HIROSHIMA MATH. J.
8 (1978), 235–253

The Enumeration of Embeddings of Lens Spaces and Projective Spaces

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(Received November 1, 1977)

Introduction

The purpose of this article is to study the enumeration problem of embeddings of the lens space $L^n(p)$ mod p (odd prime), the real projective space RP^n and the complex projective space CP^n in Euclidean spaces.

Let M be an m-dimensional closed differentiable manifold, and let $g: M^* \to RP^\infty$ (the infinite dimensional real projective space) denote the classifying map of the double covering

$$\pi: M \times M - \Delta \longrightarrow M^* = (M \times M - \Delta)/Z_2$$

over the reduced symmetric product M^* of M, where Δ is the diagonal and Z_2 acts on $M \times M - \Delta$ via $t(x, y) = (y, x)$. Also Z_2 acts on the n-dimensional sphere S^n via the antipodal map and we obtain the fiber bundle

$$p: (S^\infty \times S^n)/Z_2 \ (\simeq RP^n) \longrightarrow RP^\infty$$

which is homotopically equivalent to the natural inclusion $RP^n \subset RP^\infty$. Then the following theorem is due to A. Haefliger [7].

THEOREM. *Let $2(n+1) > 3(m+1)$. If there exists an embedding of M in R^{n+1}, then there exists a bijection between the set $[M \subset R^{n+1}]$ of isotopy classes of embeddings of M in R^{n+1} and the set $[M^*, RP^n; g]$ of (vertical) homotopy classes of liftings of $g: M^* \to RP^\infty$ to RP^n.*

The set $[M^*, RP^n; g]$ has the structure of an abelian group by J. C. Becker [2]. Thus, the set $[M \subset R^{n+1}]$ is an abelian group via the bijection of this theorem. We study the groups $[L^n(p) \subset R^{4n+2-i}]$, $[RP^n \subset R^{2n-i}]$ and $[CP^n \subset R^{4n-i}]$ for $i < 6$ and prove the theorems below.

THEOREM A. *The following statements hold for odd prime p:*

(1) $[L^n(p) \subset R^{4n+1}] = 0,$ $\quad n > 2.$

(2) $[L^n(p) \subset R^{4n}] = Z_p,$ $\quad n > 3.$

(3) $[L^n(p) \subset R^{4n-1}] = Z_p,$ $\quad n > 4.$

$$(4)\quad [L^n(p) \subset R^{4n-2}] = \begin{cases} Z_p + Z_p, & p \neq 3,\ n > 5, \\ Z_3 + Z_3 + Z_9, & p = 3,\ n \equiv 2(3),\ n > 5, \\ Z_9, & p = 3,\ n \not\equiv 2(3),\ n > 5. \end{cases}$$

$$(5)\quad [L^n(p) \subset R^{4n-3}] = Z_p, \qquad n > 6.$$

THEOREM B. *The following statements hold for even* n:

(1) *Let* $n \geq 10$. *If there is an embedding of* RP^n *in* R^{2n-3}, *then*

$$[RP^n \subset R^{2n-3}] = \begin{cases} Z_2, & n \not\equiv 6(8), \\ Z_2 + Z_2, & n \equiv 6(8). \end{cases}$$

(2) *Let* $n \geq 12$. *If there is an embedding of* RP^n *in* R^{2n-4}, *then*

$$[RP^n \subset R^{2n-4}] = \begin{cases} 0, & n \equiv 0(4), \\ Z_2, & n \equiv 2(8), \\ Z_2 + Z_2 + Z_2, & n \equiv 6(8). \end{cases}$$

(3) *Let* $n \geq 12$. *If there is an embedding of* RP^n *in* R^{2n-5}, *then*

$$[RP^n \subset R^{2n-5}] = Z_2, \qquad n \equiv 0(4),$$

$$\sharp[RP^n \subset R^{2n-5}] = \begin{cases} 4, & n \equiv 2(8), \\ 8 \text{ or } 16, & n \equiv 6(8), \end{cases}$$

where $\sharp S$ *denotes the cardinality of the set* S.

THEOREM C. *The following statements hold*:

(1) *Let* $n > 5$, $n \neq 2^r + 2^s$ $(r \geq s > 0)$. *Then*

$$[CP^n \subset R^{4n-3}] = \begin{cases} Z, & n \equiv 0(2), \\ Z + Z_2, & n \equiv 1(2). \end{cases}$$

(2) *Let* $n > 6$. *If there is an embedding of* CP^n *in* R^{4n-4}, *then*

$$[CP^n \subset R^{4n-4}] = 0, \qquad n \equiv 0(2).$$

(3) *Let* $n > 7$. *If there is an embedding of* CP^n *in* R^{4n-5}, *then*

$$[CP^n \subset R^{4n-5}] = Z + Z, \qquad n \equiv 0(2).$$

For the assumptions of the existence of an embedding in Theorems B and C, there are several known results, cf. e.g., [14] and [16]. By this time, D. R.

Bausum, L. L. Larmore, R. D. Rigdon and the author have studied $[RP^n \subset R^{2n-i}]$ for $i<3$ and $[CP^n \subset R^{4n-i}]$ for $i<3$ in [1], [9], [19], [20] and [18].

We devote § 1 to the construction of a finite decreasing filtration of the group $[X, RP^n; f]$ of homotopy classes of liftings of $f: X \to RP^\infty$ to RP^n. Next, we calculate the cohomology of $L^n(p)^*$ in §2 and prove Theorem A in § 3. In § 4, we calculate the cohomology of $(RP^n)^*$ and $(CP^n)^*$ and in § 5, we prove Theorems B and C.

§ 1. Enumeration of liftings in the fibration $RP^n \to RP^\infty$

D. R. Bausum constructed in [1, §§ 1–3] the fifth stage Postnikov factorization of the fibration $p: RP^n \to RP^\infty$ with fiber S^n and converted it into the factorization of the fibration $(RP^n)^2 \to RP^n$ which is the pullback of p by p. However, we use a somewhat modified factorization given as follows ($n \geq 8$):

$$\begin{array}{ccccccccccc} & & & & C_3 & & C_2 & & C_1 & & \\ & & & & \uparrow & & \uparrow & & \uparrow & & \\ (RP^n)^2 & \xrightarrow{q} & E_4 & \xrightarrow{p_4} & E_3 & \xrightarrow{p_3} & E_2 & \xrightarrow{p_2} & E_1 & \longrightarrow & RP^n, \end{array}$$

$$E_1 = \begin{cases} K(Z, n) \times RP^n, & n \equiv 1(2), \\ L_\phi(Z, n) \times_{RP^\infty} RP^n, & n \equiv 0(2), \end{cases}$$

$$C_1 = \begin{cases} K(Z_2, n+2) \times K(Z_2, n+4) \times K(Z_3, n+4) \times RP^n, & n \equiv 1(2), \\ K(Z_2, n+2) \times K(Z_2, n+4) \times L_{\phi'}(Z_3, n+4) \times_{RP^\infty} RP^n, & n \equiv 0(2), \end{cases}$$

$$C_2 = K(Z_2, n+3) \times K(Z_2, n+4) \times RP^n,$$

$$C_3 = K(Z_2, n+4) \times RP^n,$$

and the map q is an $(n+6)$-equivalence. Here $L_\phi(Z, n) \times_{RP^\infty} RP^n$ is the pullback of $L_\phi(Z, n) = S^\infty \times_{Z_2} K(Z, n)^{*)} \to S^\infty / Z_2 = RP^\infty$ by $p: RP^n \to RP^\infty$, where the action of Z_2 on $K(Z, n)$ is induced from the non-trivial homomorphism $\phi: Z_2 \to \mathrm{Aut}(Z)$. Also $L_{\phi'}(Z_3, n+4) \times_{RP^\infty} RP^{n *)}$ is defined in the same way by using the non-trivial homomorphism $\phi': Z_2 \to \mathrm{Aut}(Z_3)$.

