $\alpha_i\neq\alpha_j$ for $i\neq j$. We can put

$$\operatorname{div}(x-\alpha_i) = \mathrm{P}_i + \mathrm{P}'_i + \mathrm{P}''_i - D, \qquad i = 1, 2,\dots, s. \tag{40}$$

Now we shall show that, if we put

$$\operatorname{div}(y) = B' - E$$

where B' and E are zero and pole divisors of y respectively, then $4\mathrm{P}\leqq E\leqq 2D$. From (38) and $\deg\gamma_1(x)\leqq 2$, it follows that $E\leqq 2D$. Since C is non-hyperelliptic, we know $\deg E\geqq 3$. If $\deg E=3$ then $[k(\mathrm{C}): k(y)]=3$, and hence $k(x)=k(y)$ by uniqueness of the trigonal linear system. It contradicts $k(\mathrm{C})=k(x, y)$. Therefore E must be one of the following divisors;

$$2\mathrm{P} + 2\mathrm{P}',\ 3\mathrm{P} + \mathrm{P}',\ 4\mathrm{P},\ 3\mathrm{P} + 2\mathrm{P}',\ 4\mathrm{P} + \mathrm{P}' \ \text{ and } \ 4\mathrm{P} + 2\mathrm{P}'.$$

Here, $|E|$ has no fixed component. But $|2\mathrm{P}+2\mathrm{P}'|$ has a fixed component because $l(2\mathrm{P}+2\mathrm{P}')=2$ by Riemann-Roch theorem and hence $l(2\mathrm{P}+2\mathrm{P}')=l(2\mathrm{P}+\mathrm{P}')$. Therefore $E\neq 2\mathrm{P}+2\mathrm{P}'$. Similarly we have $E\neq 3\mathrm{P}+\mathrm{P}'$. Moreover, we have $E\neq 3\mathrm{P}+2\mathrm{P}'$. Indeed, if we assume $E=3\mathrm{P}+2\mathrm{P}'$, then $[k(x, y): k(y)]=5$, and hence $\deg\delta(x)=5$. Then the coefficient of x^4y must be zero because x^4y in (39) is the only term with the least $v_P(x^4y)=-11$. Then we know similarly the coefficient of x^5 must be zero. This is a contradiction. Therefore we have $4P \leqq E \leqq 2D$.

Next we seek components of B'. If $y(Q)=0$ and $Q \neq P'$, then by (39), $\delta(x(Q))=0$ and hence $x(Q)=\alpha_i$ for some $i=1, 2,\dots, s$. Therefore Q must coincide with one of P_i, P'_i and P''_i. But two of P_i, P'_i and P''_i must not be components of B'. To see this, if we assume that $\operatorname{div}(y)=P_i+P'_i+B''-2D$ where $B''>0$, then by Cor. to Th. 2, $P_i+P'_i+P''_i+D$ and $P_i+P'_i+B''$ are canonical, and hence $P''_i+D \sim B''$. By Riemann-Roch theorem, we have $l(P''_i+D)=1+l(P_i+P'_i)=2$, and hence $l(P''_i+D)=l(D)$. This implies P''_i is a fixed component of $|P''_i+D| = |B''|$. Hence P''_i is a component of B''. Therefore $\operatorname{div}(y/(x-\alpha_i)) \geqq -D$. Since $\{1, x\}$ is a base of $L(D)$, we have $y=(x-\alpha_i)(ax+b)$. This is impossible.

Therefore we have

$$\operatorname{div}(y) = mP' + \textstyle\sum_{i=1}^{s} n_i P'_i - E$$

where m is a non-negative integer which is equal to 0 unless $E=4P$. Lastly we show that $n_i=m_i$ for $i=1, 2,\dots, s$. If $P_i \neq P'_i$, then $\gamma(\alpha_i) \neq 0$ by (39), and hence we have $v_{P'_i}(x-\alpha_i)=1$. Therefore $m_i=v_{P'_i}(\delta(x))=v_{P'_i}(y^3+\gamma(x)y)=v_{P'_i}(y)+v_{P'_i}(y^2+\gamma(x))=v_{P'_i}(y)=n_i$. If $P_i=P'_i$, then $\gamma(\alpha_i)=0$, and hence we know that $\gamma(x)$ is divisible by $x-\alpha_i$ and that $y(P''_i)=0$. Since it is impossible that $B' \geqq P'_i + P''_i$, we know that $P_i=P'_i=P''_i$ and $n_i=1$. Hence $v_{P'_i}(x-\alpha_i)=3$ and $v_{P'_i}(y)=1$, and so $v_{P'_i}(\gamma(x)) \geqq 3$. Therefore from $v_{P'_i}(\delta(x))=v_{P'_i}(y^3+\gamma(x)y)$, we have $3m_i=3$, and hence $m_i=1=n_i$. Thus Proposition is proved.

Applying Prop. 2 to the case of characteristic 2, we obtain the following

COROLLARY 1. *Let notations and assumptions be as in Prop.* 2. *If the defining field* k *of* C *is of characteristic* 2 *and* $P \neq P'$, *then* $E=4P$ *or* $4P+P'$, $\deg \gamma(x)=4$, $\operatorname{div}(x-\alpha_i)=2P_i+P'_i-D$ *for all* i, *and* $\{P, P_1,\dots, P_s\}$ *is the set of all ramification points of the covering* π: $C \to \mathbf{P}^1$ *defined by* $\pi(Q)=(1, x(Q))$.

