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Finiteness Conditions for Abelian Ideals and Nilpotent Ideals in Lie Algebras

Fujio KUBO

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1. Introduction

Let us first recall the definitions of several classes of Lie algebras over a field.

$L \in \lhd \mathfrak{A}$-Fin; every abelian ideal of L is finite-dimensional.

$L \in \lhd \mathfrak{N}$-Fin; every nilpotent ideal of L is finite-dimensional.

$L \in$ Max-$\lhd \mathfrak{A}$, Max-$\lhd \mathfrak{N}$, Max-$\lhd$ E$\mathfrak{A}$; L satisfies the maximal condition for abelian, nilpotent and soluble ideals respectively.

$L \in$ Min-$\lhd \mathfrak{A}$, Min-$\lhd \mathfrak{N}$, Min-$\lhd$ E$\mathfrak{A}$; L satisfies the minimal condition for abelian, nilpotent and soluble ideals respectively.

For the above classes, R. K. Amayo and I. Stewart have asked the following among "Some open questions" at the end of their book [1]:

Question 23. *Is* $\lhd \mathfrak{N}$-Fin *equal to* $\lhd \mathfrak{A}$-Fin?

Question 24. *Are there any inclusions between* Max-$\lhd \mathfrak{A}$, Max-$\lhd \mathfrak{N}$, Max-$\lhd$ E$\mathfrak{A}$; Min-$\lhd \mathfrak{A}$, Min-$\lhd \mathfrak{N}$, Min-$\lhd$ E$\mathfrak{A}$?

The purpose of this paper is to give the negative answer to Question 23 and a partial answer to Question 24.

2. Tensorial extensions

Let $\mathfrak{k}$ be an arbitrary field. If A is a commutative associative algebra over $\mathfrak{k}$ and S is a Lie algebra over $\mathfrak{k}$, we can as in [2] define the Lie algebra $A \otimes_{\mathfrak{k}} S$ over $\mathfrak{k}$ with multiplication

$$[a \otimes s, a' \otimes s'] = (aa') \otimes [s, s'] \qquad (a, a' \in A;\ s, s' \in S).$$

If S is a Lie algebra over $\mathfrak{k}$, for any $s \in S$ we define the adjoint transformation s^* of S by $xs^* = [x, s]$ $(x \in S)$. As s varies throughout S, these transformations generate a subalgebra $\mathfrak{M}$ of the associative algebra $End_{\mathfrak{k}}(S)$ of all linear transformations of S. Now S is a right $\mathfrak{M}$-module with the above action. The centroid $\mathfrak{C}$ of $\mathfrak{M}$ is the algebra of all $\mathfrak{M}$-endomorphisms of S, these being regarded as left operaters on S. The Lie algebra S is central simple if it is simple and the centroid $\mathfrak{C}$ is the ground field $\mathfrak{k}$ with its standard action on S. I. Stewart has shown in [2] that if S is central simple and A has an identity then every ideal of $A \otimes_{\mathfrak{k}} S$ is of the form $U \otimes S$ where U is an ideal of A. From this result, we have immediately the following

LEMMA 1. *If S is a finite-dimensional central simple Lie algebra over $\mathfrak{k}$ and A is a commutative associative algebra over $\mathfrak{k}$ with an identity, then the correspondence $U \mapsto U \otimes S$ between the ideals of A and the ideals of $A \otimes_{\mathfrak{k}} S$ is one-to-one, and U is abelian (resp. nilpotent, finite-dimensional) if and only if so is $U \otimes S$.*

3. Construction of a commutative associative algebra

Let A be a commutative associative algebra over $\boldsymbol{Q}$ with basis $\{1, a, e_1, e_2, \ldots\}$ and multiplication

$$aa = ae_i = e_i a = 0, \quad e_i e_j = e_j e_i = \delta_{ij} a \qquad (i, j = 1, 2, \ldots),$$

$$x1 = 1x = x \qquad \text{for any } x \in A.$$

Let B be the ideal of A generated by $\{a, e_1, e_2, \ldots\}$. Then we have the following