Let X be a CW-complex of dimension less than $n+6$ and let $n>7$. If $g: X \to RP^\infty$ has a lifting f to RP^n, then $[X, RP^n; g] \approx [X, (RP^n)^2; f]$. By the standard exact couple argument, we can construct a spectral sequence. In this spectral

*) $L_\phi(Z, n) = K(Z, n; \phi)$ and $L_{\phi'}(Z_3, n+4) = K(Z_3, n+4; \phi')$ by Bausum's notation.

sequence, the differentials d_1 are given by the following primary operations:

Case I. $n \equiv 1(2)$.

$$\Theta^i: H^{i-1}(X; Z) \longrightarrow H^{i+1}(X; Z_2) \times H^{i+3}(X; Z_2) \times H^{i+3}(X; Z_3),$$

$$\Theta^i(a) = (Sq^2\rho_2 a + \varepsilon_1 v^2\rho_2 a,\ Sq^4\rho_2 a + \varepsilon_2 v^4\rho_2 a,\ \mathscr{P}_3^1\rho_3 a);$$

$$\Gamma^i: H^i(X; Z_2) \times H^{i+2}(X; Z_2) \times H^{i+2}(X; Z_3) \longrightarrow H^{i+2}(X; Z_2) \times H^{i+3}(X; Z_2),$$

$$\Gamma^i(a, b, c) = (Sq^2 a + \varepsilon_1 v^2 a,\ Sq^2Sq^1 a + Sq^1 b);$$

$$\Delta^i: H^{i+1}(X; Z_2) \times H^{i+2}(X; Z_2) \longrightarrow H^{i+3}(X; Z_2),$$

$$\Delta^i(a, b) = Sq^2 a + \varepsilon_1 v^2 a + Sq^1 b;$$

where

$$\varepsilon_1 = \begin{cases} 1, & n \equiv 1(4), \\ 0, & n \equiv 3(4), \end{cases} \qquad \varepsilon_2 = \begin{cases} 1, & n \equiv 3, 5(8), \\ 0, & n \equiv 1, 7(8). \end{cases}$$

Case II. $n \equiv 0(2)$.

$$\Theta^i: H^{i-1}(X; \underline{Z}) \longrightarrow H^{i+1}(X; Z_2) \times H^{i+3}(X; Z_2) \times H^{i+3}(X; \underline{Z}_3),$$

$$\Theta^i(a) = (Sq^2\rho_2 a + \varepsilon_3 v^2\rho_2 a,\ Sq^4\rho_2 a + \varepsilon_4 v^4\rho_2 a,\ \mathscr{P}_3^1\rho_3 a),$$

($\mathscr{P}_3^1$ is the reduced power operation mod 3 in local coefficients [6]);

$$\Gamma^i: H^i(X; Z_2) \times H^{i+2}(X; Z_2) \times H^{i+2}(X; \underline{Z}_3) \longrightarrow H^{i+2}(X; Z_2) \times H^{i+3}(X; Z_2),$$

$$\Gamma^i(a, b, c) = ((Sq^2 + vSq^1 + (1-\varepsilon_3)v^2)a,\ (Sq^2Sq^1 + v^2Sq^1 + \varepsilon_3 v^3)a + (Sq^1 + v)b);$$

$$\Delta^i: H^{i+1}(X; Z_2) \times H^{i+2}(X; Z_2) \longrightarrow H^{i+3}(X; Z_2),$$

$$\Delta^i(a, b) = Sq^2 a + (1-\varepsilon_3)v^2 a + Sq^1 b + vb;$$

$$\varepsilon_3 = \begin{cases} 1, & n \equiv 2(4), \\ 0, & n \equiv 0(4), \end{cases} \qquad \varepsilon_4 = \begin{cases} 1, & n \equiv 4, 6(8), \\ 0, & n \equiv 0, 2(8). \end{cases}$$

In Cases I and II, ρ_p is the mod p reduction, $v = g^*z$, where z is the generator of $H^1(RP^\infty; Z_2) = Z_2$, and $\underline{Z}$ and $\underline{Z}_3$ are the local systems on X induced by $\pi_1(X)$

$\xrightarrow{g_*} \pi_1(RP^\infty)=Z_2 \xrightarrow{\phi} \mathrm{Aut}(Z)$ and $\pi_1(X) \xrightarrow{g_*} Z_2 \xrightarrow{\phi'} \mathrm{Aut}(Z_3)$, respectively.
Further, the differentials d_2 are given by the secondary operations

$$\Phi^i\colon \mathrm{Ker}\,\Theta^i \longrightarrow \mathrm{Ker}\,\Delta^{i+1}/\mathrm{Im}\,\Gamma^i,$$

$$\Psi^i\colon \mathrm{Ker}\,\Gamma^i/\mathrm{Im}\,\Theta^{i-1} \longrightarrow \mathrm{Coker}\,\Delta^i,$$

defined by $\Gamma^{i+1}\Theta^i=0$ and $\Delta^{i+1}\Gamma^i=0$. Also, the differential d_3 is a tertiary operation

$$\chi^i\colon \mathrm{Ker}\,\Phi^i \longrightarrow \mathrm{Coker}\,\Psi^i.$$

Then the theorem of J. C. McClendon [12, Theorem 5.1] is stated as follows:

PROPOSITION 1.1. *Let X be a CW-complex of dimension less than $n+6$ and let $n>7$. If $g\colon X\to RP^\infty$ has a lifting to RP^n, then*

(1) *$[X, RP^n; g]$ has a natural abelian group structure and*

(2) *there exists a decreasing filtration of $[X, RP^\infty; g]$:*

$$[X, RP^n; g] = F_0 \supset F_1 \supset F_2 \supset F_3 \supset 0,$$

such that

$$F_0/F_1 = \mathrm{Ker}\,\chi^{n+1}, \qquad F_1/F_2 = \mathrm{Ker}\,\Psi^{n+1},$$

$$F_2/F_3 = \mathrm{Coker}\,\Phi^n, \qquad F_3 = \mathrm{Coker}\,\chi^n.$$

§2. The cohomology of $L^n(p)^*$

The purpose of this section is to study the cohomology groups $H^i(L^n(p)^*; G)$ of the reduced symmetric product $L^n(p)^*$ of the lens space $L^n(p)$ mod p, where p is an odd prime. Here the coefficient G is either Z, Z_2, Z_3 or the local systems $\underline{Z}$, $\underline{Z}_3$ induced from the double covering $\pi\colon L^n(p)\times L^n(p)-\Delta\to L^n(p)^*$. We always use the Bockstein exact sequences

$$\begin{array}{l}\cdots \longrightarrow H^{i-1}(\ ;Z_q) \xrightarrow{\delta_q} H^i(\ ;Z) \xrightarrow{\times q} H^i(\ ;Z) \xrightarrow{\rho_q} H^i(\ ;Z_q) \longrightarrow \cdots,\\ \cdots \longrightarrow H^{i-1}(\ ;\underline{Z}_q) \xrightarrow{\delta_q} H^i(\ ;\underline{Z}) \xrightarrow{\times q} H^i(\ ;\underline{Z}) \xrightarrow{\rho_q} H^i(\ ;\underline{Z}_q) \longrightarrow \cdots,\end{array} \tag{2.1}$$

associated with $0\to Z \xrightarrow{\times q} Z \xrightarrow{\rho_q} Z_q\to 0$.