PROOF. Assume $E=2D$ and then we can put

$$f(x, y) = y^3 + (\textstyle\sum_{i=0}^{4} b_i x^i)y + \sum_{i=0}^{6} c_i x^i$$

where $c_6 \neq 0$. Represent x and y by power series of a local parameter t at P;

$$x = t^{-2} + \alpha_{-1}t^{-1} + \alpha_0 + \cdots$$

$$y = \beta_{-4}t^{-4} + \beta_{-3}t^{-3} + \cdots,\ \beta_{-4} \neq 0.$$

We put these in $f(x, y)=0$. From coefficients of t^{-12} and t^{-11}, we obtain $\beta_{-4}^3 + b_4\beta_{-4}+c_6=0$ and $\beta_{-4}^2\beta_{-3}+b_4\beta_{-3}+6c_6\alpha_{-1}=(\beta_{-4}^2+b_4)\beta_{-3}=0$. Since $\beta_{-4}(\beta_{-4}^2+b_4)=c_6 \neq 0$, we have $\beta_{-3}=0$. Therefore

$$\operatorname{div}(y-\beta_{-4}x^2-(\beta_{-2}-\beta_{-4}\alpha_{-1}^2)x) \geqq -(\mathrm{P}+2\mathrm{P}').$$

This implies $l(\mathrm{P}+2\mathrm{P}')\geqq 2$ and hence we have a contradiction $\mathrm{P}+2\mathrm{P}'\in g_3^1$. $E\neq 2D$ implies $E=4\mathrm{P}$ or $4\mathrm{P}+\mathrm{P}'$ and hence $\deg\gamma(x)=4$. If $\mathrm{P}_i\neq\mathrm{P}_i''$, then since $f(\alpha_i, y) =y(y^2+\gamma(\alpha_i))=y(y+\sqrt{\gamma(\alpha_i)})^2$, we have $\operatorname{div}(y+\sqrt{\gamma(\alpha_i)})=\mathrm{P}_i+\mathrm{P}_i''+B''-E$ where B'' is a positive divisor whose components are different from those of E. This is impossible by the similar argument as in the proof of Prop. 2. Therefore we have $\operatorname{div}(x-\alpha_i)=2\mathrm{P}_i+\mathrm{P}_i'-D$. As for the last assertion, P, $\mathrm{P}_1,\dots$ and P_s are clearly ramification points of π. Let R be a ramification point other than P and put $\alpha=x(\mathrm{R})$ and $\beta=y(\mathrm{R})$. Then we have $f_y(\alpha, \beta)=0$. Since we have

$$f(x, y)-yf_y(x, y)=(x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_s)^{m_s},$$

$x(\mathrm{R})=\alpha_i$ for some $i=1, 2,\dots, s$. Therefore R must be one of P_i. Thereby Cor. 1 is proved.

Another application of Prop. 2 is the following

COROLLARY 2. *Under the same notations and assumptions as Prop.* 2, *if the covering* $\pi\colon \mathrm{C}\to\mathbf{P}^1$ *defined by* $\pi(\mathrm{Q})=(1, x(\mathrm{Q}))$ *is completely ramified at each ramification point, then* C *is birationally equivalent to a plane curve*

$$y^3+(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_5)=0 \tag{41}$$

where $\alpha_i\neq\alpha_j$ *for* $i\neq j$.

Conversely, if a curve C *over an algebraically closed field k of characteristic* $p\neq 3$ *is birationally equivalent to a plane curve* (41), *then* C *is of genus* 4 *and has a unique trigonal linear system* g_3^1. *Moreover we have* $3\mathrm{Q}\in g_3^1$ *for every* g_3^1*-special point* Q.

PROOF. Let R be a ramification point other than P, and put $\alpha=x(\mathrm{R})$ and $\beta=y(\mathrm{R})$. Then β is a triple root of

$$f(\alpha, y)=y^3+\gamma(\alpha)y+\delta(\alpha)=0,$$

and so $\beta=0$. Therefore R is one of P_i', $i=1, 2,\dots, s$, in Prop. 2 and $\gamma(\alpha)=0$. Let $\{\mathrm{P}, \mathrm{R}_1,\dots, \mathrm{R}_t\}$ be the set of all ramification points of $\pi\colon \mathrm{C}\to\mathbf{P}^1$. Then since $\operatorname{div}(x-x(\mathrm{R}_i))=3\mathrm{R}_i-3\mathrm{P}$ for $i=1, 2,\dots, t$, we have $\operatorname{div}(dx)=2\sum_{i=1}^t\mathrm{R}_i-4\mathrm{P}$. Hence from $\deg(dx)=2t-4$, we have $t=5$. Since $\deg\gamma(x)\leqq 4$ by Prop. 2 and $\gamma(x(\mathrm{R}_i))=0$, we have $\gamma(x)=0$, and hence $\mathrm{P}_1', \mathrm{P}_2',\dots, \mathrm{P}_s'$ in Prop. 2 are ramification points of π. Therefore by Prop. 2, we obtain (41).