LEMMA 2. *The ideals of A are precisely*
(1) 0, $<a>$, A, *and*
(2) *all subspaces U of A such that $<a> \subsetneqq U \subseteq B$.*

PROOF. It is easy to check that the above subspaces are ideals of A. Let U be any ideal of A which is not of type (1). Then we take an element x in $U \backslash <a>$ such that

$$x = \alpha a + \beta 1 + \sum_{i=1}^{n} \alpha_i e_i \qquad (\alpha, \alpha_i, \beta \in \boldsymbol{Q}).$$

If $\beta = 0$, then there exists a non-zero coefficient α_m and hence $xe_m = \alpha_m a \in U$. It follows that $<a> \subseteq U$. If $\beta \neq 0$, then $xa = \beta a \in U$, which implies $<a> \subseteq U$. Now suppose that $U \nsubseteq B$. Then we can take an element

$$x = \alpha a + 1 + \sum_{i=1}^{n} \alpha_i e_i$$

in U. For any positive integer j,

$$xe_j = e_j + \left(\sum_{i=1}^{n} \alpha_i \delta_{ij}\right) a.$$

Since $<a> \subseteq U$, $e_j \in U$ and therefore $B \subseteq U$. Hence $U = A$.

Now we can show the following

LEMMA 3. (1) *Every abelian ideal of A is 0 or $<a>$.*
(2) *Every proper ideal of A is nilpotent.*

PROOF. Since B is nilpotent, the assertion (2) is immediate by Lemma 2. To show the assertion (1), it is enough to show that every ideal U of type (2) in Lemma 2 is not abelian. We can take an element x in U such that $x=\alpha a+\sum_{i=1}^{n}\alpha_i e_i$ where $\alpha_n \neq 0$. Since

$$x^2 = (\sum_{i=1}^{n} \alpha_i^2)a \neq 0,$$

U is not abelian.

4. Answer to Questions 23 and 24

Let A be a commutative associative algebra over $\boldsymbol{Q}$ introduced in Section 3 and S be the split simple Lie algebra $<x, y, z>$ over $\boldsymbol{Q}$:

$$[x, y] = z,\ [x, z] = 2x,\ [y, z] = -2y.$$

Put $L=A\otimes_Q S$. Then we show that L has the following properties, which answer to Question 23 and answer partially to Question 24.

THEOREM. *$L \in \lhd\mathfrak{A}$-Fin $\cap$ Max-$\lhd\mathfrak{A}$ $\cap$ Min-$\lhd\mathfrak{A}$, $L \notin \lhd\mathfrak{N}$-Fin, $L \notin$ Max-$\lhd\mathfrak{N}$ and $L \notin$ Min-$\lhd\mathfrak{N}$.*

PROOF. Since S is finite-dimensional and central simple, we have immediately the first assertion by Lemmas 1 and 3. Since B is an infinite-dimensional nilpotent ideal of A, so is $B\otimes S$ in L by Lemma 1. Hence $L \notin \lhd\mathfrak{N}$-Fin. Put

$$H_n = <a, e_1, e_2,\dots, e_n> \otimes S.$$

Then $\{H_n\}$ is a strictly ascending chain of nilpotent ideals of L. Hence $L \notin$ Max-$\lhd\mathfrak{N}$. Next put

$$K_n = <a, e_n, e_{n+1},\dots> \otimes S.$$

Then $\{K_n\}$ is a strictly descending chain of nilpotent ideals of L. Hence $L \notin$ Min-$\lhd\mathfrak{N}$.

References

[1] R. K. Amayo and I. Stewart, Infinite-dimensional Lie algebras, Noordhoff, Leyden, 1974.

[2] I. Stewart, Tensorial extensions of central simple algebras, J. Algebra, **25** (1973), 1–14.

Department of Mathematics,
Faculty of Science
Hiroshima University