Let x and y be the generators of $H^2(L^n(p); Z)=Z_p$ and $H^1(L^n(p); Z_p)=Z_p$, respectively, such that $\delta_p y=x$. Denote $\rho_p x$ by the same symbol x. Then the mod p cohomology ring of $L^n(p)$ is given by

$$H^*(L^n(p); Z_p) = \Lambda(y)\otimes Z_p[x]/(x^{n+1}), \tag{2.2}$$

where $\Lambda(y)$ denotes the exterior algebra on y; and the integral cohomology is given by

$$(2.3)\quad H^i(L^n(p); Z) = \begin{cases} Z, & i = 0,\ 2n+1, \\ Z_p \text{ generated by } x^{i/2}, & i \equiv 0(2),\ 0 < i \le 2n, \\ 0, & \text{otherwise,} \end{cases}$$

where $H^{2n+1}(L^n(p); Z)$ is generated by the cohomology fundamental class $[L^n(p)]$, and the relation $\rho_p[L^n(p)] = yx^n$ holds.

The next lemma is an immediate result of [16, Proposition 2.9] and (2.2–3).

LEMMA 2.4. *The* mod 2 *cohomology groups of $L^n(p)^*$ are given by*

$$H^i(L^n(p)^*; Z_2) = \begin{cases} Z_2 & \text{for } 0 \le i \le 2n+1, \\ 0 & \text{otherwise.} \end{cases}$$

COROLLARY 2.5. *The cohomology groups $H^i(L^n(p)^*; Z)$ and $H^i(L^n(p)^*; \underline{Z})$ are finite and have no 2-torsions for $i > 2n+1$.*

For an automorphism σ of the group G, G^σ denotes the subgroup of the invariant elements with respect to σ. By using this corollary, the applications of the Serre spectral sequence of the fibration $L^n(p) \times L^n(p) - \Delta \xrightarrow{\pi} L^n(p)^* \to RP^\infty$ and its twisted version (see [12, § 1]) show the following

LEMMA 2.6. *Both homomorphisms*

$$\pi^*: H^i(L^n(p)^*; Z\ (or\ Z_3)) \longrightarrow H^i(L^n(p) \times L^n(p) - \Delta; Z\ (or\ Z_3))^{t^*} \quad for\ \ i > 2n+1,$$

$$\pi^*: H^i(L^n(p)^*; \underline{Z}\ (or\ \underline{Z}_3)) \longrightarrow H^i(L^n(p) \times L^n(p) - \Delta; Z\ (or\ Z_3))^{-t^*} \quad for\ \ i > 2n+1,$$

are isomorphisms, where t is the involution transposing the factors.

Hereafter we identify $H^i(L^n(p)^*; Z)$ and $H^i(L^n(p)^*; \underline{Z})$ with $H^i(L^n(p) \times L^n(p) - \Delta; Z)^{t^*}$ and $H^i(L^n(p) \times L^n(p) - \Delta; Z)^{-t^*}$ for $i > 2n+1$, respectively. Consider the Thom isomorphism

$$\phi: H^i(L^n(p); Z) \xrightarrow{\cong} H^{2n+1+i}(L^n(p) \times L^n(p), L^n(p) \times L^n(p) - \Delta; Z),$$

$$\phi(x^j) = U \cup (1 \times x^j), \quad \text{if} \quad 2j = i,\ 0 < j \le n,$$

where the Thom class $U \in H^{2n+1}(L^n(p) \times L^n(p), L^n(p) \times L^n(p) - \Delta; Z) = Z$ is the generator. The Thom isomorphism and the cohomology exact sequence of the pair $(L^n(p) \times L^n(p), L^n(p) \times L^n(p) - \Delta)$ lead to the following

LEMMA 2.7. *The homomorphism*

$$i^*: H^{2k}(L^n(p) \times L^n(p); Z) \longrightarrow H^{2k}(L^n(p) \times L^n(p) - \Delta; Z), \quad 4n+2 > 2k > 2n+1,$$

is an isomorphism and the sequence

$$0 \longrightarrow Z_p \xrightarrow{j^*} H^{2k+1}(L^n(p) \times L^n(p); Z) \xrightarrow{i^*} H^{2k+1}(L^n(p) \times L^n(p) - \Delta; Z) \longrightarrow 0, \quad 2k+1 > 2n+1,$$

is exact, where i and j are the natural inclusions.

Moreover, the action of t^* on $H^*(L^n(p)\times L^n(p), L^n(p)\times L^n(p)-\Delta; Z)$ is well-known [15, p. 305], and is given by

(2.8) $\quad t^*a = -a \quad$ *for* $\quad a \in H^*(L^n(p) \times L^n(p), L^n(p) \times L^n(p) - \Delta; Z).$

LEMMA 2.9. *For* $i<2n+1$,

$$H^{4n+2-i}(L^n(p) \times L^n(p); Z)/\mathrm{Ker}\, i^* = \begin{cases} Z_p^{2j} & \text{for } i = 4j, \\ Z_p^{2j+1} & \text{for } i = 4j+1, 4j+2, \\ Z_p^{2j+2} & \text{for } i = 4j+3, \end{cases}$$

(G^k denotes the direct sum of k-copies of G), generated by the set $A \cup B$ given as follows:

$$A = \begin{cases} \{x^{n-k} \times x^{n+1-2j+k} + x^{n+1-2j+k} \times x^{n-k} \mid 0 \le k \le j-1\}, & i = 4j, \\ \{x^{n-k} \times x^{n-2j+k} + x^{n-2j+k} \times x^{n-k},\ x^{n-j} \times x^{n-j} \mid 0 \le k \le j-1\}, & i = 4j+2, \\ \{\delta_p(yx^{n-k} \times yx^{n-2j-1+k} - yx^{n-2j-1+k} \times yx^{n-k}) \mid 0 \le k \le j\}, & i = 4j+1, \\ \{\delta_p(yx^{n-k} \times yx^{n-2j-2+k} - yx^{n-2j-2+k} \times yx^{n-k}) \mid 0 \le k \le j\}, & i = 4j+3; \end{cases}$$

$$B = \begin{cases} \{x^{n-k} \times x^{n+1-2j+k} - x^{n+1-2j+k} \times x^{n-k} \mid 0 \le k \le j-1\}, & i = 4j, \\ \{x^{n-k} \times x^{n-2j+k} - x^{n-2j+k} \times x^{n-k} \mid 0 \le k \le j-1\}, & i = 4j+2, \\ \{\delta_p(yx^{n-k} \times yx^{n-2j-1+k} + yx^{n-2j-1+k} \times yx^{n-k}) \mid 1 \le k \le j\}, & i = 4j+1, \\ \{\delta_p(yx^{n-k} \times yx^{n-2j-2+k} + yx^{n-2j-2+k} \times yx^{n-k}),\ \delta_p(yx^{n-j-1} \times yx^{n-j-1}) \mid 1 \le k \le j\}, & i = 4j+3. \end{cases}$$

If we notice that

$$j^*U = \pm(1 \times [L^n(p)] - [L^n(p)] \times 1 + \sum_{i=1}^{[n/2]} \delta_p(yx^{n-i} \times yx^{i-1} +$$

$$yx^{i-1} \times yx^{n-i}) + \{\delta_p(yx^{[n/2]} \times yx^{[n/2]})\}),$$

(the term in the bracket { } is present only when n is odd), then the proof of this lemma is a simple calculation.