As for the converse part, since $k(\mathrm{C})/k(x)$ is a cyclic extension of degree 3, the covering $\pi\colon \mathrm{C}\to\mathbf{P}^1$ defined by $\pi(\mathrm{Q})=(1, x(\mathrm{Q}))$ is completely ramified at each ramification point. Put $\operatorname{div}(x)=D'-D$ where D and D' are positive divisors of degree 3, and put $u=x^{-1}$, $v=x^{-2}y$, then

$$v^3 + u(1 - \alpha_1 u)(1 - \alpha_2 u)\cdots(1 - \alpha_5 u) = 0.$$

Hence we can put $D=3\mathrm{P}$. By (41), we can also put $\operatorname{div}(x-\alpha_i)=3\mathrm{P}_i-3\mathrm{P}$ for $i=1, 2,\dots, 5$. If R is a ramification point of π other than P, then by the same argument as in the first part of this proof, R is one of P_i. Therefore we have $\operatorname{div}(dx)=2\sum_{i=1}^5\mathrm{P}_i-4\mathrm{P}$. Since $\deg(dx)=6$, we know C is of genus 4. Since $\operatorname{div}(y^3)=\sum_{i=1}^5\operatorname{div}(x-\alpha_i)=3(\sum_{i=1}^5\mathrm{P}_i-5\mathrm{P})$, we have $\operatorname{div}(y)=\sum_{i=1}^5\mathrm{P}_i-5\mathrm{P}$, and hence $y\in L(2D)$. $1, x, x^2, y$ are clearly independent elements in $L(2D)$ over k. Hence we know $l(2D)=4$ and hence $2D$ is a canonical divisor by Riemann-Roch theorem. Therefore by Cor. to Th. 2, C must have a unique trigonal linear system.

§4. The case of genus 4 and of characteristic 2

First we prove the existence of curves with non-classical types of gap sequences.

THEOREM 3. *If a curve* C *over an algebraically closed field* k *of characteristic* 2 *is birationally equivalent to a plane curve*

$$f(x, y) = x^3y^3 + x^3 + y^3 + \lambda = 0, \qquad \lambda \neq 0, 1, \tag{42}$$

then C *is a curve of genus* 4 *whose gap sequence at ordinary points is* $\{1, 2, 3, 5\}$.

PROOF. Put $\operatorname{div}(x)=\sum_{i=1}^3\mathrm{P}_i-\sum_{i=1}^3\mathrm{Q}_i$ and $\operatorname{div}(y)=\sum_{i=1}^3\mathrm{P}'_i-\sum_{i=1}^3\mathrm{Q}'_i$. By Prop. 1, C is of genus 4, and it is easily seen that $\{1, x, y, xy\}$ is a base of the space $L(\sum_{i=1}^3\mathrm{Q}_i+\sum_{i=1}^3\mathrm{Q}'_i)$ for the canonical divisor $\sum_{i=1}^3\mathrm{Q}_i+\sum_{i=1}^3\mathrm{Q}'_i$. Since $k(\mathrm{C})$ is separable over $k(x)$, there exists an iterative higher derivation D_x^ν with respect to x. We denote $D_x^\nu y$ by $y^{(\nu)}$. Operating D_x to (42), we have

$$y' = x^2(y^3 + 1)/y^2(x^3 + 1).$$

Since, by (6), $D_x^2y^3=y^{(2)}y^2+y'^2y$ and $D_x^3y^3=y^{(3)}y^2+y'^3$, operating D_x^2 and D_x^3 to (42), we have

$$x(y^3 + 1) + x^2y^2y' + (x^3 + 1)yy'^2 + (x^3 + 1)y^2y^{(2)} = 0 \quad \text{and}$$

$$y^3 + 1 + xy^2y' + x^2yy'^2 + (x^3 + 1)y'^3 + x^2y^2y^{(2)} + (x^3 + 1)y^2y^{(3)} = 0.$$

Hence we have

$$\begin{aligned} y^{(2)} &= x(y^3 + 1)(x^3y^3 + x^3 + y^3)/y^5(x^3 + 1)^2 \\ &= \lambda x(y^3 + 1)/y^5(x^3 + 1)^2 \end{aligned}$$

and so we have

$$\begin{aligned} y^{(3)} &= (y^3+1)\{(x^3+1)y^6 + x^6(y^3+1) + \lambda x^3y^3\}/y^8(x^3+1)^3 \\ &= (y^3+1)(x^3y^3+x^3+y^3)^2/y^8(x^3+1)^3 \\ &= \lambda^2(y^3+1)/y^8(x^3+1)^3. \end{aligned}$$

Therefore we know

$$\Delta_{x;1,2,3}(1, x, y, xy) = \begin{vmatrix} 1 & x & y & xy \\ 0 & 1 & y' & y + xy' \\ 0 & 0 & y^{(2)} & y' + xy^{(2)} \\ 0 & 0 & y^{(3)} & y^{(2)} + xy^{(3)} \end{vmatrix} = (y^{(2)})^2 - y'y^{(3)} = 0.$$

This means C has a non-classical type of gap sequence at ordinary points. By (20), we know that this gap sequence is $\{1, 2, 3, 5\}$. Thus our Theorem 3 is proved.

REMARK. By the same manner as above, we have

$$y^{(4)} = x^2(y^3+1)(x^6y^6 + x^3y^3 + \lambda)/y^8(x^3+1)^4.$$

Hence, we obtain the following Wronskian determinant of C:

$$\begin{aligned} &\Delta_{x;1,2,4}(1, x, y, xy) \\ &\quad = x(y^3+1)^2(x^3y^3+\lambda)\{x^6y^6 + (\lambda+1)x^3y^3 + \lambda^2\}/y^{13}(x^3+1)^5. \end{aligned}$$

We give one more lemma for the proof of non-existence of a curve with non-classical type of gap sequence other than curves which are birationally equivalent to a plane curve (42) in Th. 3.