By identifying $H^{4n+2-i}(L^n(p) \times L^n(p); Z)/\mathrm{Ker}\, i^*$ with $H^{4n+2-i}(L^n(p) \times L^n(p) - \Delta; Z)$ by i^* for $i < 2n+1$, the integral cohomology group and the cohomology group with coefficients in $\underline{Z}$ of $L^n(p)^*$ are determined by Lemmas 2.6–9.

PROPOSITION 2.10. *Let* $i < 2n+1$. *Then*

$$H^{4n+2-i}(L^n(p)^*; Z) = \begin{cases} Z_p^j & \text{for } i = 4j, \\ Z_p^{j+1} & \text{for } i = 4j+1, 4j+2, 4j+3, \end{cases}$$

generated by A, and

$$H^{4n+2-i}(L^n(p)^*; \underline{Z}) = \begin{cases} Z_p^j & \text{for } i = 4j, 4j+1, 4j+2, \\ Z_p^{j+1} & \text{for } i = 4j+3, \end{cases}$$

generated by B.

As for the cohomology groups $H^i(L^n(p)^*; Z_3)$ and $H^i(L^n(p)^*; \underline{Z}_3)$, it follows that

LEMMA 2.11. *The following relations hold.*
(1) *If* $p \neq 3$, *then*

$$H^t(L^n(p)^*; Z_3) = 0, \qquad H^t(L^n(p)^*; \underline{Z}_3) = 0 \qquad \text{for } t > 2n+1.$$

(2) *If* $p=3$, *then*

$$H^{4n+1}(L^n(3)^*; Z_3) = Z_3 \text{ generated by } yx^n \times x^n + x^n \times yx^n,$$

$$H^{4n}(L^n(3)^*; Z_3) = Z_3 + Z_3 \text{ generated by } \{yx^n \times yx^{n-1} - yx^{n-1} \times yx^n, \; x^n \times x^n\},$$

$$H^{4n+1}(L^n(3)^*; \underline{Z}_3) = 0, \qquad H^{4n}(L^n(3)^*; \underline{Z}_3) = 0,$$

$$H^{4n-1}(L^n(3)^*; \underline{Z}_3) = Z_3 \text{ generated by } x^n \times yx^{n-1} - yx^{n-1} \times x^n$$

$$= yx^n \times x^{n-1} - x^{n-1} \times yx^n.$$

§3. Proof of Theorem A

It is known that $L^n(p)$ is embedded in R^m for $m \geq 3(2n+1)/2$, (cf. e.g., [13, Theorem 1.1]). We prove (4) and (5) for $p=3$ only. The others are obtained easily by the same way.

PROOF OF (4) FOR $p=3$. The group $[L^n(3) \subset R^{4n-2}] = [L^n(3)^*, RP^{4n-3}; g]$ in the introduction is clearly isomorphic to $[L^n(3)^*, (RP^{4n-3})^2; f]$, where $f: L^n(3)^* \to RP^{4n-3}$ is a fixed lifting of $g: L^n(3)^* \to RP^\infty$. Therefore

$$[L^n(3) \subset R^{4n-2}] \approx [L^n(3)^*, E_4; f]$$

by the dimensional reason. By Lemma 2.4, the homotopy exact sequence of fibrations p_i $(i=2, 3, 4)$ in §1 induces isomorphisms

$$[L^n(3)^*, E_4; f] \xrightarrow[\approx]{p_{4\#}} [L^n(3)^*, E_3; f] \xrightarrow[\approx]{p_{3\#}} [L^n(3)^*, E_2; f]$$

and an exact sequence

$$H^{4n-4}(L^n(3)^*; Z) \xrightarrow{\Theta^{4n-3}} H^{4n}(L^n(3)^*; Z_3) \xrightarrow{i_\#} [L^n(3)^*, E_2; f]$$

$$\xrightarrow{p_{2\#}} H^{4n-3}(L^n(3)^*; Z) \xrightarrow{\Theta^{4n-2}} H^{4n+1}(L^n(3)^*; Z_3).$$

Here $\Theta^i = \mathscr{P}_3^1\rho_3$ for $i=4n-2, 4n-3$ by Proposition 1.1.

To determine Θ^i, consider the commutative diagram

$$\begin{array}{ccc}
H^i(L^n(3)^*; Z) & \xrightarrow{\Theta^{i+1}=\mathscr{P}_3^1\rho_3} & H^{i+4}(L^n(3)^*; Z_3) \\
\approx\big\downarrow \pi^* & & \approx\big\downarrow \pi^* \\
H^i(L^n(3) \times L^n(3) - \Delta; Z)^{t^*} & \xrightarrow{\mathscr{P}_3^1\rho_3} & H^{i+4}(L^n(3) \times L^n(3) - \Delta; Z_3)^{t^*} \\
\approx\big\uparrow i^* & & \big\uparrow i^* \\
(H^i(L^n(3) \times L^n(3); Z)/\mathrm{Ker}\, i^*)^{t^*} & \xrightarrow{\mathscr{P}_3^1\rho_3} & (H^{i+4}(L^n(3) \times L^n(3); Z_3)/\mathrm{Ker}\, i^*)^{t^*}.
\end{array}$$

In this diagram, π^*'s are isomorphisms by Lemma 2.6 and i^* in the left hand side is an isomorphism by Lemma 2.7 and (2.8). By the use of this diagram, Proposition 2.10 and Lemma 2.11, a simple calculation yields that

$$\mathrm{Ker}\, \Theta^{4n-2} = \begin{cases} Z_3 + Z_3 \ \textit{generated by} \ \{\delta_3(yx^n \times yx^{n-3} - yx^{n-3} \times yx^n), \\ \qquad \delta_3(yx^{n-1} \times yx^{n-2} - yx^{n-2} \times yx^{n-1})\}, \quad n \equiv 2(3), \\ Z_3 \ \textit{generated by} \ \delta_3(yx^n \times yx^{n-3} - yx^{n-3} \times yx^n) + \\ \qquad \delta_3(yx^{n-1} \times yx^{n-2} - yx^{n-2} \times yx^{n-1}), \quad n \not\equiv 2(3); \end{cases}$$

$$\operatorname{Coker}\Theta^{4n-3}=\begin{cases} Z_3+Z_3 \text{ generated by } \{yx^n\times yx^{n-1}-yx^{n-1}\times yx^n, \\ \qquad\qquad x^n\times x^n\}, & n\equiv 2(3), \\ Z_3 \text{ generated by } yx^n\times yx^{n-1}-yx^{n-1}\times yx^n, \\ & n\not\equiv 2(3). \end{cases}$$