LEMMA 2. *Let C be a curve of genus 4 over an algebraically closed field k of characteristic 2, which has a unique trigonal linear system g_3^1. Assume the gap sequence at ordinary points on C is of non-classical type. Then there does not exist such a point P that $3P \in g_3^1$, and the number of g_3^1-special points is equal to 2 or 3.*

PROOF. Assume $3P \in g_3^1$. Let D be a divisor in g_3^1 other than 3P. Then by Cor. to Th. 2, $2D$ is canonical. By (3) and (4) there exist functions x and y rational over k such that

$$\operatorname{div}(x) = 3P - D \quad \text{and} \quad \operatorname{div}(y) = P + D' - 2D$$

where D' is a positive divisor whose components are different from P. Then

$\{1, x, x^2, y\}$ is a base of $L(2D)$ over k as easily seen. Since $v_P(y)=1$, $k(C)$ is separable over $k(y)$ and hence there exists an iterative higher derivation D_y^ν with respect to y. Since the gap sequence at ordinary points is of non-classical type, we have

$$
(43) \qquad \Delta_{y;1,2,3}(1, x, x^2, y) = \begin{vmatrix} 1 & x & x^2 & y \\ 0 & x' & 0 & 1 \\ 0 & x^{(2)} & x'^2 & 0 \\ 0 & x^{(3)} & 0 & 0 \end{vmatrix} = x^{(3)}x'^2 = 0.
$$

From $[k(C): k(x)]=3$, it follows that $k(C)$ is separable over $k(x)$, and hence $x' \neq 0$. Therefore we have $x^{(3)}=0$. On the other hand, from $\operatorname{div}(x)=3P-D$, using y as a local parameter at P, we have $x^{(3)}(P) \neq 0$. This is a contradiction, therefore $3P \notin g_3^1$.

As for the second assertion, let x be a non-constant function such that

$$\operatorname{div}(x) = D - (Q_1 + Q_2 + Q_3)$$

where D and $Q_1+Q_2+Q_3$ belong to g_3^1 and $Q_i \neq Q_j$ for $i \neq j$. We can put

$$\operatorname{div}(dx) = \textstyle\sum_{i=1}^t n_i P_i - 2(Q_1 + Q_2 + Q_3)$$

where $n_i \geqq 1$ and $\{P_1, P_2,\ldots, P_t\}$ is the set of all ramification points of the covering π: $C \to \mathbf{P}^1$ defined by $\pi(Q)=(1, x(Q))$. Let P be one of these ramification points. By (3) and (4) there exists a function y rational over k such that

$$\operatorname{div}(y) = P + D' - 2(Q_1 + Q_2 + Q_3)$$

where D' is a positive divisor whose components are different from P. Then putting $\alpha = x(P)$, we can represent x by the power series of the local parameter y at P in the following form;

$$x - \alpha = \alpha_2 y^2 + \alpha_3 y^3 + \cdots$$

where $\alpha_\nu = (D_y^\nu x)(P)$ for $\nu=2, 3,\ldots$. By the same argument as the first assertion, we have $\Delta_{y;1,2,3}(1, x, x^2, y) = x^{(3)}x'^2 = 0$ and hence $x^{(3)}=0$. Since k is of characteristic 2 and $\alpha_3 = x^{(3)}(P)=0$,

$$dx = (\alpha_5 y^4 + \alpha_7 y^6 + \cdots)dy.$$

Therefore we obtain $n_i = \gamma_{P_i}(\operatorname{div}(dx)) \geqq 4$ for $i=1, 2,\ldots, t$. By Riemann-Roch theorem, $\sum_{i=1}^t n_i = 12$ and so $1 \leqq t \leqq 3$. On the other hand, by Cor. 1 to Prop. 2, we know the number of all ramification points of π is greater than 1. Thus Lemma is proved.

REMARK. Since $x^{(7)}=D_y^4 x^{(3)}=0$, $n_i=8$ unless $n_i=4$.
Now, we shall prove the converse of Theorem 3.

THEOREM 4. *If a curve* C *of genus* 4 *over an algebraically closed field* k *of characteristic* 2 *has a non-classical type of gap sequence at ordinary points, then* C *is birationally equivalent to a plane curve*

$$x^3y^3 + x^3 + y^3 + \lambda = 0, \qquad \lambda \neq 0, 1.$$

PROOF. It is sufficient to prove that a curve C of genus 4 over k of characteristic 2 with a unique trigonal linear system g_3^1 has the classical type of gap sequence at ordinary points by Lemma 1 and Prop. 1. If the gap sequence at ordinary points on C is of non-classical type, this sequence must be 1, 2, 3, 5 by (20) and hence the orders of a Wronskian determinant of C are 0, 1, 2, 4.

According to Lemma 2, Prop. 2 and its Cor. 1, our proof is divided in four cases. We use the same notations as in Prop. 2. We note there is no point P such that $3\mathrm{P} \in g_3^1$ by Lemma 2 and hence $\mathrm{P} \neq \mathrm{P}'$.

(i) The case; the number s of finite ramification points of $\pi\colon \mathrm{C} \to \mathbf{P}_1$ defined by $\pi(\mathrm{Q})=(1, x(\mathrm{Q}))$ is equal to 1 and the pole divisor of y is $E=4\mathrm{P}+\mathrm{P}'$. Since, by some translation of x, we may assumes $\alpha_1=0$ in Prop. 2, C is birationally equivalent to a plane curve

$$y^3 + \left(\textstyle\sum_{i=0}^4 b_i x^i\right)y + x^5 = 0. \tag{44}$$

Therefore, by Cor. 1 to Prop. 2, we can put

$$\operatorname{div}(x) = 2\mathrm{P}_1 + \mathrm{P}_1' - (2\mathrm{P} + \mathrm{P}') \quad \text{and} \quad \operatorname{div}(y) = 5\mathrm{P}_1' - (4\mathrm{P} + \mathrm{P}'). \tag{45}$$

Replacing x and y by $b_4^{-3/2}x$ and $b_4^{-5/2}y$ respectively, we may assume $b_4=1$. We note that $b_0 \neq 0$ since $\mathrm{P}_1 \neq \mathrm{P}_1'$. First we show that $b_1 \neq 0$. Assume $b_1=0$. Then

$$f(x, y) = y(y + \sqrt{b_0})^2 + x^2\{(b_2 + b_3x + x^2)y + x^3\} = 0.$$

Hence, since $v_{\mathrm{P}_1}(x)=2$ and $v_{\mathrm{P}_1}(y)=0$ by (45), we have $v_{\mathrm{P}_1}(y+\sqrt{b_0}) \geqq 2$. Therefore we can put $\operatorname{div}(y+\sqrt{b_0})=2\mathrm{P}_1+B'-(4\mathrm{P}+\mathrm{P}')$ where B' is a positive divisor whose components are different from P and P′, and hence $2\mathrm{P}_1+B'+\mathrm{P}' \sim 4\mathrm{P}+2\mathrm{P}' \sim 2\mathrm{P}_1+\mathrm{P}_1'+2\mathrm{P}+\mathrm{P}'$ by (45), and so $B' \sim \mathrm{P}_1'+2\mathrm{P}$. This contradicts uniqueness of g_3^1. Hence $b_1 \neq 0$. Next we shall prove $b_2=0$ and $b_3 \neq 1$. In fact, represent x and y by power series of a local parameter t at P;

$$x = t^{-2} + \alpha_{-1}t^{-1} + \alpha_0 + \cdots \quad \text{and} \quad y = \beta_{-4}t^{-4} + \beta_{-3}t^{-3} + \beta_{-2}t^{-2} + \cdots$$

and put these in (44). From coefficients of t^{-12} and t^{-10}, we obtain $\beta_{-4}=1$ and so $\beta_{-3}^2+b_3+1=0$. If we assume $\beta_{-3}=0$, then

$$\operatorname{div}(y - x^2 - (\beta_{-2} - \alpha_{-1}^2)x) \geqq -(\mathrm{P} + 2\mathrm{P}'),$$

and hence we have $l(\mathrm{P}+2\mathrm{P}')\geqq 2$. By uniqueness of g_3^1, we have a contradiction $\mathrm{P}+2\mathrm{P}'\sim 2\mathrm{P}+\mathrm{P}'$. Therefore $\beta_{-3}\neq 0$ and $b_3\neq 1$. Hence from (45) and $dy=(-3\beta_{-3}t^{-4}-\beta_{-1}t^{-2}+\cdots)dt$, we can put

$$\operatorname{div}(dy) = 4\mathrm{P}_1' + D' - (4\mathrm{P} + 2\mathrm{P}')$$

where D' is a positive divisor of degree 8 whose components are different from P_1, P and P'. If Q is a component of D', then $x-x(\mathrm{Q})$ is a local parameter at Q because the zero divisor of $x-x(\mathrm{Q})$ belongs to g_3^1 and there is no g_3^1-special point other than P and P_1 by the assumption $s=1$. If we represent y by power series of $u=x-x(\mathrm{Q})$ at Q;

$$y - y(\mathrm{Q}) = \beta_2 u^2 + \beta_3 u^3 + \beta_4 u^4 + \cdots,$$

then

$$dy = (\beta_3 u^2 + \beta_5 u^4 + \beta_7 u^6 + \cdots)du.$$

By the assumption on gap sequence, we have $\Delta_{u;1,2,3}(1, x, x^2, y)=y^{(3)}=0$ for the base $\{1, x, x^2, y\}$ of $L(4\mathrm{P}+2\mathrm{P}')$ and hence $\beta_3=y^{(3)}(\mathrm{Q})=0$. Therefore $dy=(\beta_5 u^4+\beta_7 u^6+\cdots)du$. This implies $D'=8\mathrm{Q}$ or $4\mathrm{Q}+4\mathrm{Q}'$. Hence the number of ramification points other than P_1' and P of the covering $\pi'\colon \mathrm{C}\to\mathbf{P}^1$ defined by $\pi'(\mathrm{R})=(1, y(\mathrm{R}))$ is equal to 1 or 2. Since we have

$$f_x(x, y) = (b_1 + b_3 x^2)y + x^4 \quad \text{and}$$

$$f(x, y) - x f_x(x, y) = y(y^2 + b_0 + b_2 x^2 + x^4),$$

x-coordinates of these ramification points other than P_1' and P must consist of all roots of the equation

$$\begin{aligned}(46)\quad g(x) &= (b_1 + b_3 x^2)(\sqrt{b_0} + \sqrt{b_2}x + x^2) + x^4\\ &= (b_3 + 1)x^4 + b_3\sqrt{b_2}x^3 + (b_1 + b_3\sqrt{b_0})x^2 + b_1\sqrt{b_2}x + b_1\sqrt{b_0}\\ &= 0\end{aligned}$$

where $b_3+1\neq 0$. Hence $g(x)=0$ must have at most two roots. If $g(x)=(b_3+1)(x-\alpha)^4$ or $(b_3+1)(x-\alpha)^2(x-\beta)^2$, then we have obviously $b_2=0$. If $g(x)=(b_3+1)(x-\alpha)^3(x-\beta)$, then α must be a common root of (46),

$$(47)\qquad g_x(x) = b_3\sqrt{b_2}x^2 + b_1\sqrt{b_2} = 0 \quad \text{and}$$

$$\partial_x^2 g(x) = b_3\sqrt{b_2}x + (b_1 + b_3\sqrt{b_0}) = 0.$$

Assume $b_2\neq 0$. If $b_3\neq 0$, then $x^2=b_1 b_3^{-1}$. Putting this in $g(x)=0$, we have a contradiction $b_1=0$. If $b_3=0$, then by (47), we also have $b_1=0$. Thus we know $b_2=0$.