This result and the above exact sequence give rise to the exact sequences

$$0\longrightarrow Z_3+Z_3\xrightarrow{i_*}[L^n(3)^*, E_2; f]\xrightarrow{p_{2*}}Z_3+Z_3\longrightarrow 0, \qquad n\equiv 2(3),$$

$$0\longrightarrow Z_3\xrightarrow{i_*}[L^n(3)^*, E_2; f]\xrightarrow{p_{2*}}Z_3\longrightarrow 0, \qquad n\not\equiv 2(3).$$

To consider the group extensions of these exact sequences, let

$$\Phi(3, 1)\colon \operatorname{Ker}\Theta^{4n-2}\longrightarrow \operatorname{Coker}\Theta^{4n-3}$$

be the homomorphism defined by

$$\Phi(3, 1)(a)=b, \qquad i_*(b)=3p_{2*}^{-1}(a).$$

LEMMA 3.1. $\qquad \Phi(3, 1)=\mathscr{P}_3^1\delta_3^{-1}.$

PROOF. Let $p_2'\colon E_2'\to K(Z, 4n-3)$ be the principal fibration with classifying map $\mathscr{P}_3^1\rho_3\colon K(Z, 4n-3)\to K(Z_3, 4n+1)$ and consider the commutative diagram of fibrations in the category $\mathscr{X}_{RP^{4n-3}}$ (see [11, 1]).

$$\begin{array}{ccc}
RP^{4n-3}\times K(Z_3, 4n) & \subset & RP^{4n-3}\times K(Z_2, 4n-2)\times K(Z_2, 4n)\times K(Z_3, 4n) \\
\downarrow & & \downarrow \\
RP^{4n-3}\times E_2' & \subset & E_2 \\
\downarrow{\scriptstyle 1\times p_2'} & & \downarrow{\scriptstyle p_2} \\
RP^{4n-3}\times K(Z, 4n-3) & = & E_1 \\
\downarrow{\scriptstyle 1\times\mathscr{P}_3^1\rho_3} & & \downarrow{\scriptstyle \Theta^{4n-2}} \\
RP^{4n-3}\times K(Z_3, 4n+1) & \subset & RP^{4n-3}\times K(Z_2, 4n-1)\times K(Z_2, 4n+1) \\
 & & \times K(Z_3, 4n+1).
\end{array}$$

Since $H^i(L^n(3)^*; Z_2)=0$ for $i>2n+1$ by Lemma 2.4, the homotopy exact sequences and the five lemma yield a commutative diagram of exact sequences

$$\begin{array}{ccccc}
 & & H^{4n-4}(L^n(3)^*;Z) & & \\
 & & \downarrow & & \\
 & & H^{4n}(L^n(3)^*;Z_3) & & \\
 & \swarrow & \downarrow & \searrow & \\
[L^n(3)^*, E_2'] & \approx & [L^n(3)^*, RP^{4n-3}\times E_2';f] & \approx & [L^n(3)^*;E_2;f] \\
 & \searrow & \downarrow & \swarrow & \\
 & & H^{4n-3}(L^n(3)^*;Z) & & \\
 & & \downarrow & & \\
 & & H^{4n+1}(L^n(3)^*;Z_3). & &
\end{array}$$

Considering the left exact sequence, we can easily verify that $\Phi(3, 1)$ coincides with $\Phi(3, 1)$ in [10, 1]. By [10, Corollary 3.7. Case II], we have $\Phi(3, 1)=\mathscr{P}_3^1\delta_3^{-1}$.

This lemma shows the relations

$$\begin{aligned}
&\Phi(3, 1)(\delta_3(yx^n\times yx^{n-3}-yx^{n-3}\times yx^n)) \\
&\qquad=(n-3)(yx^n\times yx^{n-1}-yx^{n-1}\times yx^n), \\
&\Phi(3, 1)(\delta_3(yx^{n-1}\times yx^{n-2}-yx^{n-2}\times yx^{n-1})) \\
&\qquad=(n-2)(yx^{n-1}\times yx^n-yx^n\times yx^{n-1}).
\end{aligned}$$

These relations imply that

$$[L^n(3)\subset R^{4n-2}]=[L^n(3)^*, E_2;f]=\begin{cases} Z_3+Z_3+Z_9 & \text{for } \ n\equiv 2(3), \\ Z_9 & \text{for } \ n\not\equiv 2(3). \end{cases}$$

Proof of (5) for $p=3$. By the same way as in the proof of (4) for $p=3$, there are an isomorphism

$$[L^n(3)\subset R^{4n-3}]=[L^n(3)^*, E_2;f],$$

and an exact sequence

$$\begin{aligned}
&H^{4n-5}(L^n(3)^*;\underline{Z})\xrightarrow{\Theta^{4n-4}=\mathscr{P}_3^1\rho_3} H^{4n-1}(L^n(3)^*;\underline{Z}_3)\longrightarrow \\
&\qquad [L^n(3)^*, E_2;f]\longrightarrow H^{4n-4}(L^n(3)^*;\underline{Z})\xrightarrow{\Theta^{4n-3}} H^{4n}(L^n(3)^*;\underline{Z}_3).
\end{aligned}$$

Since $H^{4n-4}(L^n(3)^*;\underline{Z})=Z_3$ and $H^{4n}(L^n(3)^*;\underline{Z}_3)=0$ by Proposition 2.10 and Lemma 2.11, it is sufficient to show that $\Theta^{4n-4}=\mathscr{P}_3^1\rho_3$ is an epimorphism. Consider the diagram

$$\begin{array}{ccc} H^{4n-5}(L^n(3)^*;\ \underline{Z}) & \xrightarrow{\mathscr{P}^1_3\rho_3} & H^{4n-1}(L^n(3)^*;\ \underline{Z}_3) \\ \approx\downarrow\pi^* & & \approx\downarrow\pi^* \\ H^{4n-5}(L^n(3)\times L^n(3)-\Delta;\ Z)^{-t^*} & \xrightarrow{\mathscr{P}^1_3\rho_3} & H^{4n-1}(L^n(3)\times L^n(3)-\Delta;\ Z_3)^{-t^*} \\ \approx\uparrow i^* & & \uparrow i^* \\ (H^{4n-5}(L^n(3)\times L^n(3);\ Z)/\mathrm{Ker}\, i^*)^{-t^*} & \xrightarrow{\mathscr{P}^1_3\rho_3} & (H^{4n-1}(L^n(3)\times L^n(3);\ Z_3)/\mathrm{Ker}\, i^*)^{-t^*}. \end{array}$$

Here π^*'s are isomorphisms by Lemma 2.6 and i^* in the left hand side is an isomorphism by Lemma 2.7, and the last two $\mathscr{P}^1_3$'s are the ordinary reduced power operations mod 3 and the first $\mathscr{P}^1_3$ is the twisted one (see Proposition 1.1). By using Proposition 2.10, there are relations

$$\mathscr{P}^1_3\rho_3(\delta_3(yx^{n-1}\times yx^{n-3}+yx^{n-3}\times yx^{n-1})) = (2n-5)(x^n\times yx^{n-1}-yx^{n-1}\times x^n),$$

$$\mathscr{P}^1_3\rho_3(\delta_3(yx^{n-2}\times yx^{n-2})) = (2-n)(x^n\times yx^{n-1}-yx^{n-1}\times x^n).$$

If $n-2\equiv 0(3)$, then $2n-5\not\equiv 0(3)$. Hence Θ^{4n-4} is an epimorphism by Lemma 2.11.