Now, represent x and y by power series of a local parameter u at P′

$$x = u^{-1} + \alpha_0 + \alpha_1 u + \cdots \quad \text{and}$$

$$y = \beta_{-1}u^{-1} + \beta_0 + \beta_1 u + \cdots, \beta_{-1} \neq 0,$$

and put these in $f(x, y)=0$. Then from coefficients of u^{-5}, u^{-4} and u^{-3}, we know $\beta_{-1}=1$, $b_3+\beta_0+\alpha_0=0$ and $1+b_3(\beta_0+\alpha_0)+\beta_1+\alpha_1=0$. Hence

$$\alpha_1 + \beta_1 = 1 + b_3^2 = (1 + b_3)^2.$$

On the other hand, $\{u^2, u^2x, u^2x^2, u^2y\}$ is clearly a base of $L(4\mathrm{P}+2\mathrm{P}'-\operatorname{div}(u^2))$. Hence from the assumption on gap sequence and (13), it follows that

$$\Delta_{x;1,2,3}(u^2, u^2x, u^2x^2, u^2y)(\mathrm{P}') = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & \alpha_0 & \alpha_0^2 & \beta_0 \\ 0 & \alpha_1 & 0 & \beta_1 \end{vmatrix} = \alpha_1 + \beta_1 = 0.$$

Therefore we have a contradiction $b_3=1$. Thus this case does not occur.

(ii) The case; $s=2$ and $E=4\mathrm{P}+\mathrm{P}'$. Since, by a suitable linear transformation of x and y, we may assume $\alpha_1=0$ and $\alpha_2=1$ in Prop. 2, we can see that, by Prop. 2 and its Cor. 1, C is birationally equivalent to a plane curve

$$f(x, y) = y^3 + (\textstyle\sum_{i=0}^4 b_i x^i)y + x^{m_1}(x-1)^{m_2},$$

so that we can put

$$\operatorname{div}(x) = 2\mathrm{P}_1 + \mathrm{P}_1' - (2\mathrm{P} + \mathrm{P}') \text{ and } \operatorname{div}(y) = m_1\mathrm{P}_1' + m_2\mathrm{P}_2' - (4\mathrm{P} + \mathrm{P}'),$$

whence $m_1+m_2=5$. Here we may put $m_1\mathrm{P}_1'+m_2\mathrm{P}_2'=4\mathrm{P}_1'+\mathrm{P}_2'$. To see this, assume $\operatorname{div}(y)=3\mathrm{P}_1'+2\mathrm{P}_2'-(4\mathrm{P}+\mathrm{P}')$. Then

$$4\mathrm{P} + 2\mathrm{P}', \ \mathrm{P}_1' + 2\mathrm{P}_1 + 2\mathrm{P} + \mathrm{P}', \ 2\mathrm{P}_1' + 4\mathrm{P}_1 \text{ and } 3\mathrm{P}_1' + 2\mathrm{P}_2' + \mathrm{P}'$$

are all canonical divisors by Cor. to Th. 2 and so Hermitian P_1'-invariants are 0, 1, 2, 3 by (4). This contradicts the assumption on the gap sequence. Hence we may assume

$$f(x, y) = y^3 + (\textstyle\sum_{i=0}^4 b_i x^i)y + x^5 + x = 0. \tag{48}$$

Since $v_{\mathrm{P}_1}(y+\sqrt{b_0})=1$ by the same argument as $b_1\neq 0$ in (i), $y+\sqrt{b_0}$ is a local parameter at P_1. Expand x by a power series of $t=y+\sqrt{b_0}$ at P_1:

$$x = c_2t^2 + c_3t^3 + \cdots,\ c_2 \neq 0.$$

Putting these in (48), from coefficients of t^2 and t^3 we know $\sqrt{b_0}+b_1\sqrt{b_0}c_2+c_2=0$ and $1+b_1c_2+b_1\sqrt{b_0}c_3+c_3=0$. On the other hand, by the assumption on the gap sequence, we have

$$\Delta_{t;1,2,3}(1, x, x^2, y) = \begin{vmatrix} 1 & x & x^2 & y \\ 0 & x' & 0 & 1 \\ 0 & x^{(2)} & x'^2 & 0 \\ 0 & x^{(3)} & 0 & 0 \end{vmatrix} = x'^2x^{(3)} = 0.$$

Since $[k(C): k(x)]=3$, $k(C)$ is separable over $k(x)$, and hence $x'\neq 0$. Hence we have $x^{(3)}=0$, and so $c_3=x^{(3)}(P_1)=0$, whence $1+b_1c_2=0$. Therefore we have a contradiction $c_2=0$.

(iii) The case; $s=1$ and $E=4P$. Since we may assume $\alpha_1=0$ in Prop. 2, we may put by Prop. 2 and its Cor. 1,

$$\mathrm{div}(x) = 2P_1 + P_1' - (2P + P') \text{ and } \mathrm{div}(y) = mP' + m_1P_1' - 4P$$

where $0\leqq m\leqq 3$.

(a) $m=0$, hence $m_1=4$. C is birationally equivalent to a plane curve

$$f(x, y) = y^3 + (\textstyle\sum_{i=0}^4 b_ix^i)y + x^4 = 0 \tag{49}$$

where $b_4\neq 0$ by Cor. 1 to Prop. 2. Represent x and y by power series of a local parameter t at P':

$$x = t^{-1} + \alpha_0 + \alpha_1t + \cdots \quad \text{and}$$

$$y = \beta_0 + \beta_1t + \beta_2t^2 + \cdots,\ \beta_0 \neq 0.$$

By the assumption on the gap sequence, we have

$$\Delta_{t;1,2,3}(t^2, t^2x, t^2x^2, t^2y)(P') = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & \alpha_0 & \alpha_0^2 & \beta_0 \\ 0 & \alpha_1 & 0 & \beta_1 \end{vmatrix} = \beta_1 = 0.$$

Putting the above series in (49), from the coefficient of t^{-3}, we know $b_3\beta_0+b_4\beta_1=b_3\beta_0=0$, and hence $b_3=0$. This implies $v_P(y^3+b_4x^4y)=v_P(y(y+\sqrt{b_4}x^2)^2)\geqq -8$, and hence $v_P(y+\sqrt{b_4}x^2)\geqq -2$, and so $\mathrm{div}(y+\sqrt{b_4}x^2)\geqq -2(P+P')$.

Since $l(2P+2P')=2=l(2P+P')$ by Riemann-Roch theorem, P′ is a fixed point of $|2P+2P'|$, and hence $\operatorname{div}(y+\sqrt{b_4}x^2)\geqq -(2P+P')$. Therefore $y+\sqrt{b_4}x^2 \in L(2P+P')$. This contradicts linear independence of 1, x, x^2, y.

(b) $m=1$. We have

$$\operatorname{div}(x) = 2P_1 + P_1' - (2P + P') \text{ and } \operatorname{div}(y) = P' + 3P_1' - 4P.$$

Hence the following divisors

$$4P_1 + 2P_1',\ P' + 2P + 2P_1 + P_1',\ 2P' + 4P \text{ and } 3P' + 3P_1'$$

are all canonical by Cor. to Th. 2. Therefore Hermitian P′-invariants are 0, 1, 2, 3. This is a contradiction.

(c) $m=2$. In this case, we have $\operatorname{div}(y)=2P'+2P_1'-4P$, and hence by Prop. 2, C is birationally equivalent to a plane curve

$$f(x, y) = y^3 + (\textstyle\sum_{i=0}^4 b_i x^i)y + x^2 = 0.$$

If we put $u=x^{-1}$ and $w=yx^{-2}$, then we have

$$\operatorname{div}(u) = 2P + P' - (2P_1 + P_1'),\quad \operatorname{div}(w) = 4P' - 4P_1,$$

and

$$w^3 + (\textstyle\sum_{i=0}^4 b_i u^{4-i})w + u^4 = 0.$$

Therefore this case is reduced to the case (iii)–(a).

(d) $m=3$. By the same manner as (c), this case is reduced to the case (i).

(iv) The case; $s=2$ and $E=4P$. In this case, putting $\alpha_1=0$ and $\alpha_2=1$ in Prop. 2, by Prop. 2 and its Cor. 1 we may assume that C is birationally equivalent to a plane curve

$$f(x, y) = y^3 + (\textstyle\sum_{i=0}^4 b_i x^i)y + x^{m_1}(x-1)^{m_2} = 0,$$

and that

$$\operatorname{div}(x) = 2P_1 + P_1' - (2P + P') \text{ and } \operatorname{div}(y) = mP' + m_1P_1' + m_2P_2' - 4P$$

where $0\leqq m\leqq 2$.

(e) $m=0$. If we assume $\operatorname{div}(y)=3P_1'+P_2'-4P$, then $4P+2P'$, $P_1'+2P_1+2P+P'$, $2P_1'+4P_1$ and $3P_1'+P_2'+2P'$ are canonical divisors, and hence Hermitian P_1'-invariants are 0, 1, 2, 3. This is a contradiction. Therefore we may have $\operatorname{div}(y)=2P_1'+2P_2'-4P$ and C is birationally equivalent to

$$f(x, y) = y^3 + (\textstyle\sum_{i=0}^4 b_i x^i)y + x^4 + x^2 = 0.$$

Exactly by the same way as (iii)–(a), we know this case is impossible.

(f) $m=1$. We may have $\operatorname{div}(y)=P'+2P'_1+P'_2-4P$. Hence, putting $u=x^{-1}$ and $w=yx^{-2}$, this case is reduced to the case (e).

(g) $m=2$. By the same transformation of x and y as (f), this case is reduced to the case (ii). Thus our Theorem is proved.

§ 5. The case of genus 4 and of characteristic 5

By Th. C and Th. D in § 1, a curve of genus 4 whose gap sequence at ordinary points is of non-classical type must be defined over a field of characteristic 2 or 5. In this final section, we shall prove the uniqueness of such a curve over a field of characteristic 5.

LEMMA 3. *Let* C *be a curve of genus* 4 *over an algebraically closed field* k *of characteristic* 5. *If the gap sequence at ordinary points on* C *is of non-classical type, then* C *has a unique trigonal linear system* g_3^1, *and* $3P \in g_3^1$ *for every* g_3^1*-special point* P.

PROOF. By Lemma 1, we know C has a unique g_3^1. If we assume $2P+P' \in g_3^1$ for a g_3^1-special point P, and $P \neq P'$, then by Cor. to Th. 2, $2P+P'+D$ and $4P+2P'$ are canonical divisors where D is a divisor in g_3^1 without P as a component. Since C is non-hyperelliptic, this implies the Hermitian P-invariants are 0, 1, 2, 4 by (3) and (4). On the other hand by (22), the orders of a Wronskian determinant of C are 0, 1, 2, 5. This contradicts the formula (13). Therefore $3P \in g_3^1$.