§4. The cohomology of $(RP^n)^*$ and $(CP^n)^*$

This section is devoted to determine some cohomology groups of $(RP^n)^*$ and $(CP^n)^*$.

Let F denote the real field R or the complex field C and let d be 1 or 2 according as $F=R$ or C, and let $G_{n+1,2}(F)$ denote the Grassmann manifold of 2-planes in F^{n+1}. The cohomology ring of $G_{n+1,2}(F)$ is well-known and is given as follows:

$$\text{(4.1)}\quad H^*(G_{n+1,2}(F);\ G) = G[x,\ y]/(a_n,\ a_{n+1}) \qquad (G=Z_2 \ \text{ if } \ F=R, \ =Z \ \text{ if } \ F=C),$$

where $\deg x=d$, $\deg y=2d$ *and* $a_r=\sum_i\binom{r-i}{i}x^{r-2i}y^i$ $(r=n,\ n+1)$. *Moreover, there are relations*

$$x^{2^t}y^{n-1-i}=0 \quad \textit{if} \quad i\neq 2^t-1 \qquad \textit{for some} \quad t, \ \text{(cf. [5, Corollary 4.1])}$$

$$x^{2^{r+1}-1}=0,\ x^{2^{r+1}-2}y^s\neq 0 \qquad \textit{for} \quad n=2^r+s\ (0\leq s<2^r).$$

The mod 2 *cohomology ring of* $G_{n+1,2}(C)$ *is given by*

$$H^*(G_{n+1,2}(C);\ Z_2) = Z_2[x,\ y]/(a_n,\ a_{n+1}),$$

where x, y and a_r ($r=n$, $n+1$) are the mod 2 *reduction of the same symbols in the integral cohomology. Further, there is a relation*

$$Sq^d x = xy.$$

The last relation for $F=R$ and the induction lead to the following lemma. Details will be omitted.

LEMMA 4.2. *There are the following relations in $H^*(G_{n+1,2}(R); Z_2)$.*

(1) $Sq^2 y^t = t y^{t+1} + \binom{t}{2} x^2 y^t$.

(2) $Sq^3 y^t = \alpha_t x^3 y^t$,

$$\alpha_t = \sum_{0<i<t} \binom{i}{2} \equiv \begin{cases} 0(2) & \text{for} \quad t \not\equiv 3(4), \\ 1(2) & \text{for} \quad t \equiv 3(4). \end{cases}$$

(3) $Sq^4 y^t = \binom{t}{2} y^{t+2} + \alpha_t x^2 y^{t+1} + \beta_t x^4 y^t$,

$$\beta_t = \sum_{0<i<t} \alpha_t \equiv \begin{cases} 0(2) & \text{for} \quad t \equiv l(8), \quad 0 \leq l \leq 3, \\ 1(2) & \text{for} \quad t \equiv l(8), \quad 4 \leq l \leq 7. \end{cases}$$

Case I. $(RP^n)^*$.

The mod 2 cohomology ring of $(RP^n)^*$ is investigated by S. Feder [4], [5] and D. Handel [8] and is given as follows:

(4.3) *$(RP^n)^*$ has the homotopy type of a $(2n-1)$-dimensional closed manifold and $H^*((RP^n)^*; Z_2)$ has $\{1, v\}$ as a basis of an $H^*(G_{n+1,2}(R); Z_2)$-module, where v is the first Stiefel-Whitney class of the double covering $RP^n \times RP^n - \Delta \to (RP^n)^*$ and the ring structure is given by the relation*

$$v^2 = vx.$$

The group structure of $H^t((RP^n)^*; Z_2)$ and its basis for $2n-4 \leq t \leq 2n-1$ are determined by the Poincaré duality and are given in [19, (6.3)] and [19, (8.3)]. By the same way, we have

(4.4) *Let $n=2^r+s$, $2<s<2^r$. Then the* mod 2 *cohomology groups $H^t((RP^n)^*; Z_2)$ for $2n-8 \leq t \leq 2n-5$ are given in the table below.*

t	$H^t((RP^n)^*; Z_2)$	*basis*
$2n-5$	Z_2^5	$x^{2^{r+1}-5+2i} y^{s-i} (i=0, 1)$, $vx^{2^{r+1}-6+2i} y^{s-i} (0 \leq i \leq 2)$
$2n-6$	Z_2^6	$x^{2^{r+1}-6+2i} y^{s-i} (0 \leq i \leq 2)$, $vx^{2^{r+1}-7+2i} y^{s-i} (0 \leq i \leq 2)$
$2n-7$	Z_2^7	$x^{2^{r+1}-7+2i} y^{s-i} (0 \leq i \leq 2)$, $vx^{2^{r+1}-8+2i} y^{s-i} (0 \leq i \leq 3)$
$2n-8$	Z_2^8	$x^{2^{r+1}-8+2i} y^{s-i} (0 \leq i \leq 3)$, $vx^{2^{r+1}-9+2i} y^{s-i} (0 \leq i \leq 3)$

Now, $H^*((RP^n)^*; \underline{Z})$ and $H^*((RP^n)^*; \underline{Z}_3)$ are the cohomology with coefficients in the local system on $(RP^n)^*$ determined by $v \in H^1((RP^n)^*; Z_2)$.

(4.5) ([9, p. 481]) *The groups* $H^t((RP^n)^*; Z)$ *and* $H^t((RP^n)^*; \underline{Z})$ *are 2-primary groups for* $n<t<2n-1$.

Consider the Bockstein exact sequences (2.1) for $q=2$ and for $(RP^n)^*$. Then there are relations

$$(4.6) \qquad \rho_2\delta_2 = Sq^1, \qquad \rho_2\tilde{\delta}_2 = Sq^1 + v.$$

By (4.4–6), we can easily verify the following results.

LEMMA 4.7. *Let* $n\equiv 0(2)$, $n=2^r+s$ $(3\leq s<2^r)$. *Then we have*

$H^{2n-5}((RP^n)^*; Z) = Z_2^2$ *generated by* $\{\delta_2(vx^{2^{r+1}-5}y^{s-1}),$
$\delta_2(x^{2^{r+1}-4}y^{s-1})\}$,

$H^{2n-6}((RP^n)^*; Z) = Z_2^4$ *generated by* $\{\delta_2(vx^{2^{r+1}-8}y^s), \delta_2(x^{2^{r+1}-7}y^s),$
$\delta_2(vx^{2^{r+1}-4}y^{s-2}), \delta_2(x^{2^{r+1}-3}y^{s-2})\}$,

$\rho_2H^{2n-7}((RP^n)^*; Z) = Z_2^3$ *generated by* $\{vx^{2^{r+1}-6}y^{s-1}, x^{2^{r+1}-5}y^{s-1},$
$vx^{2^{r+1}-2}y^{s-3}\}$;

$H^{2n-4}((RP^n)^*; \underline{Z}) = Z_2^2$ *generated by* $\{\tilde{\delta}_2(x^{2^{r+1}-5}y^s),$
$\tilde{\delta}_2(x^{2^{r+1}-3}y^{s-1})\}$,