THEOREM 5. *If a curve* C *of genus* 4 *over an algebraically closed field* k *of characteristic* 5 *has a non-classical type of gap sequence at ordinary points, then* C *is birationally equivalent to the plane curve*

$$y^5 + y - x^3 = 0.$$

The gap sequence at ordinary points on the curve C *is* $\{1, 2, 3, 6\}$.

PROOF. By Lemma 3, C has a unique trigonal linear system g_3^1. Lemma 3, Prop. 2, and its Cor. 2 mean that C may be birationally equivalent to a plane curve

$$f(x, y) = y^3 + x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x = 0 \tag{50}$$

where $\delta(x)=x^5+a_1x^4+a_2x^3+a_3x^2+a_4x$ is squarefree, and that we have

$$\operatorname{div}(x-\alpha_i) = 3P_i - 3P \quad \text{for} \quad i = 1, 2, \dots, 5 \quad \text{and}$$

$$\operatorname{div}(y) = (P_1 + \cdots + P_5) - 5P$$

where $\alpha_1, \dots, \alpha_5$ are the roots of $\delta(x)=0$. We shall show $a_1=a_2=a_3=0$. Since $\{1, x, x^2, y\}$ is a base of $L(6P)$ for the canonical divisor 6P and $k(C)$ is separable over $k(x)$ as easily seen, and since the orders of a Wronskian determinant of C are 0, 1, 2, 5 by (22), we have

$$\Delta_{x;1,2,3}(1, x, x^2, y) = y^{(3)} = 0 \quad \text{and}$$

$$\Delta_{x;1,2,4}(1, x, x^2, y) = y^{(4)} = 0.$$

Operating D_x^ν for $\nu=1, 2, 3, 4$ on $f(x, y)=0$, we have

$$3y^2y' + 4a_1x^3 + 3a_2x^2 + 2a_3x + a_4 = 0, \tag{51}$$

$$3(y^2y^{(2)} + yy'^2) + a_1x^2 + 3a_2x + a_3 = 0, \tag{52}$$

$$yy'y^{(2)} + y'^3 + 4a_1x + a_2 = 0, \tag{53}$$

$$3y(y^{(2)})^2 + 3y'^2y^{(2)} + a_1 = 0. \tag{54}$$

Assume $a_1 \neq 0$. Let β_1, β_2, β_3 be the roots of

$$\delta'(x) = 4a_1x^3 + 3a_2x^2 + 2a_3x + a_4 = 0,$$

and let Q_1, Q_2, Q_3 be the points of C such that $x(Q_i)=\beta_i$. Since $\delta(x)$ is square-free, $\delta(\beta_i)\neq 0$ for $i=1, 2, 3$, and hence by (50), $y(Q_i)\neq 0$, and so by (51), $y'(Q_i)=0$. Hence by (53), $4a_1\beta_i+a_2=0$ for $i=1, 2, 3$. Therefore $\beta_1=\beta_2=\beta_3$, say, $=\beta$. This implies $\delta'(x)=4a_1(x-\beta)^3$, and hence β is also a root of $\partial_x^2\delta(x)=a_1x^2+3a_2x+a_3=0$. Hence by (52), $y^{(2)}(Q_1)=0$. Therefore by (54), we get a contradiction. Hence we have $a_1=0$. Now, by (54), we have $y^{(2)}(yy^{(2)}+y'^2)=0$. If $y^{(2)}=0$, then by (53), we know y' is a constant. This is impossible by (51). If $yy^{(2)}+y'^2=0$, then by (52) we have $a_2=a_3=0$. Therefore we have

$$y^3 + x^5 + a_4x = 0.$$

Replacing x and y by $a_4^{1/4}y$ and $-a_4^{5/12}x$ respectively, we have

$$y^5 + y - x^3 = 0.$$

The last assertion follows from (22). Thus our proof is completed.

References

[1] J. L. Coolidge, A treatise on algebraic plane curves, Dover, 1931.

[2] H. Hasse and F. K. Schmidt, Noch eine Begründung der Theorie der höheren Differentialquotienten, J. Reine Angew. Math. **177** (1937), 215–237.

[3] K. Hensel and G. Landsberg, Theorie der algebraischen Funktionen einer Variablen, Leipzig, 1902 (Repr. New York, 1965).

[4] A. Kuribayashi and K. Komiya, On Weierstrass points of non-hyperelliptic compact Riemann surfaces of genus three, Hiroshima Math. J. **7** (1977) 743–768.

[5] S. Lang, Introduction to algebraic geometry, Interscience Tracts n° 5, New York, 1958.

[6] B. A. Olson, On higher order Weierstrass points, Ann. of Math. **95** (1972), 357–364.

[7] F. K. Schmidt, Die Wronskische Determinante in beliebigen differenzierbaren Funktionenkörpern, Math. Z. **45** (1939), 62–74.

[8] ———, Zur arithmetishen Theorie der algebraischen Funktionen II. Allgemeinen Theorie der Weierstrasspunkte, Math. Z. **45** (1939), 75–96.

[9] R. J. Walker, Algebraic curves, Princeton Univ. Press, 1950.

[10] A. Weil, Foundations of algebraic geometry, A.M.S.Coll.Publ., New York, 1946 (Revised ed. 1962).

[11] ———, Sur les courbes algébriques et les variétés qui s'en déduisent, Hermann, Paris, 1948.

Department of Mathematics,
Faculty of Liberal Arts & Education,
Yamanashi University