$H^{2n-5}((RP^n)^*; \underline{Z}) = Z_2^3$ *generated by* $\{\tilde{\delta}_2(x^{2^{r+1}-6}y^s),$
$\tilde{\delta}_2(x^{2^{r+1}-4}y^{s-1}), \tilde{\delta}_2(x^{2^{r+1}-2}y^{s-2})\}$,

$H^{2n-6}((RP^n)^*; \underline{Z}) = Z_2^3$ *generated by* $\{\tilde{\delta}_2(x^{2^{r+1}-7}y^s),$
$\tilde{\delta}_2(x^{2^{r+1}-5}y^{s-1}), \tilde{\delta}_2(x^{2^{r+1}-3}y^{s-2})\}$,

$H^{2n-7}((RP^n)^*; \underline{Z}) = Z_2^4$ *generated by* $\{\tilde{\delta}_2(x^{2^{r+1}-8}y^s), \tilde{\delta}_2(x^{2^{r+1}-6}y^{s-1}),$
$\tilde{\delta}_2(x^{2^{r+1}-4}y^{s-2}), \tilde{\delta}_2(x^{2^{r+1}-2}y^{s-3})\}$;

$H^{2n-1}((RP^n)^*; Z_3) = Z_3, \qquad H^{2n-1}((RP^n)^*; \underline{Z}_3) = 0.$

Case II. $(CP^n)^*$.

The integral and the mod 2 cohomology of $(CP)^{n*}$ are investigated by S. Feder [5] and the author [18], and are given as follows:

(4.8) *$(CP^n)^*$ has the homotopy type of an unorientable $(4n-2)$-dimensional closed manifold and $H^*((CP^n)^*; Z_2)$ has $\{1, v, v^2\}$ as basis of an $H^*(G_{n+1,2}(C); Z_2)$-module and $H^*((CP^n)^*; Z)$ has $\{1, u\}$ as generators of an $H^*(G_{n+1,2}(C); Z)$-module, where v is the first Stiefel-Whitney class of the double covering $CP^n \times CP^n - \Delta \to (CP^n)^*$ and $u = \delta_2 v$. The ring structures are given by the relations*

$$v^3 = vx, \qquad u^2 = ux.$$

Then the integral and the mod 2 cohomology groups of $(CP^n)^*$ are given by the following.

(4.9) *Let $n = 2^r + s$ $(0 < s < 2^r)$. Then we have*

t	$H^t((CP^n)^*; Z_2)$	basis
$4n-2$	Z_2	$v^2x^{2^{r+1}-2}y^s$
$4n-3$	Z_2	$vx^{2^{r+1}-2}y^s$
$4n-4$	$Z_2 + Z_2$	$x^{2^{r+1}-2}y^s$, $v^2x^{2^{r+1}-3}y^s$
$4n-5$	Z_2	$vx^{2^{r+1}-3}y^s$
$4n-6$	$Z_2 + Z_2 + Z_2$	$x^{2^{r+1}-3}y^s$, $v^2x^{2^{r+1}-4}y^s$, $v^2x^{2^{r+1}-2}y^{s-1}$
$4n-7$	$Z_2 + Z_2$	$vx^{2^{r+1}-4}y^s$, $vx^{2^{r+1}-2}y^{s-1}$

$H^{4n-6}((CP^n)^*; Z) = Z + Z_2 + Z_2$ *generated by* $\{x^{2^{r+1}-3}y^s,$

$ux^{2^{r+1}-4}y^s,\ ux^{2^{r+1}-2}y^{s-1}\}$,

$H^i((CP^n)^*; Z) = 0$ *for odd* i.

Using the Poincaré duality $H^{4n-2-i}((CP^n)^*; \underline{Z}) = H_i((CP^n)^*; Z)$ and the Bockstein exact sequence (2.1), we can show the following:

(4.10) *Let $n = 2^r + s$ $(0 < s < 2^r)$.*

$H^{4n-4}((CP^n)^*; \underline{Z}) = Z$ *generated by* a *with*

$$\rho_2(a) = v^2x^{2^{r+1}-3}y^s + x^{2^{r+1}-2}y^s,$$

$H^{4n-5}((CP^n)^*; \underline{Z}) = Z_2$ *generated by* $\rho_2^{-1}(vx^{2^{r+1}-3}y^s)$,

$H^{4n-6}((CP^n)^*; \underline{Z}) = Z + Z$ *generated by* $\{b, b'\}$ *with*

$$\rho_2(b) = v^2x^{2^{r+1}-4}y^s + x^{2^{r+1}-3}y^s,$$

$$\rho_2(b') = v^2x^{2^{r+1}-2}y^{s-1},$$

$$H^{4n-7}((CP^n)^*; \underline{Z}) = Z_2 + Z_2 \text{ generated by } \{\rho_2^{-1}(vx^{2^{r+1}-4}y^s),$$
$$\rho_2^{-1}(vx^{2^{r+1}-2}y^{s-1})\};$$

$$H^{4n-2}((CP^n)^*; \underline{Z}_3) = Z_3, \qquad H^{4n-3}((CP^n)^*; \underline{Z}_3) = 0;$$

$$H^{4n-2}((CP^n)^*; Z_3) = 0, \qquad H^{4n-3}((CP^n)^*; Z_3) = 0.$$

§ 5. Proofs of Theorems B and C

PROOF OF THEOREM B. We prove (1) only. The others are similar and will be omitted. By applying Proposition 1.1 for $(RP^n)^*$ and $2n-4$ in place of X and n, respectively, there follows a decreasing filtration

$$[RP^n \subset R^{2n-3}] = F_0 \supset F_1 \supset F_2 \supset F_3 \supset 0$$

such that

$$F_0/F_1 = \mathrm{Ker}\,\chi^{2n-3}, \qquad F_1/F_2 = \mathrm{Ker}\,\Psi^{2n-3},$$

$$F_2/F_3 = \mathrm{Coker}\,\Phi^{2n-4}, \qquad F_3 = \mathrm{Coker}\,\chi^{2n-4},$$

where Φ^i, Ψ^i and χ^i are the secondary and the tertiary operations defined by the homomorphisms

$$\Theta^i\colon H^{i-1}((RP^n)^*; \underline{Z}) \longrightarrow$$
$$H^{i+1}((RP^n)^*; Z_2) \times H^{i+3}((RP^n)^*; Z_2) \times H^{i+3}((RP^n)^*; \underline{Z}_3),$$

$$\Theta^i(a) = \begin{cases} (Sq^2\rho_2 a,\ Sq^4\rho_2 a + v^4\rho_2 a,\ \mathscr{P}_3^1\rho_3 a), & n \equiv 0(4), \\ (Sq^2\rho_2 a,\ Sq^4\rho_2 a,\ \mathscr{P}_3^1\rho_3 a), & n \equiv 2(4); \end{cases}$$

$$\Gamma^i\colon H^i((RP^n)^*; Z_2) \times H^{i+2}((RP^n)^*; Z_2) \times H^{i+2}((RP^n)^*; \underline{Z}_3) \longrightarrow$$
$$H^{i+2}((RP^n)^*; Z_2) \times H^{i+3}((RP^n)^*; Z_2),$$

$$\Gamma^i(a, b, c) = ((Sq^2 + vSq^1 + v^2)a,\ (Sq^2Sq^1 + v^2Sq^1)a + (Sq^1 + v)b);$$

$$\Delta^i\colon H^{i+1}((RP^n)^*; Z_2) \times H^{i+2}((RP^n)^*; Z_2) \longrightarrow H^{i+3}((RP^n)^*; Z_2),$$

$$\Delta^i(a, b) = Sq^2 a + v^2 a + Sq^1 b + vb.$$

Using the results of § 4, we can easily verify that

$$\mathrm{Ker}\,\Theta^{2n-3} = \begin{cases} Z_2, & n \equiv 0(4), \\ 0, & n \equiv 2(4), \end{cases}$$

$$\operatorname{Im}\Gamma^{2n-3}=\operatorname{Ker}\Delta^{2n-2},\qquad \operatorname{Im}\Gamma^{2n-4}=\operatorname{Ker}\Delta^{2n-3},$$

$$\operatorname{Ker}\Gamma^{2n-3}=Z_2+Z_2+Z_2,$$

$$\operatorname{Coker}\Delta^{2n-3}=0,\qquad \operatorname{Im}\Theta^{2n-4}=\begin{cases} Z_2+Z_2+Z_2, & n\equiv 0(4),\\ Z_2+Z_2, & n\equiv 2(8),\\ Z_2, & n\equiv 6(8),\end{cases}$$

$$\operatorname{Coker}\Delta^{2n-4}=0,$$

Hence it follows that

$$\operatorname{Ker}\Phi^{2n-3}=\operatorname{Ker}\Theta^{2n-3},\qquad \operatorname{Ker}\chi^{2n-3}=\operatorname{Ker}\Phi^{2n-3},$$

$$\operatorname{Ker}\Psi^{2n-3}=\operatorname{Ker}\Gamma^{2n-3}/\operatorname{Im}\Theta^{2n-4}=\begin{cases} 0, & n\equiv 0(4),\\ Z_2, & n\equiv 2(8),\\ Z_2+Z_2, & n\equiv 6(8),\end{cases}$$

$$\operatorname{Coker}\Phi^{2n-4}=0,\qquad \operatorname{Coker}\chi^{2n-4}=0.$$

This implies that

$$[RP^n\subset R^{2n-3}]=\begin{cases} Z_2, & n\not\equiv 6(8),\\ Z_2+Z_2, & n\equiv 6(8).\end{cases}$$

REMARK OF THEOREM B. In (3) for $n\equiv 2(4)$, the secondary and the tertiary operations cannot be calculated. Therefore $[RP^n\subset R^{2n-5}]$ for $n\equiv 2(4)$ is not determined and so is $[RP^n\subset R^{2n-i}]$ $(i=3, 4, 5)$ for $n\equiv 1(2)$ by the same reason.

PROOF OF THEOREM C. We can prove (1) only. (2) and (3) are obtained by the same way. By Proposition 1.1, there is a decreasing filtration

$$[CP^n\subset R^{4n-3}]=F_0\supset F_1\supset F_2\supset F_3\supset 0$$

such that

$$F_0/F_1=\operatorname{Ker}\chi^{4n-3},\qquad F_1/F_2=\operatorname{Ker}\Psi^{4n-3},$$

$$F_2/F_3=\operatorname{Coker}\Phi^{4n-4},\qquad F_3=\operatorname{Coker}\chi^{4n-4},$$

where Φ^i, Ψ^i and χ^i are the secondary and the tertiary operations defined by the homomorphisms

$$\Theta^i\colon H^{i-1}((CP^n)^*;\underline{Z})\longrightarrow$$
$$H^{i+1}((CP^n)^*;Z_2)\times H^{i+3}((CP^n)^*;Z_2)\times H^{i+3}((CP^n)^*;\underline{Z}_3),$$

$$\Theta^i(a)=\begin{cases} (Sq^2\rho_2 a,\ Sq^4\rho_2 a+v^4\rho_2 a,\ \mathscr{P}_3^1\rho_3 a), & n\equiv 0(2),\\ (Sq^2\rho_2 a,\ Sq^4\rho_2 a,\ \mathscr{P}_3^1\rho_3 a), & n\equiv 1(2);\end{cases}$$

$$\Gamma^i\colon H^i((CP^n)^*; Z_2) \times H^{i+2}((CP^n)^*; Z_2) \times H^{i+2}((CP^n)^*; \underline{Z}_3) \longrightarrow H^{i+2}((CP^n)^*; Z_2) \times H^{i+3}((CP^u)^*; Z_2),$$

$$\Gamma^i(a, b, c) = ((Sq^2 + vSq^1 + v^2)a, (Sq^2Sq^1 + v^2Sq^1)a + (Sq^1 + v)b);$$

$$\Delta^i\colon H^{i+1}((CP^n)^*; Z_2) \times H^{i+2}((CP^n)^*; Z_2) \longrightarrow H^{i+3}((CP^n)^*; Z_2),$$

$$\Delta^i(a, b) = Sq^2a + v^2a + Sq^1b + vb.$$

By (4.1) and (4.8–10), there are the relations.

$$\operatorname{Ker}\Theta^{4n-3} = Z, \qquad \operatorname{Ker}\Theta^{4n-4} = \begin{cases} 0, & n \equiv 0(2), \\ Z_2, & n \equiv 1(2), \end{cases}$$

$$\operatorname{Im}\Theta^{4n-4} = \begin{cases} Z_2, & n \equiv 0(2), \\ 0, & n \equiv 1(2), \end{cases} \qquad \operatorname{Ker}\Gamma^{4n-3} = Z_2,$$

$$\operatorname{Im}\Gamma^{4n-3} = \operatorname{Ker}\Delta^{4n-2} = 0, \qquad \operatorname{Im}\Gamma^{4n-3} = \operatorname{Ker}\Delta^{4n-3},$$

$$\operatorname{Coker}\Delta^{4n-4} = 0, \qquad \operatorname{Coker}\Delta^{4n-4} = 0.$$

Hence it follows that

$$\operatorname{Ker}\Phi^{4n-3} = \operatorname{Ker}\Theta^{4n-3} = Z, \qquad \operatorname{Ker}\chi^{4n-3} = \operatorname{Ker}\Phi^{4n-3} = Z,$$

$$\operatorname{Ker}\Psi^{4n-3} = \operatorname{Ker}\Gamma^{4n-3}/\operatorname{Im}\Theta^{4n-4} = \begin{cases} 0, & n \equiv 0(2), \\ Z_2, & n \equiv 1(2), \end{cases}$$

$$\operatorname{Coker}\Phi^{4n-3} = 0, \qquad \operatorname{Coker}\chi^{4n-4} = 0.$$

Therefore, if $n\equiv 0(2)$, then $[CP^n \subset R^{4n-3}] = F_0 = Z$, and if $n \equiv 1(2)$, then $0 \to Z_2 \to F_0 \to Z \to 0$ is a short exact sequence. This completes the proof.

REMARK OF THEOREM C. As for $[CP^n \subset R^{4n-i}]$ $(i=4, 5)$ for $n\equiv 1(2)$, the following are verified.

(2)′ $$\#[CP^n \subset R^{4n-4}] = \begin{cases} 2 \text{ or } 4, & n \equiv 1(4), \\ 4 \text{ or } 8, & n \equiv 3(4); \end{cases}$$

(3)′ $[CP^n \subset R^{4n-5}] = Z + Z + G,$

$$\#G = \begin{cases} 1 \text{ or } 2, & n \equiv 1(4), \\ 2 \text{ or } 4, & n \equiv 3(4). \end{cases}$$

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